

Tannaka Duality: Reconstructing Groups from Representations

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- 1 What is a Representation?
- 2 Limits of Reconstruction
- 3 More Structure and the Main Result

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2 Limits of Reconstruction

3 More Structure and the Main Result

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- Is there a way to turn group theory into linear algebra?

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Remark

These objects and homomorphisms form a category

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Both have the same character table $\chi_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}$

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- Is there a way to salvage a positive result from question 4 though?

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Remark

These give the category of representations the structure of a Tannakian Category

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Theorem

Given a symmetric monoidal category of representations, $\text{Rep}(G)$, and its forgetful functor, ω , we can reconstruct G as $\underline{\text{Aut}}^{\otimes}(\omega)$

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Examples

An automorphism of ω at a representation X , on a vector space V looks like the following

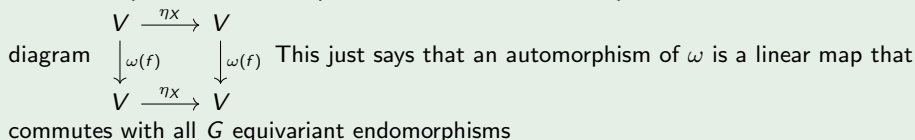
diagram
$$\begin{array}{ccc} V & \xrightarrow{\eta_X} & V \\ \downarrow \omega(f) & & \downarrow \omega(f) \\ V & \xrightarrow{\eta_X} & V \end{array}$$
 This just says that an automorphism of ω is a linear map that commutes with all G equivariant endomorphisms

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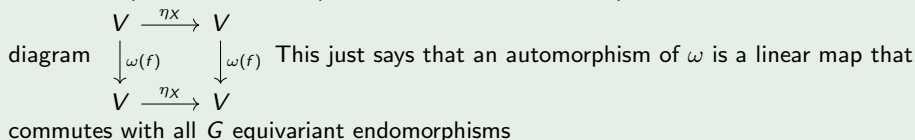
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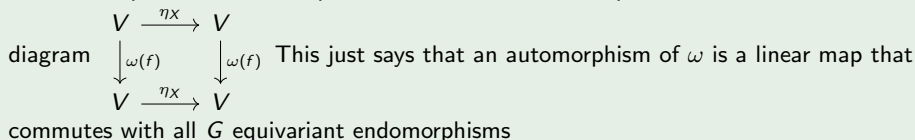
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- Why is this realistic?
- Because by definition of G equivariant, the action of G satisfies it