

Riemannian Geometry

Amy Needham

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1 Introduction

Let M be a smooth manifold (from here we omit the word smooth) and g be a Riemannian metric on M (we later say metric). A Riemannian metric is a choice of inner product on the tangent bundle TM which “varies smoothly” with the point. A choice of coordinate neighbourhood with coordinates $x = (x_i)$ gives us, in coordinates, $g = \sum_{i,j=1}^n g_{ij}(x) dx_i dx_j$ where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. The pair (M, g) is a Riemannian metric.

Example 1.1.

1. $\mathbb{R}^n$ with the Euclidean metric, ie $g_{ij} = \delta_{ij}$ Whitney embedding theorem asserts every smooth manifold embeds into one of these (in fact, for some $n \leq 2 \dim M$) and this induces a choice of Riemannian metric. More generally, if (N, h) is a Riemannian manifold and $F: M \rightarrow N$ is an immersion, then we get an induced Riemannian metric by pullback on M . In particular, this works if F is a diffeomorphism $F: M \rightarrow N$.
2. Let G be a Lie group. For every $x \in G$ we get two diffeomorphisms of G , $L_x: y \mapsto xy$ and $R_x: y \mapsto yx$. The associated Lie algebra is $\mathfrak{g} = T_e G$. A vector field X on G is left-invariant if for every $x \in G$, $dL_x(X) = X$. Any such X arises from tangent vector to the identity, as $X_x = dL_x(X_e)$. This is a bijection (exercise). A Riemannian metric on G is *left invariant* if L_x is an isometry. We can similarly define one of these by transporting an inner product on the fibre of TM over the identity. If the metric is left and right invariant, we say it is *bi-invariant*. It is a fact that every compact Lie group has a bi-invariant metric. We shall see later that we can't drop the compact assumption, for example $SL(2, \mathbb{R})$.

Definition 1.1. Let (M, g) and (N, h) be Riemannian manifold. $f: M \rightarrow N$ is an *isometry* if it is a diffeomorphism and $g = f^*h$

For (M, g) there is a preferred choice of connection on M , namely the Levi Civita connection $\nabla: \Omega^0(TM) \rightarrow \Omega_M^1(TM)$ which is the unique connection such that for all $X, Y, Z \in \text{Vect}(M)$,

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

The first can be written as $dg(X, Y) = g(\nabla X, Y) + g(X, \nabla Y)$ and we can write Γ_{jk}^i for ∇ in local coordinates (x_i) ie

$$\nabla_{\frac{\partial}{\partial x_j}} \left(\frac{\partial}{\partial x_k} \right) = \sum \Gamma_{jk}^i \frac{\partial}{\partial x_i}$$

and the second requirement says this is symmetric in its lower indices. ∇ induces a unique connection on T^*M such that $X\langle\alpha, Y\rangle = \langle\nabla_X \alpha, Y\rangle + \langle\alpha, \nabla_X Y\rangle$. More generally, on a tensor $(TM)^{\otimes p} \otimes (T^*M)^{\otimes q} = \mathcal{T}^{p,q}$ we can induce a unique connection ∇ on $\mathcal{T}^{p,q}$ mapping into $\mathcal{T}^{p,q+1}$ such that

$$\nabla(s_1 \otimes s_2) = (\nabla s_1) \otimes s_2 + s_2 \otimes (\nabla s_2)$$

Therefore the first condition to be compatible with the metric is $\nabla g = 0$.

[LECTURE MISSING]

Lemma 1.1. (*Sectional curvature determines the full Riemannian Curvature*) Let V be a real vector space of dimension at least 2 and $R', R'': V \times V \times V \times V \rightarrow \mathbb{R}$ are linear in each of the four arguments and satisfy the symmetries (i)-(iv) of the Riemannian curvature. Let $K', K'': \text{Gr}(2, V) \rightarrow \mathbb{R}$ be defined in the same way as the sectional curvature, and assume $K' = K''$. Then $R' = R''$.

Proof. By hypothesis,

$$R'(X + Z, Y, X + Z, Y) = R''(X + Z, Y, X + Z, Y)$$

for any $X, Y, Z \in V$. Therefore we have by linearity and symmetries (i), (iii)

$$R'(X, Y, Z, Y) = R''(X, Y, Z, Y)$$

Repeating the same trick with the Y argument, we get

$$R'(X, Y, Z, T) + R'(X, T, Z, Y) = R''(X, Y, Z, T) + R''(X, T, Z, Y)$$

$$R'(X, Y, Z, T) - R''(X, Y, Z, T) = R''(Y, Z, X, T) - R''(Y, Z, X, T)$$

by symmetries (i) and (iii). The left hand side is invariant under cyclic permutation of X, Y, Z , so symmetry (ii) implies

$$3(R'(X, Y, Z, T) - R''(X, Y, Z, T)) = 0$$

so we are done. $\square$

Corollary 1.1.1. (*Constant sectional curvature*) Assume that for any $p \in M$, K_p is constant. Let $R_p^0(X, Y, Z, T) = g_p(X, Z)g_p(Y, T) - g_p(X, T)g_p(Y, Z)$. Then $R_p = K_p R_p^0$ and K_p is smooth in $p \in M$

Proof. This follows from the lemma applied to $R' = KR^0$ and $R'' = R$. K is smooth as it can be written in terms of R and the metric $\square$

Remark 1.1. We can further show that K is constant on all M independent of p if $\dim M \geq 3$. (This requires the second Bianchi identity)

Let $\{e_i\}$ be an orthonormal basis of $T_p M$. Put $R_{ij,kl} = -g(R(e_i, e_j)e_k, e_l)$. Then $K_p = k \in \mathbb{R}$ is independent of $\Pi \in \text{Gr}(2, T_p M)$ holds if and only if $R_{ij,kl} = -k(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$, ie if and only if $R_{ij,ji} = -R_{ij,ij} = k$ and all other components are 0.

It is useful to extract from $R_{ij,kl}$ some simpler quantities

Definition 1.2. The *Ricci curvature* of g at $p \in M$ can be defined as $\text{Ric}_p(X, Y) = \text{tr}(v \mapsto R_p(X, v)Y)$ where $v, X, Y \in T_p M$. Locally, $\text{Ric}_{ij} = \sum_q R_{j,iq}^q = \sum_{p,q} g^{pq} R_{pj, iq}$

Remark 1.2. The third symmetry implies the Ricci curvature is symmetric, and hence equivalent to a quadratic form $\text{Ric}_p(X) = \frac{1}{n-1} \text{Ric}_p(X, X)$

Definition 1.3. $s \in C^\infty(M)$, the *scalar curvature* of g is the trace of Ric with respect to g . If $g_{ij}(p) = \delta_{ij}$, then $s = \sum \text{Ric}_{ii} = \sum R_{ji, ij}$. In general, $s = g^{ij} \text{Ric}_{ij}$

Remark 1.3. Sometimes, one uses the convention $s/n(n-1)$

Remark 1.4. In low dimensions,

- $n = 2$ (surfaces) we have $g = Edu^2 + 2Fdudv + Gdv^2$ and $R_{12,21} = \frac{1}{2}s(EG - F^2)$, and so $\frac{1}{2}s$ is the sectional curvature and we can show this is also the Gaussian curvature, so $R_{12,21}$ is the determinant of the second fundamental form
- $n = 3$. The curvature tensor $R(g)$ is determined by $\text{Ric}(g)$
- $n = 4$. We will consider this case later

If we have constant sectional curvature at p , then using the tensor K from before, $\text{Ric}_p = (n-1)K_p g_p$ and $s(p) = n(n-1)K_p$.

2 Geodesics

Let $E = TM$ and let $\pi: E \rightarrow M$ be the projection. The fibre is $\mathbb{R}^n$. Let $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ be a curve through $p \in M$. We wish to lift γ to E as $\tilde{\gamma}: (-\varepsilon, \varepsilon) \rightarrow E$ which projects down to γ . In coordinates, if $\gamma(t) \in U$ and $\gamma(t) = (x_1(t), \dots, x_n(t))$ are coordinates, then $\tilde{\gamma}(t) = (x(t), \alpha(t))$ where $\alpha: I \rightarrow \mathbb{R}^n$.

Now let ∇ be a connection on M , with coefficients $\Gamma_{jk}^i(x)$ with respect to the local coordinates.

Definition 2.1. A *parallel lift* of γ from $a_0 \in T_{\gamma(0)} M$ is determined locally in U by the linear ODE

$$\dot{a}_i + \sum_{j,k=1}^n \Gamma_{jk}^i a_j \dot{x}_k = 0$$

with initial condition $a(0) = a_i$

A solution to this exists and is unique, by usual ODE theorems. Geometrically, the left hand side of the ODE defines the covariant derivative of a vector field along a curve,

$$\frac{\nabla \tilde{\gamma}}{dt} = (x(t), \dot{a} + \sum \Gamma_{jk}^i a_j \dot{x}_k) : I \rightarrow TM$$

Remark 2.1. It is a fact that $\frac{\nabla}{dt}$ as defined above is a well defined map between vector fields along curves in M , uniquely determined by the following conditions:

1. $\frac{\nabla}{dt}$ is real-linear
2. $\frac{\nabla(f\tilde{\gamma})}{dt} = \frac{\partial f}{\partial t} \tilde{\gamma} + f \frac{\nabla \tilde{\gamma}}{dt}$ where $f : I \rightarrow \mathbb{R}$
3. If $\tilde{\gamma}(t) = X(\gamma(t))$ for some vector field X on U and $\dot{\gamma} = Y(\gamma(t))$, then $\frac{\nabla \tilde{\gamma}}{dt}(t) = (\nabla_Y X)(\gamma(t))$

Now let ∇ be the Levi-Civita connection of some Riemannian metric g on M . Each $\gamma(t)$ has a *canonical lift* $\dot{\gamma}(t) = (x(t), \dot{x}(t))$

Definition 2.2. A curve $\gamma(t)$ in (M, g) is a *geodesic* if the canonical lift is parallel with respect to the Levi-Civita connection

Remark 2.2. In coordinates, $\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0$. This is a non-linear second order ODE, so a unique solution exists locally given initial position and tangent vector

Remark 2.3. We can extend $\dot{\gamma}$ to a C^∞ vector field in some neighbourhood of $p = \gamma(0)$. WLOG $\dot{x}(0) = (1, 0, \dots, 0)$. We assume that $x(p) = 0$. $x_1(t)$ is invertible around $t = 0$ by IFT, so $t = t(x_1)$, and so we can write $x_k = x_k(t(x_1))$ for $k \geq 2$, and we put $\dot{\gamma}(x) = \dot{\gamma}(x_1, x_2(x_1), \dots, x_n(x_1))$. Then $\nabla_{\dot{\gamma}} \dot{\gamma}|_{\gamma(t)} = 0$ if γ is a geodesic.

Proposition 2.1. $|\dot{\gamma}(t)|_g = \text{const}$ for any geodesic γ .

Proof. Use the extension of the vector field $\dot{\gamma}$ around $p = \gamma(0)$. Then $\dot{\gamma}(g(\dot{\gamma}, \dot{\gamma})) = g(\nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma}) + g(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma})$. Then at any $q = \gamma(t)$, $0 = \dot{\gamma}g(\dot{\gamma}, \dot{\gamma}) = \sum \dot{x}_k \frac{\partial}{\partial x_k} g(\dot{\gamma}, \dot{\gamma}) = \frac{\partial}{\partial t} |\dot{\gamma}(t)|_g$, so we are done. $\square$

Example 2.1.

1. $(\mathbb{R}^n, d_2)$ has $\Gamma_{jk}^i = 0$, so the geodesic equations are $\ddot{x}_k = 0$, and geodesics are therefore straight lines.
2. For S^n with the round metric, geodesics are great circles

Fix $p \in M$. If $\gamma(t, a)$ is a geodesic, then $\frac{\partial}{\partial t} \gamma(\lambda t, a) = \lambda \dot{\gamma}(\lambda t, a)$ and the second derivative acquires another λ , so $\gamma(\lambda t, a) = \gamma(t, \lambda a)$ also solves the geodesic equations.

Solutions of ODEs depend smoothly on initial conditions. In particular, the domain of definition depends continuously on a , so $\varepsilon_a \geq \varepsilon' > 0$ where $|a|_g \leq \varepsilon''$, hence $\gamma_p(1, a) = \gamma_p(\varepsilon', \frac{a}{\varepsilon'})$ is well defined for $|a|_g \leq \varepsilon'' \varepsilon' = \varepsilon$.

Definition 2.3. Let (M, g) be a Riemannian manifold and $p \in M$. The *exponential map* at p is

$$\exp_p: a \mapsto \gamma(1, a)$$

whenever defined

Remark 2.4. This is well defined in a neighbourhood of 0 by the previous discussion.

Proposition 2.2. $(d\exp_p)_0 = \text{id}_{T_p M}$

Proof.

$$\begin{aligned} (d\exp_p)_0 a &= \left. \frac{\partial}{\partial t} \exp_p(ta) \right|_{t=0} \\ &= \left. \frac{\partial}{\partial t} \gamma(1, ta) \right|_{t=0} \\ &= \left. \frac{\partial}{\partial t} \gamma(t, a) \right|_{t=0} \\ &= a \end{aligned}$$

□

Corollary 2.0.1. $\exp_p: B_g(0, \varepsilon) \subseteq T_p M \rightarrow U \subseteq M$ is a diffeomorphism for some small $\varepsilon > 0$

Proof. Inverse Mapping Theorem. □

Thus $(\exp_p)^{-1}$ is a chart near $p \in M$ and defines local coordinates around p called the *geodesic coordinates* at p . In these coordinates, $\gamma(t, a) = ta$ for sufficiently small t . Consider the linear isometry $(T_p M, g) \cong (\mathbb{R}^n, d_2)$.

$$f: (r, v) \in (0, \delta) \times S^{n-1} \rightarrow \exp_p(rv) \in M$$

Then $\Sigma_r = f(\{r\} \times S^{n-1}) \subseteq M$ is a *geodesic sphere about p*

Theorem 2.1. (*Gauss Lemma*) $\gamma_p(t, a)$ meets Σ_r orthogonally. Thus $g = dr^2 + h(r, v)$ where for all r , $h(r, v) = g|_{\Sigma_r}$

Proof. We can assume $\delta > 1$. Choose some $X(v) \in \text{Vect}(S^{n-1} \subseteq T_p M)$. Extend $X(v)$ to a vector field on $B \setminus \{0\}$ (the unit ball) independent of r . We define $\tilde{X}(r, v) = rX(v)$ on $B \setminus \{0\}$. We define $Y(f(r, v)) = (d\exp_p)_{rv} \tilde{X}(r, v)$ a vector field on $B' \setminus \{p\}$ where $B' = \exp_p(B)$. We want to show $Y \perp \frac{\partial}{\partial r}$ at each point where Y is defined.

$$\frac{\partial}{\partial r} = \dot{\gamma}_p(t, a) \frac{1}{|a|_{g(p)}}$$

as $\gamma(t, a) = ta$ in geodesic coordinates. $\dot{\gamma}_p(t, a)$ for $|a| = 1$ determines a smooth vector field on $B' \setminus \{p\}$. Recall $\nabla_{\dot{\gamma}}(\dot{\gamma}) = 0$ as γ is a geodesic, so

$$\frac{\partial}{\partial t} g \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = \frac{\partial}{\partial t} g(\dot{\gamma}, \dot{\gamma}) = 0$$

In the limit as $r \rightarrow 0$, we get $g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1$ so one part is done.
Now

$$\nabla_{\dot{\gamma}} Y - \nabla_Y \dot{\gamma} = (df) \left(\nabla_{\frac{\partial}{\partial r}} \tilde{X} - \nabla_{\tilde{X}} \frac{\partial}{\partial r} \right) = (df) \left[\frac{\partial}{\partial r}, \tilde{X} \right] = (df) \frac{\partial}{\partial r} \tilde{X} = Y/r$$

Now

$$\begin{aligned} \frac{\partial}{\partial r} g(Y, \dot{\gamma}) &= g(\nabla_{\dot{\gamma}} Y, \dot{\gamma}) + g(Y, \nabla_{\dot{\gamma}} \dot{\gamma}) \\ &= g(\nabla_{\dot{\gamma}} Y, \dot{\gamma}) \\ &= g(\nabla_Y \dot{\gamma} + Y/r, \dot{\gamma}) \\ &= \frac{1}{r} g(Y, \dot{\gamma}) \end{aligned}$$

using $g(\nabla_Y \dot{\gamma}, \dot{\gamma}) = \frac{1}{2} Y g(\dot{\gamma}, \dot{\gamma}) = 0$. Now if $G = g(Y, \dot{\gamma})$, G satisfies $\frac{\partial}{\partial r} G = G/r$, so G is linear in r , and the derivative with respect to r is therefore independent of r . But $\lim_{r \rightarrow 0} \frac{\partial}{\partial r} G = \lim g(X, \frac{\partial}{\partial r}) = 0$ as $d\exp_p = \text{id}$ at the origin and so $g(Y, \dot{\gamma}) = 0$ $\square$

Let $a \in T_p M$ and assume $|a| < \delta$. Then $g((d\exp_p)_a a, (d\exp_p)_a a) = |a|_g^2 g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = |a|_g^2$

Corollary 2.1.1. $g((d\exp_p)_a a, (d\exp_p)_a a) = g(a, w)$ for any $w \in T_p M$

Proof. By noting that $w = g(w, a)a + v$ where $v \perp a$ $\square$

3 Jacobi Fields

We choose a tangent vector $a_0 \in T_p M$ such that $|a_0|_g < \delta$ is in the domain of the exponential map. We define a curve $a(s) : (-\varepsilon, \varepsilon) \rightarrow T_p M$ such that $a(0) = a_0$ and $\dot{a}(0)$ is linearly independent to $a(0)$. We define $f(t, s) = \exp_p(ta(s))$ for $0 \leq t \leq 1$ and $|s| < \varepsilon$. This defines an embedded local surface around $p \in M$. Then $(d\exp_p)_{a_0} w = \frac{\partial f}{\partial s}(1, 0)$ and more generally, $(d\exp_p)_{ta_0} tw = \frac{\partial f}{\partial s}(t, 0)$