

# Symplectic Topology

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## 0.1 Outline

After the introduction, the course will follow the following structure

1. Symplectic Linear Algebra and Vector Bundles
2. Neighbourhood Theorems for Symplectic Manifolds
3. Constructions and Examples
4.  $J$ -Holomorphic Curves, Quantum Cohomology and (maybe) Floer Theory

## 1 Symplectic Topology

**Definition 1.1.** A **symplectic manifold** is a pair  $(M, \omega)$  where

1.  $M$  is a smooth manifold
2.  $\omega \in \Omega^2(M)$  satisfies  $d\omega = 0$  and  $\omega|_{T_p M}: T_p M \times T_p M \rightarrow \mathbb{R}$  is a perfect pairing

The perfect pairing gives an isomorphism  $\tilde{\omega}: T_p^* M \rightarrow T_p M$  for all  $p \in M$

**Example 1.1.** We have a couple easy examples

1.  $(\mathbb{R}^{2n}, \omega_{\text{std}} = \sum_{i=1}^n dx_i \wedge dy_i)$
2.  $(\Sigma_g, \omega = d \text{ vol})$

The historical motivation for symplectic manifolds is classical mechanics. If  $M$  is the phase space of some physical system, we want, given a function  $H: M \rightarrow \mathbb{R}$  encoding the energy, a vector field  $X_H \in \Gamma(TM)$  such that trajectories satisfy  $\dot{x}(t) = X_H$ . We set  $X_H = \tilde{\omega}(dH)$  (this solves  $\omega(X_H, \cdot) = dH$ ). The non-degeneracy says that this is possible and the closedness condition says that the mechanics are time independent. We want  $X_H \in \ker dH$  as that is the condition for energy to not change

**Example 1.2.** If we take a particle on a spring,  $M = (\mathbb{R}^2, dp \wedge dq = r dr \wedge d\theta)$  is our phase space with coordinates  $p$  being momentum and  $q$  being position.  $H = \frac{1}{2}(q^2 + p^2)$  (for unit mass). Then  $dH = r$  so  $X_H = \frac{\partial}{\partial \theta}$  so our trajectories orbit at constant speed (in phase space)

**Theorem 1.1.** (Darboux) All symplectic manifolds are locally modeled on the standard structure on even dimensional real space,  $(\mathbb{R}^{2n}, \omega_{std})$

*Proof.* Deferred □

**Theorem 1.2.** (Non-squeezing theorem)

If  $(B_{\mathbb{R}^n}(R), \omega_{std}) \hookrightarrow (B_{\mathbb{R}^2}(r) \times \mathbb{R}^{n-2}, \omega_{std})$  is a symplectic embedding, then  $r > R$

*Proof.* Deferred □

## 2 Symplectic Linear Algebra

Let  $V$  be a (finite dimensional) vector space over  $\mathbb{R}$ .

**Definition 2.1.**  $\omega \in \bigwedge^2(V^*)$  is **symplectic** if  $\omega: V \times V \rightarrow \mathbb{R}$  is a perfect pairing

Given  $W \leq V$  we define  $W^{\omega\perp} = \{v \in V \mid \omega(v, w) = 0, \forall w \in W\}$

**Example 2.1.** If we have  $(\mathbb{R}^{2n}, \sum e_i^* \wedge f_i^*)$  for some basis  $\{e_1, \dots, e_n, f_1, \dots, f_n\}$  and  $W = \mathbb{R}^2 \times \{0\}$  where the  $\mathbb{R}^2$  is spanned by  $e_1, f_1$  then  $W^{\omega\perp} = \{0\} \times \mathbb{R}^{2n-2}$

**Definition 2.2.**  $W \leq V$  is **isotropic** if  $W \leq W^{\omega\perp}$ . It is **coisotropic** if  $W^{\omega\perp} \leq W$  and **Lagrangian** if  $W = W^{\omega\perp}$ . It is **symplectic** if  $W^{\omega\perp} \cap W = \{0\}$

**Remark.** If  $W$  is symplectic then  $(V, \omega) \cong (W, \omega|_W) \oplus (W^{\omega\perp}, \omega|_{W^{\omega\perp}})$

**Example 2.2.** In the situation of example 2.1,  $\text{Span}\{e_i \mid 1 \leq i \leq j\}$  is isotropic for  $j \leq n$ , and is Lagrangian for  $j = n$ .

$W$  is symplectic.

$\text{Span}\{e_1, \dots, e_n, f_1\}$  is coisotropic

**Exercise 2.1.** There are two exercises now

1.  $(W^{\omega\perp})^{\omega\perp} = W$

2.  $\dim W^{\omega\perp} + \dim W = \dim V$

**Theorem 2.1.** If  $(V, \omega)$  is a symplectic vector space, there is some basis  $\{e_1, \dots, e_n, f_1, \dots, f_n\}$  for  $V$  such that in this basis,  $\omega = \sum_{i=1}^n e_i^* \wedge f_i^*$

*Proof.* We induct on  $\dim V$ . Choose a vector  $0 \neq e_1 \in V$ . By non-degeneracy and linearity, there is some  $f_1$  such that  $\omega(e_1, f_1) = 1$ . By the induction hypothesis, there is a basis  $\{e_2, \dots, e_n, f_2, \dots, f_n\}$  for  $\text{Span}(\{e_1, f_1\})^{\omega\perp}$ , which is symplectic. Then

$$(V, \omega) = \left( \text{Span}(\{e_1, e_2\}), e_1^* \wedge f_1^* \right) \oplus \left( \text{Span}(\{e_1, e_2\})^{\omega\perp}, \sum_{i=2}^n e_i^* \wedge f_i^* \right)$$

so we are done □

**Corollary 2.1.1.** *The dimension of a symplectic vector space is even*

*Proof.* Trivial □

We will denote  $\omega_0 = \sum_{i=1}^n e_i^* \wedge f_i^*$  for the form on  $\mathbb{R}^{2n}$ . Note that for  $x, y \in \mathbb{R}^{2n}$  we can write  $\omega(x, y) = \langle J_0 x, y \rangle$  for  $J_0 \in \text{End}(\mathbb{R}^{2n})$  where we define  $J_0(e_i) = f_i, J_0(f_i) = -e_i$  so that  $J_0^2 = -I$  which is the standard complex structure on  $\mathbb{C}^n$ , and the inner product,  $g_0$ , is the usual inner product. It is true that  $g_0(x, y) = \omega(x, J_0 y)$

For a vector space  $V$  we define  $\mathcal{J}(V) = \{J \in \text{End}(V) \mid J^2 = -I\}$ , the set of complex structures.

$H_0: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  is given by  $(u, v) \mapsto \bar{u} \cdot v$  is the Hermitian pairing, and is equal to  $g_0 + i\omega_0$

**Definition 2.3.**  $J \in \mathcal{J}(V)$  is **compatible with**  $\omega$  if  $g_J = \omega(\cdot, J\cdot)$  is an inner product

Recall that  $GL_{2n}(\mathbb{R})$  acts transitively on the set of symplectic forms on  $\mathbb{R}^{2n}$  (by the theorem above, similarly to its transitive action on Hermitian forms). Then  $J$  is compatible with  $\omega$  if and only if  $g_J + i\omega$  is a Hermitian form

**Definition 2.4.** Given  $A: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  we get a map on  $\bigwedge^2(\mathbb{R}^{2n})^*$  called  $A^*$  be  $A^*\omega(x, y) = \omega(Ax, Ay)$

We define

$$Sp(2n) = \{A \in GL_{2n}(\mathbb{R}) \mid A^*\omega = \omega\} = \{A \in GL_{2n}(\mathbb{R}) \mid A^T J_0 A = J_0\}$$

We therefore know that the set of symplectic forms 'is'  $GL_{2n}(\mathbb{R})/Sp(2n)$ .

**Proposition 2.1.**  $Sp(2n) \subset SL_{2n}(\mathbb{R})$

*Proof.*  $\omega^{\wedge n} = n!e_1^* \wedge \dots \wedge e_n^* \wedge f_1^* \wedge \dots \wedge f_n^* = n! \text{vol}_{std}$  so it follows as preservation of the symplectic form preserves the volume form □

**Remark.** We have

$$Sp(2n) \cap O(2n) = Sp(2n) \cap GL_n(\mathbb{C}) = GL_n(\mathbb{C}) \cap O(2n) = U(n)$$

as  $Sp(2n), O(2n), GL_n(\mathbb{C})$  are respectively the maps preserving  $\omega_0, g_0, J_0$   
 $U(n) \subset Sp(2n)$  is a maximal compact Lie group

**Definition 2.5.**  $\text{Lag}(V)$  the **Lagrangian Grassmanian** is

$$\{L \leq V \mid \dim L = n, L^{\omega^\perp} = L\}$$

(the first condition is redundant)

**Example 2.3.**  $\text{Lag}(\mathbb{R}^n) = \mathbb{RP}^1 \simeq S^1$  as  $\omega$  is alternating, so  $\omega(ax, cx) = 0$  for all  $a, c \in \mathbb{R}$

**Proposition 2.2.**  $U(n)$  acts transitively on  $\text{Lag}(\mathbb{R}^{2n})$

*Proof.* For  $L \in \text{Lag}(\mathbb{R}^{2n})$  fix an orthonormal basis  $e'_1, \dots, e'_n$  for  $L$  and define a map  $e_i \mapsto e'_i, 1 \leq i \leq n$  which determines an element of  $U(n)$  by complexification  $\square$

The set of orthonormal bases for  $L$  can be identified with  $O(n)$  and so  $\text{Lag}(V) \simeq U(n)/O(n)$

**Example 2.4.**  $U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$ ,  $O(1) = \{\pm 1\}$  and  $U(1)/O(1) = S^1$  with the quotient in the same manner as for  $\mathbb{RP}^1$

**Lemma 2.2.**  $J_\omega = \{J \in \mathcal{J}(\mathbb{R}^{2n}) \mid J \text{ is compatible with } \omega\}$  is homotopy equivalent to a point

*Proof.*  $\phi: J_\omega \rightarrow \{\text{Inner products on } \mathbb{R}^{2n}\} = G$  by  $J \mapsto \omega(\cdot, J\cdot)$ . The set of inner products is convex, and hence is homotopic to a point. We claim  $\phi$  is a homotopy equivalence. ie there is some  $\psi: G \rightarrow J_\omega$  with  $\phi \circ \psi \simeq I$  and  $\psi \circ \phi \simeq I$   $\psi$  is defined by polar decomposition. If  $g \in G$ , let  $A_g \in \text{End}(\mathbb{R}^{2n})$  be such that  $g = \omega(\cdot, A_g \cdot)$ . This is true as everything in sight is non-degenerate. We check that  $A^T = -A$  and  $AA^T$  is symmetric and positive definite. Define  $J_g = (AA^T)^{-1/2}A$ . We may then check that  $J_g \in J_\omega$  and if  $A_g \in J_\omega$  then  $A_g = J_g$ . Then we define  $\psi(g) = J_g$  and this works  $\square$

**Remark.**  $A_g = \sqrt{A_g^T A_g} J_g$  is a product of a symmetric and an orthogonal matrix. This is called the **polar decomposition** of  $A_g$ , which exists for any  $M \in GL_{2n}(\mathbb{R})$ .

**Exercise 2.2.** Alternatively,  $G \cong Sp(2n)/U(n)$  then show that the fibres are contractible

$\Omega_J = \{\text{symplectic structures compatible with } J\}$  is homotopic to a point

Recall that  $\text{LGr}(V) = \{\text{Lagrangians in } V\}$ . It is a fact that there is a distinguished class  $\mu \in H^1(\text{LGr}(V); \mathbb{Z})$ , called the **Maslov class** which is represented by  $\text{LGr}(n) = U(n)/O(n) \rightarrow S^1$  via  $\det(\cdot)^2$  (and then pulling back the generator of  $H^1(S^1, \mathbb{Z})$ ). The square is necessary as  $\det(O(n)) = \{\pm 1\}$

**Remark.** In fact  $\pi_1(\text{LGr}(n)) \cong \mathbb{Z}$  is generated by  $\mu$

### 3 Symplectic Manifolds

**Definition 3.1.** Recall that for  $M$  a smooth manifold of dimension  $2n$ , and  $\omega \in \Omega^2(M) = \Gamma(\wedge^2 T^*M)$ , we say  $(M, \omega)$  is **symplectic** if

1.  $(T_p M, \omega_p)$  is symplectic for all  $p \in M$
2.  $d\omega = 0$

**Remark.** The first condition is equivalent to  $\omega^{\wedge n}$  being non-vanishing everywhere

**Example 3.1.** We assume  $\mathbb{P}^n$  is over  $\mathbb{C}$ .

$(\mathbb{P}^n, \omega_{fs})$ , has the **Fubini-Study** form where  $\omega_{fs}$  is defined so  $\pi: S^{2n+1} \rightarrow \mathbb{P}^n$  pulls back  $\omega_{fs}$  to  $\omega_{std}$ ,  $\pi^*\omega_{fs} = \omega_{std}|_{S^{2n+1}}$  where we consider  $S^{2n+1} \subset \mathbb{C}^{n+1}$ . In coordinates on  $\mathbb{P}^1$ , we have  $\omega_{fs} = \frac{i}{2} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$ , normalised so that  $\int_{\mathbb{P}^1} \omega_{fs} = 1$  on an affine open, this will be proven (more generally) after Kahler manifolds are discussed (this example will be continued)

**Example 3.2.** (Cotangent Bundles) Let  $Q^n$  be a smooth manifold. We define the symplectic manifold  $(T^*Q, \omega_{can} = d\lambda_{can})$  where the **canonical one form**,  $\lambda_{can}$  is characterised as follows:  $\beta \in \Omega^1(Q) \mapsto \tilde{\beta}: Q \hookrightarrow T^*Q$  via  $q \mapsto (q, \beta_q)$ . For all  $\beta \in \Omega^1(Q)$ , we have  $\tilde{\beta}^*\lambda_{can} = \beta$ . On  $U \subset Q$  with coordinates given by  $q_1, \dots, q_n$  and  $T^*U \subset T^*Q$ ,  $\lambda_{can} = \sum p_i dq_i$  where  $p_i$  are the dual coordinates. It remains to check that  $\lambda_{can}$  is well defined in the coordinate form, but it is

**Remark.** The sign convention for  $\lambda_{can}$  varies. In some sense, we can think of this as the "only" reasonable 1-form

It is an open problem if the following holds:  $T^*Q \cong T^*Q'$  as symplectic manifolds if and only if  $Q \cong Q'$  as smooth manifolds

**Remark.** Properties of Symplectic Manifolds  $(M, \omega)$

1.  $\omega^{\wedge n}$  is a volume form, so we get a canonical orientation on  $M$ .
2. If  $M$  is compact, then  $H_{dR}^{2k}(M) \cong H^{2k}(M, \mathbb{R}) \neq 0$  for all  $k$ , as otherwise some  $\bigwedge^k \omega$  would be 0, so the volume form would be 0:

$$\langle [\omega^{2n}], [M] \rangle = \int_M \omega^{2n} > 0$$

Hence, for example, there is no symplectic structure on  $S^{2k}$  for  $k > 1$

3.  $TM$  has an almost complex structure which is compatible with  $\omega$  and unique up to homotopy (a vector bundle map  $J: TM \rightarrow TM$  which squares to  $-I$ , alternatively,  $J \in \Gamma(T^*M \otimes TM)$  which is pointwise given by polar decomposition - take  $g$  a metric and set  $J = \sqrt{A_g^T A_g}^{-1} A_g$ . An example of such is  $J_{std}$  on  $(\mathbb{R}^{2n}, \omega_{std}) \cong (\mathbb{C}^n, \omega_{std})$

**Proposition 3.1.** *If  $N^k \subset M$  is a submanifold and  $J(TN) = TN$  (as a subbundle of  $TM$ ) for  $J$  an almost complex structure compatible with  $\omega$ , then  $(N, \omega|_{TN})$  is a symplectic manifold*

*Proof.* Non-degeneracy follows because  $\omega(v, Jv) = g_J(v, v) > 0$ , and closure is immediate  $\square$

**Corollary 3.0.1.** *All quasi-projective varieties admit symplectic forms*

*Proof.* Trivial  $\square$

### 3.1 Characteristic Classes

What information can we get from the homotopy class of  $J$  for a complex structure  $J$ ?

Suppose  $X$  is a topological space and we have a complex vector bundle  $V \rightarrow X$  of complex rank  $k$ . We do not assume a manifold structure on  $X$ . How do we classify such  $V$  up to isomorphism?

From now on,  $\text{rk}(V) = 1$ , so  $V$  is a line bundle. Define  $\mathbb{CP}^\infty = \varinjlim (\mathbb{CP}^n)$  to be the 'Grassmanian of lines in  $\mathbb{C}^\infty$ ', which is the increasing union of the Grassmanians of lines in  $\mathbb{C}^n$ .

It is a fact that  $\pi_i(\mathbb{CP}^\infty)$  is  $\mathbb{Z}$  in  $i = 0, 2$  and 0 otherwise. So  $\mathbb{CP}^\infty$  is a  $K(\mathbb{Z}, 2)$ . It is a fact that, for sufficiently nice  $X$ ,  $H^n(X; G) = [X, K(G, n)]$  by a similar pullback to the case of  $K(\mathbb{Z}, 1) = S^1$ , using

$$e \in H^n(K(G, n); G) = \text{Hom}(H_n(K(G, n); G), G) \cong \text{Hom}(G, G)$$

**Example 3.3.** We saw  $K(\mathbb{Z}, 1) = S^1$  and  $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$  (the latter using the fibration  $S^1 \rightarrow S^{2\infty+1} \rightarrow \mathbb{CP}^\infty$  where the middle space is contractible). We also know  $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{RP}^2$

How far does this go? - Do we have the ring structure on cohomology induced like this?

In fact, all structures on cohomology have incarnations in this story (the theory of Spectra)

We have the tautological line bundle  $\mathcal{L}_{\text{taut}} \rightarrow \mathbb{CP}^\infty$ . We wish to classify complex bundles on  $X$  up to isomorphism. We can consider maps  $[X, \mathbb{CP}^\infty]$  and map  $f \in [X, \mathbb{CP}^\infty]$  to  $f^*\mathcal{L}_{\text{taut}}$ . It is a fact that this map is a bijection. We also know from the previous discussion that  $[X, \mathbb{CP}^\infty] \longleftrightarrow H^2(X, \mathbb{Z})$ . If we say that  $H^2(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}$  is generated by  $[H]$ , then the map is given by sending  $f \in [X, \mathbb{CP}^\infty] \mapsto f^*[-H]$ . In this way we have associated a canonical class in  $H^2(X, \mathbb{Z})$  to every line bundle on  $X$ . We call this the **Chern class** of  $\mathcal{L}$  and denote it as  $c_1(\mathcal{L})$

**Remark.** This is not a classification of *holomorphic* vector bundles when  $X$  is a complex manifold

In general, for  $V \rightarrow X$  a complex vector bundle of rank  $k$ , we define the **Chern class** of the vector bundle,  $c_1(V) = c_1(\bigwedge^k V)$ . For example we could take  $TM$  for a symplectic manifold  $(M, \omega)$

**Remark.** Some properties of  $c_1$  are as follows:

1. It is invariant under homotopy of  $J$ , the complex structure on  $V$
2.  $c_1(V^*) = -c_1(V)$
3. If we have short exact sequence  $0 \rightarrow F_1 \rightarrow E \rightarrow F_2 \rightarrow 0$  of vector bundles on  $X$ , then  $c_1(E) = c_1(F_1) + c_1(F_2)$

4.  $c_1(E \otimes F) = \text{rk}(E)c_1(F) + \text{rk}(F)c_1(E)$

5. A key property is that  $c_1(\mathcal{L}) = \text{Poincare dual of the vanishing locus of a generic section of } \mathcal{L}$

**Exercise 3.1.** *Verify the above properties when the bundles in question are direct sums of line bundles*

**Definition 3.2.** If  $M$  has an almost complex structure, denote  $c_1(M) = c_1(TM)$ . In particular we may say this when  $M$  is symplectic

**Example 3.4.**  $c_1(\Sigma_g) = \text{number of zeros counted with sign of a vector field on } \Sigma_g$ , which is  $2 - 2g$

$c_1(\mathcal{L}_{\text{taut}} \rightarrow \mathbb{CP}^n) = [-H]$  and  $c_1(\mathbb{CP}^n) = (n+1)[H]$  as  $\Gamma(\bigwedge^n \mathbb{CP}^n) \ni \frac{dz_1 \dots dz_n}{z_1 \dots z_n}$  has poles along  $\prod_{i=1}^n z_i = 0$

We have an adjunction formula:

**Proposition 3.2.** *Let  $C$  be a complex curve within  $(X^4, J)$ . Then*

$$2 \text{ genus}(C) - 2 = -c_1(X) \cdot i_*[C] + i_*[C] \cdot i_*[C]$$

, where  $i: C \rightarrow X$  is the embedding (we have a pairing of  $H^2(X, \mathbb{Z})$  with itself via Poincare duality)

*Proof.* We have a short exact sequence

$$0 \rightarrow TC \rightarrow TX|_C \rightarrow N_{C/X}$$

where the latter is the normal bundle, then take  $c_1$ . We get that, by the properties of the Chern class,  $c_1(TX|_C) = c_1(TC) + c_1(N_{C/X})$ . We can identify  $H^2(C, \mathbb{Z}) \cong \mathbb{Z}$  canonically as we have an orientation on  $C$  from the complex structure. Now we get

$$\langle i^*c_1(X), [C] \rangle = \langle e(TC), [C] \rangle + (i_*[C])^2$$

where  $e(TC)$  is the Euler class (= the Chern class) and the latter comes from the number-of-zeros interpretation of  $c_1$ . We have that  $\langle i^*c_1(X), [C] \rangle = \langle c_1(X), i_*[C] \rangle$  and we are done  $\square$

**Corollary 3.0.2.** *(Genus of a planar curve). Suppose  $C \subset \mathbb{CP}^2$  is a degree  $d$  curve, that is  $[C] = d[H]$  where  $[H]$  is the hyperplane class. Then the genus is  $g(C) = \frac{(d-1)(d-2)}{2}$*

*Proof.* By 3.2,  $2g(C) - 2 = -c_1(\mathbb{CP}^2) \cdot d[H] + d^2[H]^2 = -3d + d^2$   $\square$

**Example 3.5.** An elliptic curve has genus 1

**Remark.** If  $(X, \omega)$  is symplectic then  $X$  has an almost complex structure, but the converse doesn't hold. For example, the Hopf surface  $(\mathbb{C}^2 \setminus \{0\})/(z \sim 2z)$  has a complex structure (induced by the quotient), but is diffeomorphic to  $S^3 \times S^1$ , which is compact with 0 second cohomology so can't have a symplectic structure

## 4 Neighbourhood Theorems

Throughout this section  $(M, \omega)$  is a symplectic manifold

**Definition 4.1.** We define  $\text{Symp}(M) = \{\phi \in \text{Diff}(M) \mid \phi^*\omega = \omega\}$

**Example 4.1.** Taking  $(T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega_{std})$ , we get that  $T^{2n} \hookrightarrow \text{Symp}(T^2)$  by translations

**Definition 4.2.**  $\text{Diff}_0(M)$  is the set of diffeomorphisms which are isotopic to the identity

Recall that, if we have  $\{\phi_t\}_{t \in [0,1]}$  an isotopy of  $M$ , that is  $\phi_t \in \text{Diff}_0(M)$  and  $\phi_0 = \text{Id}$ , we can associate a time dependent vector field  $X_t$  given by the derivative in time. This is a bijective correspondence, by integrating the vector field (as compactness ensures no singularities). If  $\alpha \in \Omega^*(M)$ , then we have  $\mathcal{L}_{X_t}\alpha = \frac{d}{dt}\phi_t^*\alpha$

**Theorem 4.1.** (*Cartan's Magic Formula*) Given a vector field  $X$  and some  $\alpha \in \Omega^*(M)$ ,  $\mathcal{L}_X(\alpha) = i_X d\alpha + di_X\alpha$  where  $i_X$  is just inserting  $X$  in one of the slots of  $\alpha$

*Proof.* See III Differential Geometry □

Given an isotopy  $\phi_t$ , when is  $\phi_t \in \text{Symp}(M)$  for all  $t$ ?  
When  $\mathcal{L}_X\omega = di_X\omega = 0$  (As the starting point is the identity and using Cartan's Magic Formula) which is if and only if  $i_X\omega$  is a closed 1-form

**Example 4.2.** If  $i_X\omega = dH$  for some Hamiltonian  $H: M \rightarrow \mathbb{R}$ , then the condition holds

So we have that for any  $\phi \in \text{Diff}_0(M)$ ,  $[\phi^*\omega] = [\omega]$

**Theorem 4.2.** If  $(M, \omega)$  is compact and  $\{\omega_t\}_{t \in [0,1]}$  is a path of symplectic forms with  $\omega_0 = \omega$  and  $[\omega_t]$  constant, then there is  $\phi \in \text{Diff}_0(M)$  such that  $\phi^*\omega_0 = \omega_1$

*Proof.* First note that because  $[\omega_t]$  is constant, there is  $\sigma$  such that  $\sigma \in \Omega^1(M)$  and  $\frac{d}{dt}\omega_t = d\sigma_t$  (this relies on compactness). Find  $X_t$  a vector field such that  $\sigma_t + i_{X_t}\omega_t = 0$ . Let  $\phi_t$  be the flow along  $X_t$ , which exists by compactness. Then  $\frac{d}{dt}\phi_t^*\omega_t = \frac{d\omega_t}{dt} + \mathcal{L}_{X_t}\omega_t = d(\sigma_t + i_{X_t}\omega_t) = 0$  so  $\phi_t^*\omega_t$  is constant, so  $\phi_1^*\omega_1 = \omega_0$ . This method of proof is called Moser's trick □

We have the following two corollaries

**Exercise 4.1.** Symplectic forms on  $\Sigma_g$  modulo isotopy are in bijection with  $\mathbb{R}_{>0}$  by  $\omega \mapsto \int_{\Sigma_g} \omega$  (this defines the cohomology classes and we can interpolate between two forms in the same class)

**Exercise 4.2.** If  $X_1, X_2$  are projective varieties which are deformation equivalent, then  $(X_1, \omega_{fs}) \cong (X_2, \omega_{fs})$ . In particular, any two smooth hypersurfaces of degree  $d$  in  $\mathbb{CP}^n$  are symplectomorphic

**Remark.** The theorem fails to work without an interpolating path - the map from symplectic forms modulo isotopy to  $H^2(M; \mathbb{R})$  is not necessarily injective. We can use Chern classes to show this

**Theorem 4.3.** (*Darboux Theorem*) Let  $p \in (M, \omega)$ . Then there is a neighbourhood  $B(\epsilon) \hookrightarrow M$  with  $0 \mapsto p$  such that  $\omega$  pulls back to  $\omega_{std}$

*Proof.* The proof is by Moser's trick. Fix some chart  $g: B(\epsilon) \rightarrow M$ . Consider  $g^*\omega = \omega_1$  and call  $\omega_{std} = \omega_0$ . Without loss of generality,  $g^*\omega|_0 = \omega_{std}|_0$  by changing basis at the origin. Consider  $(1-t)\omega_0 + t\omega_1$ . This is symplectic within some  $0 \in D' \subset B(\epsilon)$  as it is symplectic at 0. Note that  $[\omega_t]$  is constant as  $H^1(X, \mathbb{R}) = 0$  and define  $\sigma_t \in \Omega^1(d')$  such that  $\frac{d}{dt}\omega_t = d\sigma_t$ . Let  $X_t$  solve  $i_{X_t}\omega = \sigma_t$  and assume  $\sigma_t|_0 = 0$ . Note that  $X_t(0) = 0$  and for some  $D'' \subset D'$ , the flow of  $X_t$  stays within  $D'$  for all  $t \in [0, 1]$ . Setting  $f = g \circ \phi_1$ , we are done as  $f^*\omega = \phi_1^*g^*\omega = \phi_1^*\omega_1 = \omega_0$ .  $\square$

**Definition 4.3.**  $V \rightarrow X$  is a **symplectic vector bundle** if there is a smooth family  $\{\omega_x \mid x \in X\}$  of symplectic forms on the fibres  $V_x$

**Remark.**  $TM \rightarrow M$  admits a symplectic structure, but this does not imply  $M$  admits a symplectic structure.

It is a fact that every symplectic vector bundle corresponds to a complex vector bundle, whose classification is equivalent to that of Hermitian vector bundles

**Theorem 4.4.** Suppose that  $Q_i \hookrightarrow (M_i, \omega_i)$  for  $i = 1, 2$  and we have a symplectomorphism  $\phi: (Q_1, \omega_1|_{TQ_1}) \rightarrow (Q_2, \omega_2|_{TQ_2})$  and a morphism of symplectic normal bundles  $\Phi: \nu_{Q_1/M_1} \rightarrow \nu_{Q_2/M_2}$  over  $\phi$ . Then there is a symplectomorphism  $\tilde{\phi}: U(Q_1) \rightarrow U(Q_2)$  where  $U(Q_i)$  is a neighbourhood of  $Q_i$

*Proof.*

**Lemma 4.5.** If  $Q \subset M$  is a smooth submanifold and  $\alpha_1, \alpha_2 \in \Omega^k(M)$  such that  $\alpha_1|_Q = \alpha_2|_Q$  then there is some  $\beta \in \Omega^{k-1}(U(Q))$  such that  $d\beta = \alpha_1 - \alpha_2$  and  $\beta = 0$  on  $Q$

*Proof.* Trivial  $\square$

After choosing a metric, we have embeddings

$$\text{nbhd}(Q_i \subset \nu_{Q_i/M_i}) \hookrightarrow U(Q_i) \subset M_i$$

by the exponential map. Now consider  $\tilde{\phi}'$  given by

$$\exp_2 \circ \Phi \circ \exp_1^{-1}: U(Q_1) \rightarrow U(Q_2)$$

. Now you can check  $\tilde{\phi}'^*\omega_2 = \omega_1'$  agrees with  $\omega_1$  on  $Q_1$ . Let  $\sigma \in \Omega^1(U'(Q_1))$  (possibly shrinking neighbourhoods) be such that  $\sigma|_{Q_1} = 0, d\sigma_1 = \omega_1 - \omega_2'$ . Now apply Moser's trick to the path  $t\omega_1 + (1-t)\omega_2$  and we are done  $\square$

**Corollary 4.5.1.** If  $C^2 \hookrightarrow (M^4, \omega)$  then a neighbourhood of  $C$ ,  $(U(C), \omega|_{U(C)})$  is determined by

1.  $\int_X \omega|_C$
2.  $[C] \cdot [C]$
3.  $c_1(M) \cdot [C]$

**Definition 4.4.** Let  $L^n \subset (M^{2n}, \omega)$  be a submanifold of half the dimension of  $M$ .  $L$  is **Lagrangian** if  $\omega|_{TL} = 0$

**Remark.** "Everything is Lagrangian"

**Example 4.3.** We have examples of Lagrangian submanifolds

1. Any real curve in  $\Sigma_g$  (for dimension reasons)
2.  $(S^1)^n \subset (\mathbb{C}^n, \omega_{std})$ , the **Clifford torus**. So there are locally Lagrangians everywhere (in Darboux charts)
3. In  $(T^*Q, \omega_{can})$ ,  $\Gamma_\beta$ , the graph of  $\beta \in \Omega^1(Q)$  such that  $d\beta = 0$ . The condition is needed as, for  $Q \hookrightarrow T^*Q$  by  $\iota_\beta$ , we have  $\iota_\beta^* \lambda_{can} = \beta$ . Take derivatives on both sides (We need  $\iota_\beta^* \omega = 0$  so  $d\beta = d\iota_\beta^* \lambda_{can} = 0$ )
4. In particular,  $Q \hookrightarrow TQ$  by the zero section is Lagrangian

Suppose  $L \subset M^{2n}$  is Lagrangian and  $J$  is an  $\omega$ -compatible almost complex structure on  $M$ . Take  $[\beta] \in \pi_2(M, L)$  and define

$$M_\beta = \{U : (D \subset \mathbb{C}, \partial D) \rightarrow (M, L) \mid du \circ j_{std} = J \circ du\}$$

where  $[u] = [b] \in \pi_2(M, L)$ . Then trivialising  $u^*TM$ , we get a loop  $\gamma$  in  $LGr(2n)$ . We have

$$\dim M_{[\beta]} = \text{constant} + \mu([\gamma])$$

where  $\mu$  is the Maslev class

**Theorem 4.6.** (*Weinstein Neighbourhood Theorem*) If  $L^n \subset (M^{2n}, \omega)$  is Lagrangian then there is a symplectomorphism between a neighbourhood of  $L$  in  $(T^*L, \omega_{can})$  and a neighbourhood of  $L$  in  $M$

*Proof.* Choose  $J$  an almost complex structure on  $TM$  compatible with  $\omega$ , and denote the associated metric as  $g$ . We get an isomorphism  $\Phi : T^*M \rightarrow TM$  induced by  $g$  which satisfies  $g(\Phi(f), v) = f(v)$ . Define the function  $\phi : T^*L \rightarrow M$  by  $(q, f) \mapsto \exp_q(J\Phi_q(f))$ . Then if we restrict to some neighbourhood of  $L$  in  $T^*L$ , this is injective. We want to check that  $\phi^* \omega|_L = \omega_{can}|_L$  in  $T^*L$ . Using that  $T(T^*L)_{q,0} = TL \oplus T^*L$

$$\begin{aligned} (\phi^*(\omega))_{q,0}((v, f), (v', f)) &= \omega(v + J\Phi f, v' + J\Phi f') \\ &= \omega(v, J\Phi f) - \omega(v', J\Phi f) \\ &= g(v, \Phi f') - g(v', \Phi f) \\ &= f'(v) - f(v') \\ &= d\lambda_{can}((v, f), (v', f')) \end{aligned}$$

So we have checked it. Now run Moser as in the symplectic neighbourhood theorem with  $(1-t)\omega_{can} + t\phi^*\omega$   $\square$

**Remark.** There is a short exact sequence  $0 \rightarrow TL \rightarrow TM|_L \rightarrow \nu_{L/M} \rightarrow 0$  and  $J: TL \rightarrow \nu_{L/M}$  is an isomorphism of unoriented vector bundles. If you keep track of the orientation, this tells us that  $[L]^2 = (-1)^{n+1}\chi(L)$ . If  $L$  is compact, then as  $\chi(L) = 0$  for odd dimensional  $L$  by Poincare duality,  $[L]^2 = -\chi(L)$

**Corollary 4.6.1.** *If  $L$  is not essential (ie  $[L] = 0 \in H_n(M; \mathbb{Z})$ , then  $\chi(L) = 0$ . In particular this holds for Clifford tori in Darboux charts*

*Proof.* Trivial □

**Example 4.4.** There are a few further examples

1. If  $f: M \rightarrow M$  is a diffeomorphism then  $f \in \text{Symp}(M)$  if and only if  $\Gamma_f = \{(x, f(x)) \mid x \in M\} \subset (M \times M, \omega \oplus -\omega)$  is Lagrangian
2. If  $f: M \rightarrow M$  satisfies  $f^*\omega = -\omega$  (ie  $f$  is **anti-symplectic**) then the set of fixed points of  $f$  is Lagrangian where it is smooth (as at such points,  $\omega = -\omega = 0$  - need to check dimension)
3. In particular, if  $(X, \omega_{fs}|_X) \subset (\mathbb{CP}^n, \omega_{fs})$  is cut out by elements of  $\mathbb{R}[x_1, \dots, x_{n+1}]$ , then complex conjugation on  $\mathbb{CP}^n$  is anti-symplectic and preserves  $X$  so that  $X(\mathbb{R})$  is Lagrangian

**Corollary 4.6.2.** *If  $(M, \omega)$  is compact, and  $H_{dR}^1(M) = 0$ , then every function  $f \in \text{Symp}(M)$  which is  $C^1$  close to the identity has at least 2 fixed points*

*Proof.* Consider  $\Gamma_f \subset M \times M$ . This is contained in some neighbourhood of the diagonal, which is symplectomorphic to  $U(M \subset T^*M)$  so we reduce to the case of  $\Gamma_f \subset T^*M$  the graphical Lagrangian of a 1-form.

We therefore have  $\Gamma_f = \text{Graph}(\beta \in H^1(M))$  with  $d\beta = 0$  and hence  $\beta = dh$  for some small  $h: M \rightarrow \mathbb{R}$  and  $\text{crit}(h) = \Gamma_f \cap M = \text{fix}(f)$  so we are done by extreme value theorem □

## 5 Kahler Manifolds

Let  $M^n$  be a complex manifold, with  $J$  an almost complex structure pulled back from an atlas of holomorphic charts  $\phi: \mathbb{C}^n \rightarrow M$ . Choose coordinates  $\{x_i, y_i\}_{1 \leq i \leq n}$  on  $M$ . We can write  $TM \otimes \mathbb{C} = \mathbb{C}\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \rangle$ . Define a complex basis of  $TM \otimes \mathbb{C}$  by  $\frac{\partial}{\partial z_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right)$  and  $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right)$ . We also set  $dz_i = dx_i + i dy_i$  and  $d\bar{z}_i = dx_i - i dy_i$  and further define the complexified cotangent bundle  $T^*M \otimes \mathbb{C} = (T^*M)^{1,0} \oplus (T^*M)^{0,1}$  where the first component is  $\mathbb{C}\langle dz_j \rangle$  and the second is  $\mathbb{C}\langle d\bar{z}_j \rangle$

To motivate why we do this, suppose we have  $f: M \rightarrow \mathbb{C}$ . Where does  $df$  live? It should live in  $\text{Hom}(T_p M, \mathbb{C}) = T_p^* M \otimes \mathbb{C}$ . We have  $d: C^\infty(M, \mathbb{C}) \rightarrow \Omega^1(M, \mathbb{C})$  splits to give  $d = \partial + \bar{\partial}$  (cf III Complex Manifolds)

Set  $\Omega^{l,m} = \{ \bigoplus_{|J|=l, |K|=m} \alpha_{J,K} dz_J \wedge d\bar{z}_K \}$ . For example we have the form  $dz_1 \wedge dz_2 \wedge d\bar{z}_3 \in \Omega^{2,1}(\mathbb{C}^3)$ . If we say  $J \in \text{End}(TM)$ ,  $J^2 = -\text{Id}_{TM}$ , then

$J\left(\frac{\partial}{\partial z_i}\right) = i\frac{\partial}{\partial \bar{z}_i}$  and  $J\left(\frac{\partial}{\partial \bar{z}_i}\right) = -i\frac{\partial}{\partial z_i}$  so that  $\Omega^{1,0}, \Omega^{0,1}$  are the  $\pm i$  eigenspaces of  $J$

**Lemma 5.1.**  *$f$  is holomorphic if and only if  $\bar{\partial}f = 0$ . Moreover we have  $d(\Omega^{p,q}) \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1}$*

*Proof.* See Complex Manifolds □

**Exercise 5.1.** *This relies on the fact that  $J$  is integrable. A priori we could have a component of  $d$  in  $\Omega^{p-1,q+2}$*

**Corollary 5.1.1.**  $d^2 = 0$  so  $\bar{\partial}^2 = 0$

*Proof.* See Complex Manifolds □

We can therefore define, for fixed  $p$ ,  $H_{\bar{\partial}}^{p,j}(M)$  as the cohomology of the complex  $(\Omega^{p,j}, \bar{\partial})$  which are complex (not smooth) invariants of  $M$ . This is the **Dolbeault Cohomology**

**Remark.** Suppose we fix a smooth manifold  $M$  and vary the complex and Kahler structures on  $M$ , with structures  $\{J_x, \omega_x\}_{x \in X}$ , eg on  $T^2 = \mathbb{C}/\{1, \lambda\}$  with  $\lambda$  in the upper half plane and varying  $\lambda$ . We have fixed descriptions of  $H^j(M, \mathbb{Z})$  for all  $j$ . Where is  $H^{1,0}(X) \subset H_{dR}^1(X; \mathbb{C}) \cong H^1(X, \mathbb{Z}) \otimes \mathbb{C}$ . This is a line in  $\mathbb{C}^n$ , and this is often a useful way to describe variations of complex structures

**Definition 5.1.** We say a complex manifold  $M$  is **Kahler** if  $M$  is equipped with some  $\omega \in \Omega^{1,1}(M)$  such that  $\partial\Omega = \bar{\partial}\Omega = 0$  and locally  $\omega = \frac{i}{2} \sum h_{j,k} dz_j \wedge d\bar{z}_k$  where  $(h_{j,k})_p$  is positive definite for all  $p \in M$

**Example 5.1.**  $\omega_{std}$  on  $\mathbb{C}$  is  $\frac{i}{2}(dz \wedge d\bar{z})$

**Theorem 5.2.** (Hodge) *If  $M$  is Kahler then there is a canonical isomorphism  $H_{dR}^k(M) \cong \bigoplus_{i+j=k} H_{\bar{\partial}}^{i,j}(M)$ . Moreover  $H_{\bar{\partial}}^{i,j} \cong H_{\bar{\partial}}^{j,i}$  for the natural conjugation map*

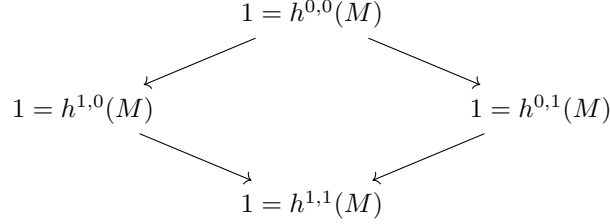
**Corollary 5.2.1.** *If  $M$  is Kahler then the odd Betti numbers are even for all  $K$*

**Theorem 5.3.** (Lefschetz) *If  $M$  is Kahler then  $\wedge[\omega]^k: H_{dR}^{n-k}(M) \rightarrow H_{dR}^{n+k}(M)$  is an isomorphism*

Lefschetz + Hodge tell us we have a "Hodge diamond" Eg for  $M$  compact of complex dimension 1, we get

$$\begin{array}{ccc}
 & H^{0,0}(M) & \\
 \swarrow & & \searrow \\
 H^{1,0}(M) & & H^{0,1}(M) \\
 \searrow & & \swarrow \\
 & H^{1,1}(M) &
 \end{array}$$

with ranks in the diagram



because of all the symmetry in the diagram.

We will assume  $M$  is Kahler until further notice. We recall that deformations of  $(M, J)$  which fix the smooth structure on  $M$  give us variations of Hodge structures, recording the embeddings  $H_{\bar{\partial}}^{j,k}(M) \hookrightarrow H_{dR}(M; \mathbb{C})$  in an intrinsic way

**Proposition 5.1.**  $\omega$  is a symplectic form

*Proof.* Check that  $h_{ij}$  being Hermitian implies  $\omega \in \Omega^{1,1}(M) \cap \Omega^2(M, \mathbb{R})$  and that positive definite implies  $\omega^n = n! \left(\frac{i}{2}\right)^n \det(h_{ij}) dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$   $\square$

**Remark.** The definition of a Kahler manifold is often a triple  $(M, \omega, J)$  with  $\omega$  symplectic and  $J$  a complex structure compatible with  $\omega$ , which is equivalent

How do we produce Kahler forms on complex manifolds  $M$ ?

**Definition 5.2.** A function  $\rho: M \rightarrow \mathbb{R}$  is **plurisubharmonic** if  $\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)_{j,k}$  is everywhere positive definite. In this case,  $\omega_p = \frac{i}{2} \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)_{j,k}$  is Kahler

**Remark.** If this is the case,  $M$  is not compact

**Example 5.2.**  $\rho = |z|^2$  defines  $\omega_{std}$  on  $\mathbb{C}^n$   
 $\rho_{fs} = \log(|z|^2 + 1)$  on  $\mathbb{C}^n$ . It is not immediately clear this is plurisubharmonic. We can check that at  $(1, 0, \dots, 0)$ ,

$$\omega_\rho = \frac{i}{2} \left( \frac{dz_1 d\bar{z}_1}{1 + \sum z_j \bar{z}_j} - \frac{(\sum \bar{z}_j dz_j)(\sum z_j d\bar{z}_j)}{(1 + \sum dz_j \bar{z}_j)^2} \right)$$

. Then note that  $\rho_{fs}$  is invariant under the action of  $U(n)$  so is plurisubharmonic everywhere

**Theorem 5.4.** Any complex submanifold on  $X \subset \mathbb{CP}^n$  is Kahler

*Proof.*  $\mathbb{CP}^n$  is covered in the affine charts  $U_i = \{[z_1 : \dots : z_{n+1}] \mid z_i \neq 0\}$ . We then check that  $\frac{i}{2} \partial \bar{\partial} \rho = \frac{1}{\pi} \omega_p$  is invariant under the change of coordinates. This defines the Fubini-Study form. For  $X = V(z_1) \cap \dots \cap V(z_j)$ ,  $\rho_{fs}|_X$  remains plurisubharmonic, and we can always reduce to this case (exercise)  $\square$

## 6 Constructions of Symplectic Manifolds

### 6.1 Blowing Up

In the complex setting, let  $Z = \{(z, l) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} \mid z \in l\}$  be the total space of the tautological line bundle over  $\mathbb{CP}^{n-1}$

**Remark.** We define  $\mathcal{O}(k)$  to be the line bundle on  $\mathbb{CP}^n$  with  $c_1(\mathcal{O}(k)) \cdot [L] = k$  where  $L$  is the hyperplane class

We have projection maps

$$\begin{array}{ccc} Z & \xrightarrow{p} & \mathbb{CP}^{n-1} \\ \downarrow \pi & & \\ \mathbb{C}^n & & \end{array}$$

and  $Z$  is the **blow up** of  $\mathbb{C}^n$  at 0. Away from  $E = \pi^{-1}(\{0\})$ ,  $\pi$  is a diffeomorphism.  $E$  is called the **exceptional divisor**