

Topics in Infinite Groups

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1 Introduction

1.1 (Bieri-Neumann-Strebel, BNS, or Sigma, invariants)

Throughout this course G is a finitely generated group, and then $(G, \mathbb{R}) = \mathbb{R}^n$ where n is the torsion free rank of G/G'

Definition 1.1. G is *metabelian* if G' is abelian

Remark 1.1. G acts by conjugation on G' , so G' is a G/G' module, or a $\mathbb{Z}(G/G')$ module

Since G/G' is a finitely generated abelian group, it is finitely presented and so G' is a finitely generated G/G' -module.

For $\chi \in \text{Hom}(G, \mathbb{R})$ we define $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$. This is a monoid.

Definition 1.2. (Bieri-Strebel, 1980)

$$\Sigma(G) = \{0 \neq \chi \in \text{Hom}(G, \mathbb{R}) \mid G \text{ is fg over a fg submonoid of } G_\chi\} \subseteq \text{Hom}(G, \mathbb{R}) \setminus \{0\}$$

Remark 1.2. The positive reals act on $\text{Hom}(G, \mathbb{R}) \setminus \{0\}$ by scalar multiplication. $\Sigma(G)$ is closed under positive scalar multiples, and Bieri-Strebel originally defined $\Sigma(G)$ as being the orbits under this action, $S(G) = (\text{Hom}(G, \mathbb{R}) \setminus \{0\})/\mathbb{R}^+$, which is an $n - 1$ -sphere. We will use $[\Sigma(G)]$ for this version.

Theorem 1.1. (*Br-St, 1980*) Suppose we have a finitely generated metabelian group. Then $[\Sigma(G)]$ is an open subset of $S(G)$

Proof. No Proof □

Theorem 1.2. (*Br-St, 1980*) If G is finitely generated metabelian, then G is finitely presented if and only if $[\Sigma(G)] \cup -[\Sigma(G)] = S(G)$

Proof. No Proof □

The proof of Theorem 1.2 uses that $S(G)$ is compact to establish finite presentability.

Theorem 1.3. (*Br-St*) Let G be finitely generated metabelian and $N \trianglelefteq G$ with G/N abelian. Then N is finitely generated if and only if $\{\chi \neq 0 \mid \chi(N) = 0\} \subseteq \Sigma(G)$

Proof. No Proof □

Remark 1.3. In particular, if $N = G'$, G' is finitely generated if and only if $\Sigma(G) = \text{Hom}(G, \mathbb{R}) \setminus \{0\}$

What do these subsets look like?

Definition 1.3. χ is *discrete* if the image of χ is infinite cyclic.

Theorem 1.4. (*Bieri-Groves, 1984*) If G is finitely generated metabelian then $\text{Hom}(G, \mathbb{R}) \setminus \Sigma(G)$ is a closed rational polytope

Proof. No Proof □

Remark 1.4. The proof is commutative algebra - links via Toric and Tropical geometry. The calculation of Σ relates to whether the exterior square of the module G' is finitely generated as a G/G' module

In 1987, Bieri-Neumann-Strebel extended the definition to all finitely generated groups. The definition still works with minor modification, as do theorems 1.1, 1.3, but theorem 1.2 fails in one direction. We instead have

Theorem 1.5. (*BNS, 1987*) If G is finitely presented and has no non-abelian free subgroups, then $[\Sigma(G)] \cup -[\Sigma(G)] = S(G)$

KS Brown (1987) showed that $\Sigma(G)$ is defined by using the existence or non-existence of certain tree actions of G . Given χ , G acts on the reals by translation by χ . Then (roughly speaking) $\chi \notin \Sigma(G)$ if there is an action of G on an $\mathbb{R}$ -tree with a fixed left end point, and infinite orbits on the 'right' end which projects equivariantly onto the action of shifting by χ . We can 'glue' two of these together to prove the theorem above.

1.2 Homological finiteness conditions

Definition 1.4. A group or monoid G is of type FP_m if the trivial $\mathbb{Z}G$ -module $\mathbb{Z}$ has a free resolution finitely generated below and at degree m

Remark 1.5. Type FP_1 is the same as being finitely generated. Type FP_2 is implied by being finitely presented (but the other direction is false)

Definition 1.5. (Bieri-Penz) $\Sigma^m(G, \mathbb{Z}) = \{\chi \mid G_\chi \text{ of type } FP_m\} \subseteq \text{Hom}(G, \mathbb{R})$

Remark 1.6. For $m = 1$, we get $\Sigma(G) \cup \{0\}$

If G is a finitely generated group and $\chi: G \rightarrow \mathbb{R}$. Consider the infinite sums $\lambda = \sum n_g g$ where $n_g \in \mathbb{Z}$.

Definition 1.6. The *Novikov ring* of G with respect to χ denoted $\widehat{\mathbb{Z}G}_\chi$ consists of all $\sum n_g g$ such that $\{g \in \text{supp } \lambda \mid \chi(g) < r\}$ is finite for all $r \in \mathbb{R}$ with the obvious addition/multiplication

Remark 1.7. This contains the ordinary group ring, and so is a G -bimodule.

Theorem 1.6. (Sikorav) $\chi \in \Sigma^m(G, \mathbb{Z})$ if and only if $H_k(G, \widehat{\mathbb{Z}G}_\chi) = 0$ for $k = 0, \dots, m$ (this homology is called the Novikov homology)

Proof. No Proof □

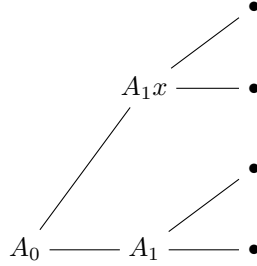
Remark 1.8. There is a clash of notation here: G_χ is the monoid of χ -positive elements, so $\mathbb{Z}G_\chi$ this is a subring of $\mathbb{Z}G$, and we can complete this, but this is not what we mean. Some authors write $\widehat{\mathbb{Z}G}^\chi$ to avoid this clash.

To use Sikorav's theorem, we want to calculate the homology group $H_k(G, \widehat{\mathbb{Z}G}_\chi)$. To do this, take a free resolution of $\mathbb{Z}$. If F_i, F_{i-1} are finitely generated free $\mathbb{Z}G$ modules in this resolution, then the boundary map $F_i \rightarrow F_{i-1}$ can be represented as a matrix with entries in $\mathbb{Z}G$. Kielak (2020) introduced Σ for such matrices and showed as part of the definition that they are integral polytopes. To calculate Novikov homology, we are applying $\cdot \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}_\chi$ to our resolution and taking the homology. The vanishing of the homology groups is related to the question as to when the matrices are right invertible as $\widehat{\mathbb{Z}G}_\chi$ -matrices. We have been doing things for right modules. These can be converted into left modules by $g \cdot m = m \cdot g^{-1}$. This switch to inverses corresponds to the antipodal map on $S(G)$ and so the left theorem is the antipodal image of the right theorem.

2 Metabelian Groups

Example 2.1. Let $G = \langle x, y \mid y^{-1}xy = x^2 \rangle = BS(1, 2)$ be a Baumslag-Solitar group. We write $x^y = y^{-1}xy$. The subgroup A generated by $\{\dots, x^{y^{-1}}, x, x^y, \dots\}$ is abelian. It is isomorphic to $\mathbb{Z}[\frac{1}{2}]$. It is normal in G and conjugation by y is multiplication by 2. Any element of G can be written as $x^q y^n$ where

$q \in \mathbb{Z}[\frac{1}{2}]$ and $n \in \mathbb{Z}$. The derived group is A as $x = [x^{-1}, y^{-1}]$. The conjugation action of G on G' factors through G/G' , which is infinite cyclic generated by $\bar{y}$. $\text{Hom}(G, \mathbb{R}) = \mathbb{R}$ and if $\chi: G \rightarrow \mathbb{R}$ is given by $y \mapsto 1$, then $G_\chi = \{x^q y^n \mid n \geq 0\}$ and $G_{-\chi} = \{x^q y^n \mid n \leq 0\}$. $\chi \notin \Sigma(G)$ since $\mathbb{Z}[\frac{1}{2}]$ is not finitely generated as a $\mathbb{Z}[\bar{y}]$ module, but $-\chi \in \Sigma(G)$ as it is finitely generated as a $\mathbb{Z}[\bar{y}^{-1}]$ module. Thus $[\Sigma(G)]$ consists of one point, $[-\chi]$. We now form a coset tree with vertices of the form $A_i a$ in A where $A_i = \{x^q \mid 2^i \mid q\}$, so $A_0 = \langle x \rangle, A_1 = \langle x^2 \rangle, A_2 = \langle x^4 \rangle, A_{-1} = \langle x^{1/2} \rangle \dots$. The edges connect $A_i a$ and $A_{i-1} a$ for each $a \in A$ (ie, inclusion of cosets). Here is a small section



We get bifurcation at each level as the index of A_i in A_{i-1} is 2. A acts on the tree by multiplication, with a_0 sending $A_i a \mapsto A_i a a_0$. y acts on the tree by conjugation, corresponding to squaring so $A_i a \mapsto A_{i+1} a^2$. This gives an action of $A \rtimes \langle y \rangle$, and in this case this is isomorphic to G . This tree maps onto the real line, with y acting by translation. There is one (fixed) end on the left, and infinitely many on the right, with infinitely many G -orbits. We know $\chi \notin \Sigma(G)$

We fix some notation of finitely generated metabelian G . We let $A = G'$ and $Q = G/A$, which we will assume is free abelian. A is a finitely generated Q -module. For $\lambda \in \mathbb{Z}Q$, write $\lambda = \sum n_i q_i$ and if $\chi: Q \rightarrow \mathbb{R}$ is a homomorphism, suppose $\min \chi(q_i)$ is attained for $i = 1, \dots, t$. Then we denote λ_χ to be the element of $\mathbb{Z}Q$ given by $\sum_{1 \leq i \leq t} n_i q_i$. Observe that if $\lambda, \mu \in \mathbb{Z}Q$, then $(\lambda\mu)_\chi = \lambda_\chi \mu_\chi$.

For $r \in \mathbb{R}$ denote $Q_{\chi, r} = \{q \mid \chi(q) \geq r\}$. Then $Q_{\chi, 0} = Q_\chi$.

Definition 2.1. If X is a finite generating set for the $\mathbb{Z}Q$ -module A , then the natural (χ, X) -filtration of A is $\{A_r\}$ where $A_r = X\mathbb{Z}Q_{\chi, r}$. These are $\mathbb{Z}Q_\chi$ -submodules and $A_i q = A_{i+\chi(q)}$. The filtration is *trivial* if for some r , $A = A_r$. Note that if $\chi \neq 0$ and $A = A_r$, then $A = A_s$ for all $s \in \mathbb{R}$.

Proposition 2.1. *The following are equivalent*

1. A is generated by X over $\mathbb{Z}Q_\chi$
2. The natural filtration is trivial
3. A is generated by X over $\mathbb{Z}M$ where M is a finitely generated submonoid of Q_χ
4. For each $x \in X$, there is a $\lambda \in \mathbb{Z}Q$ such that $x\lambda = 0$ and $\lambda_\chi = 1$

Proof. (1) and (2) are trivially equivalent.

For (2) implies (3), suppose that (2) holds. Then $A = A_0$ so take a finite generating set Y of Q with $Y = Y^{-1}$. For each $x_i \in X, y_j \in Y$ we have $x_i y_j = \sum x_j \lambda_{ji}$ for some $\lambda_{ji} \in \mathbb{Z}Q_\chi$. Let M be the monoid generated by the supports of the λ_{ji} . This is a finite set. Then $X\mathbb{Z}M y = X y \mathbb{Z}M \subseteq X\mathbb{Z}M$ for all $y \in Y$, so $X\mathbb{Z}M$ is a $\mathbb{Z}Q$ module containing X , and hence is A , and so (3) holds.

For (3) implies (2), Take the natural filtration $\{A_r \mid r \in \mathbb{R}\}$. 3 implies A is a Noetherian module over the Noetherian ring $\mathbb{Z}M$, but each A_i is a $\mathbb{Z}Q_\chi$ module, and hence a $\mathbb{Z}M$ module, done by ACC.

Suppose any of 1-3 hold. Then using 3 implies 2, $x\mathbb{Z}Q = x\mathbb{Z}Q_{\chi,r}$ for some $r > 0$, so $x \in x\mathbb{Z}Q_{\chi,r}$ and so $x = x\mu$ for some $\mu \in \mathcal{O}_{\chi,r}$. Set $\lambda = 1 - \mu$ and observe this is what we want for 4.

Finally assume (4) holds. Thus $x = x\mu$ for some $\mu \in \mathbb{Z}Q_{\chi,r}$ and $r > 0$. Hence $x = x\mu^n$ for all $n \geq 0$ and so $x \in X\mathbb{Z}Q_{\chi,nr}$ for all $n > 0$, and so $x \in \bigcap X\mathbb{Z}Q_{\chi,m}$. But this intersection is a $\mathbb{Z}Q$ submodule of A which contains X , so it must be A and the natural filtration is trivial. $\square$

Remark 2.1. These give alternative definitions of Σ

1. Set $S_\chi = 1 + \mathbb{Z}Q_{\chi>0}$. Then S_χ is multiplicatively closed. We can localise with respect to S_χ and 2.1(4) says that the localisation A_{S_χ} is 0 if and only if $\chi \in \Sigma$
2. If we form the Novikov ring, (2) says that $A \otimes_{\mathbb{Z}Q} \widehat{\mathbb{Z}Q_\chi} = 0$ is equivalent to $\chi \in \Sigma$

Note that these are independent of generating set X

Corollary 2.0.1. Σ is an open subset of $\text{Hom}(G, \mathbb{R}) \setminus \{0\}$ for such metabelian G

Proof. Immediate from 2.1(4), as small perturbations of χ preserve this condition. $\square$

Proposition 2.2. (One relator modules) Suppose $A = x\mathbb{Z}Q$ and $\text{Ann}(x) = r\mathbb{Z}Q$ so that $A = \mathbb{Z}Q/r\mathbb{Z}Q$. Then $\chi \in \Sigma$ is equivalent to $r_\chi = \pm q$ for some $q \in Q$.

Proof. From Proposition 2.1(4), $\chi \in \Sigma$ if there is λ such that $x\lambda = 0$ and $\lambda_\chi = 1$, which is equivalent to $\lambda = rt$ for some $t \in \mathbb{Z}Q$ and $\lambda_\chi = 1$. But $\lambda_\chi = r_\chi t_\chi$, so if $\chi \in \Sigma$, r_χ must be a unit in $\mathbb{Z}Q$ and hence is $\pm q$. Conversely if $r_\chi = \pm q$, then $r(\pm q^{-1}) = \lambda$, $\lambda_\chi = 1$ and $x\lambda = 0$ $\square$

Example 2.2. Let Q be free abelian with generators $\bar{x}, \bar{y}$ and $A = \mathbb{Z}Q/(\bar{x} + \bar{y} + 1)\mathbb{Z}Q$. Now $r = \bar{x} + \bar{y} + 1$. If we plot the support in a lattice in the plane, and take the convex hull we get a triangle with vertices at $1, \bar{y}, \bar{x}$. We want to compute $\Sigma^c \subseteq \text{Hom}(Q, \mathbb{R})$. Drawing this, we get a line where $r_\chi = \bar{x} + \bar{y}$, one where $r_\chi = 1 + \bar{x}$, a line where $r_\chi = 1 + \bar{y}$. Bieri and Strebel tells us that any G with $G/A = Q$ and A as in the example is finitely presented, as Σ^c has no pair of antipodal points

Example 2.3. If we now have relator $2 + \bar{x} + \bar{y}$, B-S tells us any such G is not finitely presented

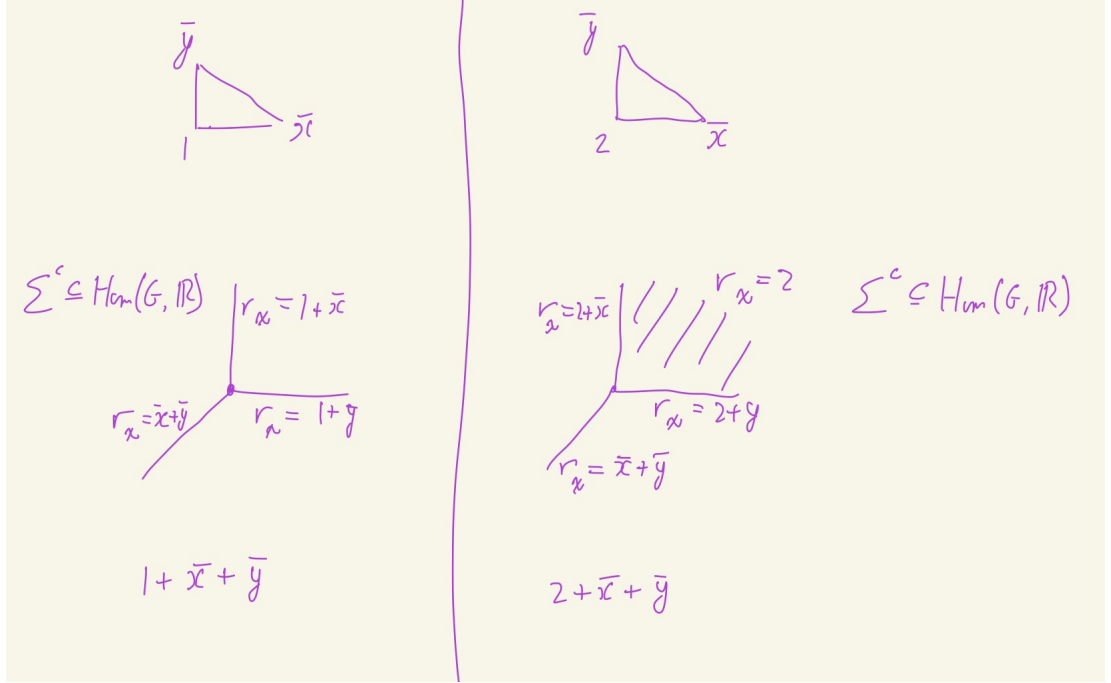


Figure 1: Examples 2.2 (left) and 2.3 (right)

Example 2.4. Let $A = \mathbb{Z}Q/(1 + \bar{x} + \bar{y} + \bar{x}^{-1} + \bar{y}^{-1})$. Then we get the diagram below. Note that 1 in the support does not affect Σ^c , so only the relation matters

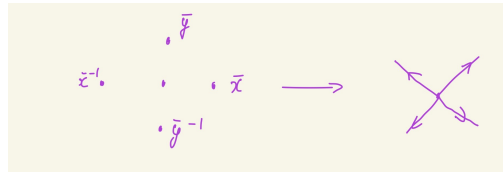


Figure 2: Example 2.4

Remark 2.2.

1. Extending to one generator modules with multiple relations means considering $A = \mathbb{Z}Q/I$ with I being a (necessarily finitely generated) ideal of $\mathbb{Z}Q$. It is immediate from our definition that if $r \in I$, then $\Sigma^c(\mathbb{Z}Q/I) \subseteq$

$\Sigma^c(\mathbb{Z}Q/r\mathbb{Z}Q)$. The Bieri-Groves polyhedrality theorem comes from seeing that we can pick a ‘nice’ generating set $r_1, \dots, r_m$ for I such that $\Sigma^c(\mathbb{Z}Q/I) = \bigcap \Sigma^c(\mathbb{Z}Q/r\mathbb{Z}Q)$. This is reminiscent of Gröbner bases in computer algebra - there is a well known algorithm for producing such bases. Similarly, it is possible to come up with an algorithm for the basis we want here, and so Bieri-Groves is an effective theorem to calculate Σ^c and Σ

2. Extending to modules A requiring more than one generator is straightforward via Lemma 2.1 below

Lemma 2.1. *Given an exact sequence of modules*

$$0 \rightarrow A_1 \rightarrow A \rightarrow A_2 \rightarrow 0$$

we have $\Sigma^c(A) = \Sigma^c(A_1) \cup \Sigma^c(A_2)$

Proof. Immediate from Proposition 2.1, namely Remark 2.1(1) and the exactness of localisation. $\square$

Now we consider the special case of Theorem 1.3. As usual $A = G'$, $G/A = Q$ is free abelian.

Theorem 2.2. *A is finitely generated abelian if and only if $\Sigma(G) = \text{Hom}(G, \mathbb{R}) \setminus \{0\}$*

Proof. The forward direction is trivial. For the backwards direction, assume A is a cyclic $\mathbb{Z}Q$ module generated by x . The argument uses the compactness of the sphere $S(G)$. For each $\chi \neq 0$, there is $\lambda \in \mathbb{Z}Q$ such that $x\lambda = 0$ and $\lambda_\chi = 1$ by Proposition 2.1. There is an open subset $O_\chi \subseteq S(G)$ containing $[\chi]$ such that the equations above hold. As we are assuming $[\Sigma(G)] = S(G)$, the sphere is covered by these open sets, and so there is a finite subcover. We hence have finitely many $\lambda_1, \dots, \lambda_t$ such that $x\lambda_j = 0$ and for any $\chi \neq 0$ there is some λ_j such that $(\lambda_j)_\chi = 1$. Write $\lambda_j = 1 - \mu_j$ for each j . Embed $Q \subseteq \mathbb{R}^n$ as lattice points and let B_ρ be the open ball centred at 0 of radius ρ . Set $Y_\rho = Q \cap B_\rho$. This is finite for each ρ , and forms an ascending chain. If k is a positive integer, then $Y_{\sqrt{k}} = Y_\rho$ for any $\sqrt{k-1} < \rho \leq k$, so $\rho \rightarrow Y_\rho$ is constant on half-open intervals. There is a radius ρ_0 and a function $\varepsilon: \{\rho \in \mathbb{R}^+ \mid \rho > \rho_0\} \rightarrow \mathbb{R}^+$ such that for every $\rho > \rho_0$ and $q \in Y_{\rho+\varepsilon(\rho)}$, one can find some $\lambda_j = 1 - \mu_j$ such that $q\mu_j \in \mathbb{Z}Y_\rho$ (only the ‘outermost point’ of the support of $q\lambda_j$ is in the outer shell). Assuming this, $0 = xq\lambda_j = xq(1 - \mu_j) = xq - xq\mu_j$ and $xq \in x\mathbb{Z}Y_\rho$. Varying $q \in Y_{\rho+\varepsilon(\rho)}$, we have that $x\mathbb{Z}Y_{\rho+\varepsilon(\rho)} = x\mathbb{Z}Y_\rho$ for every $\rho > \rho_0$. Applying for $\rho = \sqrt{k} > \rho_0$, we have $\mathbb{Z}Q = xY_\rho$ for every $\rho > \rho_0$, so we’re done as $x\mathbb{Z}Y_\rho$ is finitely generated abelian. $\square$

Remark 2.3. The proof of the more general case of Theorem 1.3 with $N \trianglelefteq G$, $N \geq A$ and G/N abelian is essentially the same argument

We prove Theorem 1.2

Proof. We prove a special case of the backwards direction where $G/A = Q$ is free abelian and $A = x\mathbb{Z}Q$ is a cyclic $\mathbb{Z}Q$ module with generator x , and $A \rightarrow G \rightarrow Q$ split. Suppose $[\Sigma(G)] \cup -[\Sigma(G)] = S(G)$. Then for each $\chi \neq 0$, there is either λ with $x\lambda = 0, \lambda_\chi = 1$ or λ with $x\lambda = 0, \lambda_{-\chi} = 1$. By the same compactness argument as before, there is a finite set $\lambda_1, \dots, \lambda_t$ such that for each χ there is some λ_j with $x\lambda_j = 0, (\lambda_j)_\chi = 1$ or the negative. As before we write $\lambda_j = 1 - \mu_j$ with $\mu_j = \sum n_{jk} q_{jk}$ where $n_{jk} \in \mathbb{Z}$ and $q_{jk} \in Q$. Q is freely generated as an abelian group by $q_1, \dots, q_n$ and

$$Q = \langle q_1, \dots, q_n \mid [q_i, q_j] = 1 \rangle$$

The aim is to construct a presentation for $A \rtimes Q$, and so we need to ensure all the conjugates $\{x^q \mid q \in Q\}$ commute (writing A multiplicatively). Let $H_\rho = \langle x, q_1, \dots, q_n \mid [q_i, q_j] = 1, [x, x^q] = 1, q \in Y_\rho \rangle$ (where Y_ρ is as in the previous proof) where q is expressed in terms of the q_i . For each j , we also add the relation $x = x^{n_{j1}q_{j1}} x^{n_{j2}q_{j2}} \dots x^{n_{jk}q_{jk}}$. We use another key geometric fact: there is a radius ρ_0 and a function $\varepsilon: \{\rho > \rho_0\} \rightarrow \mathbb{R}^+$ such that for every $\rho > \rho_0$ and each $q \in Y_{\rho+\varepsilon(\rho)}$ we can find some $\lambda_j = 1 - \mu_j$ with $q\mu_j \in \mathbb{Z}Y_\rho$ or $q^{-1}\mu_j \in \mathbb{Z}Y_\rho$. But in H_ρ , all conjugates x^q with $q \in Y_\rho$ commute with x and if $q \in Y_{\rho+\varepsilon(\rho)}$ we have either $xq = xq\mu_j \in x\mathbb{Z}Y_\rho$ or the same with q^{-1} , so either x commutes with x^q or $x^{q^{-1}}$, but in the latter case, conjugating by q gives that x commutes with x^q . So $H_\rho = H_{\rho+\varepsilon(\rho)}$. Also using that $H_\rho = H_{\sqrt{k}}$ for $\sqrt{k-1} < \rho \leq \sqrt{k}$, so we have that all H_ρ are equal for $\rho > \rho_0$, and we denote this H_∞ , which is finitely presented. We have that x commutes with x^q for all $q \in Q$ in H_∞ , and so any conjugate x^q commutes with $x^{q'}$ for $q, q' \in Q$, so the conjugates of x form a normal abelian subgroup of H_∞ , called A_∞ , where $H_\infty/A_\infty = Q$. We have a natural map $\phi: H_\infty \rightarrow A \rtimes Q$. Consider $\ker \phi$. This is a $\mathbb{Z}Q$ submodule of the cyclic $\mathbb{Z}Q$ module A_∞ . $\mathbb{Z}Q$ is Noetherian so $\ker \phi$ is finitely generated as a $\mathbb{Z}Q$ module. Take a finite set of generators and add them to the presentation of H_∞ . We are then done. $\square$

For the converse, we will prove

Theorem 2.3. *If G is finitely presented with no non-abelian free subgroups, then $[\Sigma(G)] \cup -[\Sigma(G)] = S(G)$*

Proof. We use Cayley graphs. For $\chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\}$ and $r \in \mathbb{R}$, we define $\Gamma_{\chi, r}$ to be the full subgraph of the Cayley graph on G with vertices $\{g \in G \mid \chi(g) \geq r\}$. We form a graph with vertices the components of $\Gamma_{\chi, r}$ (varying r) and edges given by inclusion. If χ is discrete, this forms a simplicial tree, but is only an $\mathbb{R}$ -tree in general. There is a natural action of G on this tree, and this equivariantly projects to $\mathbb{R}$ with G acting as $\chi(g)$.

There is a map from the Cayley graph to the component tree sending the vertex g to the component of g in $\Gamma_{\chi, \chi(g)}$, and the edge (g, g_1g) gets mapped to the only inclusion it could be.

Observe that each $\Gamma_{\chi, r}$ is acted on by $K = \ker \chi$, and so we have an induced action of K on the components of $\Gamma_{\chi, r}$ for each $r \in \mathbb{R}$.

We set $K_r = \{k \in K \mid k \text{ is connected to } 1 \text{ in } \Gamma_{\chi,r}\}$ for $r \leq 0$. This gives a filtration of K with $K_s \supseteq K_r$ if $r \leq s \leq 0$, and the filtration is exhaustive as the Cayley graph is connected. Much as in Proposition (2.1), we have $K = K_r$ for some $r < 0$ if and only if K is finitely generated over a finitely generated monoid of G_χ .

Now suppose G is finitely presented. We define the Cayley 2-complex to be the universal cover of the usual presentation complex. We have a finite number of G -orbits of G cells, and any this has trivial fundamental group. Note that in the map from the Cayley graph to the component tree, the boundary of a 2-cell in this complex is mapped to a subtree.

Now we consider the map $-\chi$ and the subgraphs $\Gamma_{-\chi,-s}$. We note that this is $\{g \in G \mid \chi(g) \leq s\}$. Once again we get a component tree for $-\chi$ which branches to the left this time (instead of the right). Pick $r < 0$ and s sufficiently large that each boundary of a 2-cell lies in $\Gamma_{\chi,0}$ or $\Gamma_{-\chi,-s}$ (or both), which is possible as we only have finitely many G -orbits of such boundaries.

We form another graph with vertices the components of $\Gamma_{\chi,0}$ or of $\Gamma_{-\chi,-s}$, with edges between components if they intersect. This is acted on by K , and is a tree. We deduce that $K = L_1 *_{L_{12}} L_1$ is an amalgamated free product. This has a non-abelian free subgroup unless $L_{12} = L_1$ or $L_{12} = L_2$, or it has index 2 in both. If $\chi \notin \Sigma$ and $-\chi \notin \Sigma$, then $L_{12} \neq L_1$ (and similar for L_2). Index 2 can be ruled out by enlarging s . $\square$

We return to metabelian groups, and let $Q = Q_1 \times Q_2$, so that $\text{Hom}(Q, \mathbb{R}) = \text{Hom}(Q_1, \mathbb{R}) \times \text{Hom}(Q_2, \mathbb{R})$, where Q_1, Q_2 are free abelian by our usual simplification, and we have a finitely generated $\mathbb{Z}Q_i$ module called A_i for $i = 1, 2$.

We have an extension $A_i \rightarrow G_i \rightarrow Q_i$ giving up $\Sigma(G_i)$, and we let $A = A_1 \otimes A_2$, where Q acts by $(q_1, q_2)(a_1 \otimes a_2) = q_1 a_1 \otimes q_2 a_2$. The immediate problem is that $A_1 \otimes A_2 = 0$ if $A_1 p_1 = 0, A_2 p_2 = 0$ for $p_1 \neq p_2$ different primes.

Theorem 2.4. (*Bieri-Groves*) *In the situation above, if A_i are annihilated by the same prime $p \in \mathbb{Z}$, then $\Sigma^c(G) = \Sigma^c(G_1) \times \Sigma^c(G_2)$, where G has a normal subgroup isomorphic to A with quotient Q .*

Remark 2.4. If we regard $\text{Hom}(Q_1, \mathbb{R})$ as a subspace of $\text{Hom}(Q, \mathbb{R})$ with $\chi(Q_2) = 0$ (and similarly with Q_2), we write $\Sigma^c(G) = \Sigma^c(G_1) + \Sigma^c(G_2)$, which is how it appeared in the original paper.

Proof. Take a generating set X_i of A_i as an $\mathbb{F}_p Q_i$ module for $i = 1, 2$. We write χ_i for $\chi|_{Q_i}$, and form the natural χ_i -filtration for A_i . From 2.1 we know that $\chi_i \notin \Sigma(G_i)$ if and only if the filtrations are non-trivial. We recall that $A_{i,s} = X_i \mathbb{F}_p Q_{i,\chi_i,s}$ and set $X = X_1 \otimes X_2$ to be an $\mathbb{F}_p Q$ generating set for $A_1 \otimes A_2$. We then form $X \mathbb{F}_p Q_{\chi,r} = \sum_{s_1+s_2=r} A_{1,\chi_1,s_1} \otimes A_{2,\chi_2,s_2} \subseteq A_1 \otimes A_2$.

These are $\mathbb{F}_p$ -vector spaces and if $\chi_i \notin \Sigma(G_i)$, then in particular $A_i \neq A_{i,\chi_i,0}$ by non-triviality. Take $a'_i \in A_i \setminus A_{i,\chi_i,0}$. Then $a'_1 \otimes a'_2 \notin X \mathbb{F}_p Q_{\chi,0}$, as everything is an $\mathbb{F}_p$ vector space, so we have non-triviality.

If one of the χ_i is 0, this is easy. $\square$

Now consider the special case of 2.4 where $Q_1 = Q_2, A_1 = A_2$ with Q_1^2 acting on $A_1^{\otimes 2}$. Consider the diagonal subgroup of Q_1^2 . This acts on $A_1^{\otimes 2}$ via the diagonal action. We have two questions, which are equivalent:

- When is $A_1^{\otimes 2}$ finitely generated under the diagonal action?
- When is N finitely generated, where $A_1^{\otimes 2} \subseteq N \trianglelefteq G$ and $N/A_1^{\otimes 2}$ is the diagonal subgroup of Q_1^2 .

Theorem 2.5. *Suppose A_1 is a finitely generated $\mathbb{F}_p Q_1$ module. Then $A_1^{\otimes 2}$ is a finitely generated Q_1 module under the diagonal action if and only if $[\Sigma(G_1)] \cup -[\Sigma(G_1)] = S(G_1)$*

Proof. With $Q = Q_1 \times Q_1$, 1.3 says N is finitely generated as a group if and only if $\{\chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \chi(N) = 0\} \subseteq \Sigma(G)$. We consider those χ with $\chi(N) = 0$. Then $\chi(q_1, q_1) = 0$ for all $q \in Q$. Writing $\chi = (\chi_1, \chi'_1)$, we have $\chi_1(q_1) + \chi'_1(q_1) = 0$, so $\chi_1 = -\chi'_1$.

From 2.4, $\chi \in \Sigma(G)$ if and only if either χ, χ'_1 are in $\Sigma(G_1)$. But for those χ vanishing on N , these are negatives of each other, and so one of them lies in $\Sigma(G_1)$ and we are done. $\square$

Remark 2.5. The theorem is about finite generation of a tensor square under the diagonal action. When we look at the homology of abelian groups, exterior powers arise, so in particular, the second homology group is the exterior square. The condition for this to be finitely generated as a Q_1 module is the same as 2.5.

Remark 2.6. If A is a finitely generated $\mathbb{Z}Q$ module, then $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finitely generated $\mathbb{Q}Q$ module. We can make the same definition for $A_{\mathbb{Q}}$ being finitely generated over $\mathbb{Q}M$ for some finitely generated submodule $M \leq Q_{\chi}$ and define $\Sigma_{\mathbb{Q}}^c(G)$ where $A \rightarrow G \rightarrow Q$. Note that $\Sigma_{\mathbb{Q}} \supseteq \Sigma(G)$

3 Polyhedrality

Let D be a division ring and Q a finitely generated free abelian group. DQ can be taken to be a *skew group algebra*. We suppose A is a finitely generated DQ module and define

$$\Sigma_D(A) = \{\chi \in \text{Hom}(Q, \mathbb{R}) \setminus \{0\} \mid A \text{ fg over } DM \text{ for some fg submodule of } Q_{\chi}\}$$

Theorem 3.1. *(Wadsley, yes that one, 2005) $\Sigma_D^c(A)$ is a closed rational polyhedron.*

Proof. Deferred. $\square$

As before we have an analogue of 2.1, giving an alternative description in terms of filtrations.

Definition 3.1. The *associated graded ring*, $\text{gr}^X(DQ) = \bigoplus_{r \in \mathbb{R}} DQ_{\chi, r} / DQ_{\chi, r}^+$ where $DQ_{\chi, r}^+ = \bigcup_{s > r} DQ_{\chi, s}$ with the usual multiplication for graded rings.

For A generated by X as a DQ module,

$$\text{gr}^X(A) = \bigoplus_{r \in \mathbb{R}} XDQ_{\chi, r} / XDQ_{\chi, r}^+$$

which is a graded module for $\text{gr}^X(DQ)$

We observe that $\text{gr}^X DQ = DQ$ with a grading.

Definition 3.2. For a subset $\Gamma \subseteq \mathbb{R}^n$ and $\gamma \in \Gamma$, the *local cone* of Γ at γ is

$$LC_\gamma(\Gamma) = \{\gamma_1 \in \mathbb{R}^n \mid \exists \varepsilon_1 > 0, \forall \varepsilon \in (0, \varepsilon_1], \gamma + \varepsilon \gamma_1 \in \Gamma\}$$

Theorem 3.2. (*Wadsley, Bieri-Groves*)

$$LC_\chi(\Sigma_D^c(A)) = \Sigma^c(\text{gr}^X(A))$$

Proof. Deferred. □

If Q is free abelian of rank n with free generating set $\{q_1, \dots, q_n\}$, then there is a map $D[x_1, \dots, x_{2n}] \rightarrow DQ$ by $x_i \mapsto q_i$ if $1 \leq i \leq n$ and $x_i \mapsto -q_{i-n}$ otherwise. Therefore A is a module over $R = D[x_1, \dots, x_{2n}]$. Set $m = 2n$. We use the notation x^α where $\alpha \in \mathbb{N}^m$, which grades R . Suppose $<$ is a total ordering on $\mathbb{N}^m$. Then we can define:

Definition 3.3. For $r = \sum \lambda_\alpha x^\alpha \in R$, the *Newton diagram* is $N(r) = \{\alpha \mid \lambda_\alpha \neq 0\}$, the support of r . If $r \neq 0$, the *privileged exponent* is the maximal element of $N(r)$, denoted by $\exp_<(r)$.

Given a right ideal $I \trianglelefteq R$ we define $\exp_<(I) = \{\exp_<(r) \mid 0 \neq r \in I\}$. This is closed under addition from $\mathbb{N}^m$.

If $I = 0$, then $\exp(I) = -\infty$.

For Gröbner bases, we need to be dealing with well orderings, so we can mimic Euclid's algorithm for computing highest common factors in $\mathbb{Z}$. The orderings arising from $\chi \in \text{Hom}(\mathbb{N}^m, \mathbb{R})$ are not necessarily well orderings. The trick is to use $R[t]$ instead of R .

Definition 3.4. Given $\alpha \in \mathbb{N}^m$, we write $|\alpha|$ or the *total degree* of α . A total ordering $<$ on $\mathbb{N}^m$ is *compatible with sums* if $\alpha < \bullet$ implies $\alpha + \gamma < \beta + \gamma$. For such a $<$, we define a total well ordering $<^h$ on $\mathbb{N}^{n+1}$ which is also compatible with respect to sums where $(\alpha, k) <^h (\beta, l)$ if $|\alpha| + k < |\beta| + l$, or they are equal and $\alpha < \beta$.

Observe that $R[t]$ has a notion of Newton diagram and $\exp$ with respect to $<^h$, and a grading with respect to total degree, and we define homogeneous components with respect to this.

Definition 3.5. We define $h: R \rightarrow R[t]$ by $\sum \lambda_\alpha x^\alpha \mapsto \sum \lambda_\alpha x^\alpha t^{k-|\alpha|}$ and $H: R[t] \rightarrow R$ by $t \mapsto 1$.

We note that $H \circ h = \text{id}_R$, and the image of h is homogeneous.

Thus if $I \trianglelefteq R$ is a right ideal, then $h(I)R[t]$ is graded in $R[t]$, and $H(h(I)R[t]) = I$.

Proposition 3.1. (*Division algorithm*) *If we have $<$ on $\mathbb{N}^m$ which is compatible with sums and $\{r_1(t), \dots, r_k(t)\} \in R[t]$ are homogeneous, then for each $p(t) \in R[t]$, there are $q_1(t), \dots, q_n(t)$ and $r(t)$ such that*

1. $p(t) = r(t) + \sum q_i(t)r_i(t)$
2. $\exp_{<^h}(q_i(t)r_i(t)) \leq \exp_{<^h}(p(t))$
3. $N(r(t)) \cap \bigcup (\exp(r_i(t) + \mathbb{N}^{n+1})) = \emptyset$

Proof. What you expect. $\square$

Definition 3.6. Given a graded right ideal J of $R[t]$, $r_1(t), \dots, r_k(t) \in J$ is a *Gröbner basis* with respect to $<^h$ if $\exp_{<^h}(J) = \bigcup \exp_{<^h}(r_i(t) + \mathbb{N}^{n+1})$.

It is *reduced* if for all $i \neq j$, $N(r_i(t)) \cap \exp_{<^h}(r_j(t) + \mathbb{N}^{n+1}) = \emptyset$ and is *homogeneous* if it is homogeneous with respect to total degree.

Theorem 3.3. *Every non-zero graded right ideal $J \trianglelefteq R[t]$ has a homogeneous reduced Gröbner basis.*

Proof. Use the division algorithm. $\square$

Theorem 3.4. *For such a J , the set of possible $\exp_{<^h}(J)$ is finite, ranging over $<$ compatible with sums.*

Proof. Let $E = \exp_{<^h}$. Then $E + \mathbb{N}^{n+1} = E$ and $\dim_D(R[t]_l/J_l) = |\{\alpha \in \mathbb{N}^{n+1} \setminus E \mid |\alpha| = l\}|$, so there are only finitely many choices. $\square$

We now consider $\chi \in \text{Hom}(\mathbb{N}^m, \mathbb{R})$. We define $<_\chi$, a total ordering compatible with sums, on $\mathbb{N}^m$ by $\alpha <_\chi \beta$ if $\chi(\alpha) > \chi(\beta)$ (note that this is not a mistake, and allows us to convert our discussion about maxima into minima), or $\chi(\alpha) = \chi(\beta)$ and $\alpha < \beta$.

For $\chi \in \text{Hom}(\mathbb{N}^m, \mathbb{R})$, we have a χ -filtration $R_{\chi,s} = D\{x^\alpha \mid \chi(\alpha) \geq s\}$, and so for a right ideal $I \trianglelefteq R$ we have a filtration by intersecting with $R_{\chi,s}$, and we can form the associated graded ideal $\text{gr}^\chi(I)$.

Note this is induced by intersections, not by module generating sets. This is so we have $\text{gr}^\chi(R/I) = \text{gr}^\chi(R)/\text{gr}^\chi(I)$.

Given this filtration of R , we have a filtration $R_{\chi,s}[t]$ of $R[t]$, and we can form the associated graded ring. If $p(t) \in R[t]$ such that $p(t) \in R_{\chi,s}[t] \setminus R_{\chi,s}^+[t]$, the *symbol* of $p(t)$ is $\sigma^\chi(p(t)) = p(t) + R_{\chi,s}^+[t] \in \text{gr}^\chi(R[t])$.

Lemma 3.5. *For a right ideal $I \trianglelefteq R$,*

$$H(\text{gr}^\chi(h(I)R[t])) = \text{gr}^\chi(I)$$

Proof. Trivial $\square$

Proposition 3.2. *If $0 \neq J \trianglelefteq R[t]$ is a right ideal, then the set of associated graded rings $\{\text{gr}^\chi(J) \mid \chi \in \text{Hom}(\mathbb{N}^m, \mathbb{R})\}$ is finite. Moreover, for each $\chi \in \text{Hom}(\mathbb{N}^m, \mathbb{R})$, we write $E[\chi] = \{\chi' \in \text{Hom}(\mathbb{N}^m, \mathbb{R}) \mid \text{gr}^{\chi'}(J) = \text{gr}^\chi(J)\}$ is a convex polyhedron.*

Proof. To sketch the proof, we show that for a subset E such that $E + \mathbb{N}^m = E$, we have $\{\text{gr}^\chi(J) \mid \exp_{<_\chi^h}(J) = E\}$ is finite, and taking a homogeneous Gröbner basis for J with respect to $<_\chi^h$, we claim we have $E[\chi]$ equal to the set of all χ' with the same symbol for all of the basis as χ has. Assuming this, $E[\chi]$ is an intersection of hyperplanes and open half spaces (giving the conditions where χ' has the correct term, and that it has no larger symbol. $\square$

Definition 3.7. Given a right ideal of R , $\chi, \chi' \in \text{Hom}(\mathbb{N}^m, \mathbb{R})$ are *equivalent* with respect to I if $\text{gr}^\chi(I) = \text{gr}^{\chi'}(I)$

Theorem 3.6. *For any non-zero $I \trianglelefteq R$, $\text{Hom}(\mathbb{N}^m, \mathbb{R})$ has finitely many equivalence classes with respect to I and each class is a polyhedral cone.*

Proof. Follows from 3.5 and 3.2. $\square$

Recall that Σ corresponds to those χ such that $\text{gr}^\chi(I) = R$ (equivalently, $\text{gr}^\chi(R/I) = 0$). Recall also 3.2 linking the local cone of Σ at χ to Σ of the graded module. To establish this, notice that 3.2 shows that there is ε_0 such that $\{\chi + \varepsilon\chi' \mid \varepsilon \in (0, \varepsilon_0]\}$ is contained in one equivalence class with respect to $h(I)R[t]$. We can prove

Lemma 3.7. *For $I \trianglelefteq R$ a right ideal and $\chi, \chi' \in \text{Hom}(\mathbb{N}^m, \mathbb{R})$ and each sufficiently small ε , we have*

$$\text{gr}^{\chi'} \text{gr}^\chi(I) = \text{gr}^{\chi + \varepsilon\chi'}(I)$$

Proof. No Proof $\square$

This provides the link between $\Sigma^c(\text{gr}^\chi(R/I))$ and the local cone of χ in $\Sigma^c(R/I)$.

3.1 Relationship with toric varieties

Suppose Q is a lattice in $\mathbb{R}^n = V$ and $\text{Hom}(Q, \mathbb{Z})$ is the dual lattice in $\text{Hom}(Q, \mathbb{R})$. Given a convex rational polyhedral cone C generated by a finite number of vectors in $\text{Hom}(Q, \mathbb{R})$, we have the dual cone $C^\vee = \{v \in V \mid \chi(v) \geq 0, \forall \chi \in C\}$. Intersecting with Q we get a monoid $M_c \leq Q$.

We have a *fan* of cones, $\mathcal{F}$ such that any face of $C \in \mathcal{F}$ lies in $\mathcal{F}$ and $0 \in \mathcal{F}$. For each $C \in \mathcal{F}$, we form $\text{maxspec } \mathbb{C}M_c$, which correspond to maps $M_c \rightarrow \mathbb{C}$. If $C_1 \subseteq C_2$, then $M_{C_2} \subseteq M_{C_1}$, we get a map $\text{maxspec } \mathbb{C}M_{C_1} \rightarrow \text{maxspec } \mathbb{C}M_{C_2}$ by contraction.

Example 3.1. If $C = \{0\}$, then $\text{maxspec } \mathbb{C}M_0 \cong (\mathbb{C}^\times)^n$. If C is generated by linearly independent vectors, the maxspec is $\mathbb{C}^n$.

For a given $\mathcal{F}$, we can glue all the maxspec together via the inclusion maps to give a toric variety $X_{\mathcal{F}}$. Given $I \trianglelefteq \mathbb{C}Q = CM_0$, we have a subvariety $Y_I = V(I) \subseteq \maxspec \mathbb{C}Q \cong (\mathbb{C}^\times)^n$, which is closed under the Zariski topology and also in the Euclidean topology.

Question: Can we take $X_{\mathcal{F}}$ so that the closure of Y_I in $X_{\mathcal{F}}$ is compact?

Answer: Yes. Certainly if $X_{\mathcal{F}}$ is compact itself (and this happens if $\bigcup_{C \in \mathcal{F}} C = \text{Hom}(Q, \mathbb{R})$). But we can compactify Y_I inside $X_{\mathcal{F}}$ as long as $\bigcup_{C \in \mathcal{F}} C \supseteq \Sigma^c(\mathbb{C}Q/I)$. The polyhedrality of $\Sigma^c(\mathbb{C}Q/I)$ gives a good choice of fan achieving equality here, and the closure of Y_I inside $X_{\mathcal{F}}$ with this choice is compact. (Tevelev, 2007).

Example 3.2. Suppose $\text{rk}(Q) = 1$ and $\mathcal{F} = \{0\}, \rightarrow, \leftarrow$. Then $X_{\mathcal{F}} = \mathbb{C}^\infty$

Recall that $X_{\mathcal{F}}$ is compact if and only if $\bigcup_{C \in \mathcal{F}} C = \text{Hom}(Q, \mathbb{R})$. We consider compactness using limit points, and aim to prove only the forwards direction here. We suppose for simplification that the cone C is generated by k linearly independent vectors that are a subset of a basis of the lattice $\text{Hom}(Q, \mathbb{Z})$. Then $\maxspec \mathbb{C}M_c \cong \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$. Take $(m_1, \dots, m_n) \in \text{Hom}(Q, \mathbb{Z})$ and $z \in \mathbb{C}$. As $z \rightarrow 0$, $(z^{m_1}, \dots, z^{m_n})$ has a limit point if and only if all $m_i \geq 0$, and $m_i = 0$ for $i > k$, which happens if and only if $(m, \dots, m_n) \in C$.

If the union of cones is not all of $\text{Hom}(Q, \mathbb{R})$, take a discrete point of the lattice not inside the union. Then $(z^{m_1}, \dots, z^{m_n})$ does not have a limit point in $X_{\mathcal{F}}$, so we are done.

4 3-manifold groups

Example 4.1. A 3-manifold M which fibres over the circle S^1 with fibres being a compact surface F can be viewed as a mapping torus. Taking fundamental groups, we have $\pi_1(F) \rightarrow \pi_1(M) \rightarrow \mathbb{Z}$.

Theorem 4.1. (*Stallings, 1961*) *Let M be a compact irreducible 3 manifold and suppose $0 \neq \phi \in \text{Hom}(\pi_1(M), \mathbb{Z})$ with finitely generated kernel not equal to $\mathbb{Z}/2\mathbb{Z}$ then we have a fibration $M \rightarrow S^1$ with map on fundamental groups given by ϕ , with $\ker \phi = \pi_1(F)$*

Comparing this with 1.3 generalised to all finitely generated groups, we know $\ker \phi$ is finitely generated if and only if $\phi, -\phi \in \Sigma(\pi_1(M))$, so $\Sigma(\pi_1(M))$ contrains information about fibrations over S^1 , (See Thurston, 1986 ‘A norm for the homology of 3-manifolds’). By Poincaré duality, we have $H_2(M, \mathbb{R}) \cong H^1(M, \mathbb{R}) \cong \text{Hom}(\pi_1(M), \mathbb{R})$ (and similar for the case with boundary). Assuming we have no boundary (the other case being similar), if $c \in H_2(M, \mathbb{Z})$, it may be represented by a compact embedded smooth surface S without boundary.

Definition 4.1. We define $\|c\|_T = \min_S \sum_{i=1}^n \chi_-(S_i)$ where the minimum is over all embedded surfaces representing c , the S_i are the components of S , and $\chi_-(F) = \max(0, -\chi(F))$.

We note that $\|kc\|_T = |k| \cdot \|c\|_T$ for $k \in \mathbb{Z}$ and the triangle inequality holds, so $\|\cdot\|_T$ extends to a function on $H_2(M, \mathbb{Q})$, and hence to a seminorm on $H_2(M, \mathbb{R})$.

Remark 4.1. Gabai showed that $\|\cdot\|_T$ is equal to $\frac{1}{2} \cdot$ Gromov seminorm.

We now switch to discussing $H^1(M, \mathbb{R})$. Thurston showed that the unit ball in $H^1(M, \mathbb{R}) \cong \text{Hom}(\pi_1(M), \mathbb{R})$ with respect to this seminorm is a polytope such that if $\phi: \pi_1(M) \rightarrow \mathbb{Z}$ is induced by a fibration, then ϕ lies in a maximal open face of the polytope, and every other ϕ' in the same cone also comes from a fibration, so we can talk of *fibred faces*.

We can assume that our 3-manifolds are smooth and we can use deRham cohomology to compute $H^1(M, \mathbb{R})$.

Definition 4.2. We define $\Sigma(M)$ to consist of those $\phi \neq 0$ such that ϕ is given by a nowhere vanishing closed 1-form which is non-vanishing on boundary.

Theorem 4.2. (*Tischler, 1970*) *A discrete (maps to $\mathbb{Z}$) map ϕ is in $\Sigma(M)$ if and only if it corresponds to a fibration.*

Proof. No Proof □

Theorem 4.3. (*Bi-N-St, 1987*) *For a smooth compact 3-manifold M , $\Sigma(M) = \Sigma(\pi_1(M))$*

Proof. No Proof □

Remark 4.2. In fact the original statement included a couple of exceptions arising from potential counterexamples to the Poincaré conjecture, but we now know this to be true.

Corollary 4.3.1. (*Bi-N-St, 1987*) *If $G = \pi_1(M)$, then $\Sigma(G)$ is symmetric and is a disjoint union of finitely many cones over convex rational polytopes of the fibred faces.*

Proof. Trivial □

Observe that for discrete ϕ , if $\phi, -\phi \in \Sigma(G)$, we know $\ker \phi$ is finitely generated, so ϕ corresponds to a fibration, so $\phi, -\phi \in \Sigma(M)$ so we have $\Sigma(G) \cap -\Sigma(G) \subseteq \Sigma(M)$.

We will see that $\Sigma(M) \subseteq \Sigma(G) \cap -\Sigma(G)$, and so we are additionally proving that $\Sigma(G)$ is symmetric in this case. For more discussion about the relationship between closed 1 forms and $\Sigma(G)$ in higher dimensions, see ‘Closed 1 forms in topology and geometric group theory’ by Farber, Geoghegan, Schutz (2008). Latour showed that for $\phi \in \Sigma(M)$, we need $\phi \in \Sigma^{(2)}(G)$, which is a generalisation of $\Sigma(G)$.

We make 3 observations before we start.

1. When we considered Cayley graphs Γ , we constructed a tree using the intersection of Γ with half spaces. $\phi \notin \Sigma(G)$ corresponds to a component tree with infinitely many ends to the right and one to the left. In other words, we can only connect components in $\Gamma_{\phi,0}$ by using a path that tracks via $\Gamma_{\phi,s}$ for some $s < 0$. Therefore in Γ/K where $K = \ker \phi$ or G' , considering the vertex $1 \in \Gamma$, this can only connect to a vertex k via $\Gamma_{\phi,s}$ for some $s < 0$, then in Γ/K we have a loop in $\Gamma_{\phi,s}/K$ based at $*$ (the image of 1) that doesn't exist in $\Gamma_{\phi,0}/K$, so the map $\pi_1(\Gamma_{\phi,0}/K) \rightarrow \pi_1(\Gamma/K)$ is not onto.
2. If G is a free group, $\Sigma(G)$ is empty unless $G = \mathbb{Z}$.
3. If G is a HNN extension of B , and ϕ is the map sending $B \rightarrow 0$ and $t \rightarrow 1$, then $\phi \in \Sigma(G)$ implies $B = B_1$ and we have an ascending HNN extension.

Proof. $\phi \in \Sigma(G)$ says there are no strict ϕ -filtrations in $\ker \phi$ $\square$

We now prove $\Sigma(M) \subseteq \Sigma(G)$.

Proof. Let X be the universal abelian cover of M . $\phi \in \Sigma(M)$ is represented by a closed 1 form not vanishing on M or ∂M . We call this ω . This lifts to an exact form, df , on X .

Set $X_f = f^{-1}([0, \infty))$ and let ζ_0 be a vector field on M tangent to ∂M with $\omega(\zeta_0) = 1$. We lift this to ζ on X . ζ_0 is complete on M as M is compact (ie, we have a global flow), so ζ is also complete. By construction, ζ is a lift via f of $\frac{d}{dt}$ on $\mathbb{R}$.

For any $x \in X$ let $g(x) \in f^{-1}(0)$ be the intersection of the flow line through x with $f^{-1}(0)$. Then $(f, g): X \rightarrow \mathbb{R} \times f^{-1}(0)$ is a homeomorphism mapping X_f to $[0, \infty) \times f^{-1}(0)$.

The inclusion $X_f \rightarrow X$ is an isomorphism of fundamental groups, and so by the first observation we deduce $\phi \in \Sigma(G)$. Note this does not depend on the dimension of M .

The converse is harder. From Thurston we know $\Sigma(M)$ is a rational polyhedral subset arising from a unit ball with respect to $\|\cdot\|_T$. To show that $\Sigma(G) \subseteq \Sigma(M)$, it suffices to show that no discrete $\phi \in \Sigma(G)$ is in $\Sigma^c(M)$. We may suppose that ϕ is surjective. We suppose that $(F, \partial F)$ is a properly embedded incompressible surface dual to ϕ . We assume this has a minimal number of components.

Cutting M along F represents M as a union of pieces M_i glued along components F_j of F . Then $G = \pi_1(M)$ is the fundamental group of the graph of groups corresponding to this decomposition with underlying graph Γ . ϕ factors as $\pi_1(M) \rightarrow \pi_1(\Gamma) \xrightarrow{\alpha} \mathbb{Z}$. Now $\phi \in \Sigma(G)$ implies $\alpha \in \Sigma(\pi_1(\Gamma))$ (easy lemma). But $\pi_1(\Gamma)$ is free, so $\pi_1(\Gamma) = \mathbb{Z}$ by observation 2.

So F is connected, and so cutting M along F gives M_0 two copies F_0, F_1 of F on the boundary of M_0 , with M the result of gluing these together. By incompressibility, $\pi_1(F_i) \rightarrow \pi_1(M_0)$ is an injection and G is a HNN extension.

Stallings showed that (M_0, F_1) is homeomorphic to $(F \otimes [0, 1], F \times \{1\})$. In particular, defining $F_2 = \partial M_0 \setminus F_1^\circ$, then $F_2 \cong F_1$ and the inclusion $F_0 \subseteq F_2$ is an inclusion of homeomorphic compact connected surfaces, and induces an injection of fundamental groups. Now $F_2 \setminus F_0^\circ$ has no closed components as F_2 is connected, and no disk components as $\pi_1(F_0) \rightarrow \pi_1(F_2)$ is injective, so it is a union of components with non-positive Euler characteristic, but it has Euler characteristic 0 as $F_0 \cong F_2$, so its components have zero Euler characteristic, so are Möbius bands or annuli. In the former case, F_0 has more components than F_2 , contradiction so it is a union of annuli.

So $(M_0, F_0 \cup F_1)$ is homeomorphic to $(F \times [0, 1], F \times \{0, 1\})$, so M is diffeomorphic to the total space of a fibration, so $\phi \in \Sigma(M)$. $\square$

4.1 HNN Extensions

We consider Condition (*) for finite sets $S \subseteq G, X \subseteq G'$ given by requiring that $[s_1, s_2]$ of elements $s_1, s_2 \in S^{\pm 1}$ acts on G' by conjugation by $c(s_1, s_2) \in \langle X \rangle$.

Lemma 4.4. *Suppose S, X satisfy the above condition and $\mathcal{A}, \mathcal{B}$ of sets of $S^{\pm 1}$ -words closed under taking subwords. Then $\langle X^{\mathcal{A}\mathcal{B}} \rangle = \langle X^{\mathcal{B}\mathcal{A}} \rangle$ and $\langle X^{\langle \mathcal{A}\mathcal{B} \rangle} \rangle = \langle X^{\langle \mathcal{A} \rangle \langle \mathcal{B} \rangle} \rangle$, where all angle brackets in the exponents are interpreted as generation as monoids.*

Proof. Trivial $\square$

Proposition 4.1. *Suppose $\phi: G \rightarrow \mathbb{Z} \subseteq \mathbb{R}$ and $t \in G$ with $\phi(t) = 1$. Then $\phi \in \Sigma(G)$ if and only if there is a finitely generated subgroup $Y \subseteq \ker \phi$ and a finitely generated Y -subgroup $B \leq G'$ such that $\dots \subseteq B^{t^{-1}} \subseteq B \subseteq B^t \subseteq \dots$ and the union of the B^{t^j} is G'*

Proof. The backwards direction is trivial. For the forwards direction, since $G = \langle \ker \phi, t \rangle$, take $S = \mathcal{Y}^{\pm 1} \cup \{t\}$ and $\mathcal{Y} \subseteq \ker \phi$. Setting M to be the monoid generated by S and let X be a finite set generating G' as an M -group.

We note that for a given S , we can achieve condition (*) by adding finitely many elements to X , so we can assume X, S satisfy (*).

Let $Y = \langle \mathcal{Y} \rangle$ and $B_0 = \langle X^Y \rangle$. Then G is generated by the $\langle t \rangle$ -conjugates of B_0 by (*) and 4.4. Since $\mathcal{Y}$ is finite, there is some k such that $\mathcal{Y}^{t^{-1}} \subseteq B = \langle B_0, \dots, B_0^{t^k} \rangle$. Using 4.4 again, we get that B is generated as a Y group by $X, \dots, X^{t^k}$. Also, since B is Y -invariant, $X^{t^{-1}Y} \subseteq B$, so again we get $\langle X^{t^{-1}Y} \rangle = B^{t^{-1}}$, so the latter is a subset of B , and so we are done. $\square$

Remark 4.3. If $N = \ker \phi$ and $\phi \in \Sigma(G)$, then as N/G' is finitely generated abelian, it is finitely generated as an M -group, and so it is finitely generated as an M_t group, where $M_t = \{1, t, \dots\}$.

A similar argument to 4.1 gives

Proposition 4.2. *If ϕ as above is discrete and $N = \ker \phi$ then $\phi \in \Sigma(G)$ implies G is an ascending HNN extension with finitely generated base B whose t conjugates are ascending with union N .*

Proof. Trivial □

We now turn observation 3 into a proposition

Proposition 4.3. *If $G = \langle B, t \mid B_1^t = B_2 \rangle$ and B is finitely generated, then if $\phi: G \rightarrow \mathbb{Z}$ has $\phi(B) = 0, \phi(t) = 1$ we have $\phi \in \Sigma(G)$ if and only if $B = B_2$, so $\dots \subseteq B \subseteq B^t \subseteq \dots$*

Proof. From the remark, $\phi \in \Sigma(G)$ implies $N = \ker \phi$ is finitely generated by S over the monoid generated by t , called M_t . Since $B^{\langle t \rangle}$ generates $\ker \phi$, there is an m_0 such that $X_1 \subseteq L_+ = \langle B^{t^j} \mid j > m_0 \rangle$. WLOG, $m_0 = 0$. Then $N = \langle X_1^{M_t} \rangle \subseteq \langle L_+^{M_t} \rangle = L^+$, so $N = L^+$.

On the other hand, N is an amalgamated free product $L_- *_{B_2^{t^{-1}}=B_1} B *_{B_2=B_1^t} L_+$, so as $L_+ = N$ we see that $B_2 = B$, so B embeds in B^t . □

5 Homological Definitions - Novikov Homology

We recall that a group G is of type FP_n if the trivial $\mathbb{Z}G$ module $\mathbb{Z}$ has a $\mathbb{Z}G$ free resolution which is finitely generated in all degrees at most n .

We recall that $\Sigma^m(G, \mathbb{Z}) = \{\chi \mid G_\chi \text{ is of type } FP_m\}$. For $m = 1$, this is equivalent to saying $\mathbb{Z}$ is finitely presented as a $\mathbb{Z}G_\chi$ module.

Proposition 5.1. *For a finitely generated group G , $\{0\} \cup \Sigma(G) = \Sigma^1(G, \mathbb{Z})$.*

Proof. Recall from the discussion about component trees the definition of $\Gamma_{\chi, r}$ and of component trees. We will prove that $\Gamma_{\chi, 0}$ is connected if and only if $\chi \in \Sigma^1(G, \mathbb{Z})$.

We suppose $\Gamma_{\chi, 0}$ is connected. If necessary by replacing x with x^{-1} in a generating set X of G , we have $\chi(X) \geq 0$. For $g \in G_\chi$, for g to be connected to 1, there is a word w with letters in $Y = X \cup X^{-1}$ with every prefix of w being at least 0 under χ . We call this the χ -track.

Now $g - 1 = (y_k \dots y_1) = 1 = (y_1 - 1) + \sum (y_j - 1)y_{j-1} \dots y_1$. We define the augmentation map $\varepsilon_\chi: \mathbb{Z}G_\chi \rightarrow \mathbb{Z}$ as expected with kernel I_χ . We have shown that the right ideal I_χ is generated by $\{y - 1 \mid y \in X\}$.

Then we have a partial resolution

$$X\mathbb{Z}G_\chi \rightarrow \mathbb{Z}G_\chi \rightarrow \mathbb{Z}$$

where the first map is given by $x \rightarrow x - 1$ so we have finished the forwards direction.

Now suppose G_χ is of type FP_1 and $\mathbb{Z}$ is finitely presented as a right $\mathbb{Z}G_\chi$ module. Then we have

$$F_1 \rightarrow F_0 \rightarrow \mathbb{Z}$$

where F_1, F_0 are finitely generated and free. Then the image of the first map embeds inside F_0 as a finitely generated submodule, and we also have

$$I_\chi \rightarrow \mathbb{Z}G_\chi \rightarrow \mathbb{Z}$$

Schanuel's lemma says that $\text{im } \phi \oplus \mathbb{Z}G_\chi = I_\chi \oplus F_0$, so I_χ is finitely generated.

Now as I_χ is finitely generated, say by λ_j . Then $\varepsilon(\lambda_i) = 0$, so λ_j is a sum of elements of the form $x_i - 1$, and the finite set of such elements generates I_χ . We prove by induction on the number of summands required to produce an arbitrary $g - 1$ that g can be represented by a word in the x_i , and so $\Gamma_{\chi,0}$ is connected for this generating set. $\square$

We recall that the Novikov ring $\widehat{\mathbb{Z}G_\chi}$ with respect to $\chi \neq 0$ is the set of all those formal sums with finitely many terms with $\chi(g) < r$ for every $r \in \mathbb{R}$.

Theorem 5.1. (*Sikorav*) Suppose G is of type FP_m , and $\chi \neq 0$. Then $\chi \in \sum^m(G, \mathbb{Z})$ if and only if $H_k(G, \widehat{\mathbb{Z}G_\chi}) = 0$ for $k = 0, \dots, m$ (this homology is called the Novikov homology)

Definition 5.1. $v_x: \widehat{\mathbb{Z}G_\chi} \rightarrow \mathbb{R} \cup \{\infty\}$ is a *valuation* if for $\lambda = \sum n_g g = \sum \lambda_r$, where λ_r is the component with $\chi(g) = r$ (the r -component). We write λ_χ for the smallest homogeneous component and say $v_x(\lambda) = \inf\{\chi(g) \mid n_g \neq 0\}$ if $\lambda \neq 0$ and ∞ otherwise.

Then $\widehat{\mathbb{Z}G_\chi}$ is a topological ring, with a basis of neighbourhoods of 0 given by the sets where v_x is larger than r . The Novikov ring is the completion of $\mathbb{Z}G$ with respect to the filtration this induces.

Writing $\lambda[m] = \sum_{r \leq m} \lambda_r \in \mathbb{Z}G$, we have $\lambda[m] \rightarrow \lambda$ and so $\mathbb{Z}G$ is dense in the Novikov ring.

Definition 5.2. A ring R is *von Neumann finite* if any surjective endomorphism of a finitely generated free R module is also injective.

Theorem 5.2. (*Koshlouwoua 2006 (discrete case), Bieri 2007 (general case)*) $\widehat{\mathbb{Z}G_\chi}$ is von Neumann finite for $\chi \neq 0$.

Proof. No Proof $\square$

Definition 5.3. (Kielak, 2020) Let A be a not necessarily square matrix over $\mathbb{Z}G$. Then we define $\Sigma(A) \subseteq \text{Hom}(G, \mathbb{R})$ to consist of those χ such that $A \otimes \mathbb{Z}G_\chi$ admits a right inverse over $\widehat{\mathbb{Z}G_\chi}$.

Remark 5.1. Kielak makes the definition more generally, for twisted group algebras.

Definition 5.4. A G chain complex $\mathcal{C} = \bigcup \mathcal{C}_D$ is *essentially m -acyclic* if there is $D \in \mathbb{R}$ such that for all $i \leq m$, the inclusion $\mathcal{C}_{r+D} \rightarrow \mathcal{C}_r$ is trivial on homology below degree $m + 1$.

Proof. (of 5.1, sketch) Recall that we filtered the Cayley graph Γ by subgraphs $\Gamma_{\chi,r}$. Suppose G is of type FP_m and $\chi \neq 0$. Let $\mathcal{F}$ be a free resolution with the first m terms finitely generated.

We filter the chain complex $\mathcal{F}$ in a similar fashion to get $\hat{\mathcal{F}} = \bigcup \mathcal{F}_{\chi,r}$. We wish to prove that

Theorem 5.3. $\mathcal{F}_{\chi,r}$ is essentially $(m-1)$ -acyclic for all r is equivalent to $H_k(G, \widehat{\mathbb{Z}G_\chi}) = 0$ for $k = 0, \dots, m$.

To motivate this theorem, take a partial free resolution of $\mathbb{Z}$ arising from a finite presentation $G = \langle X \mid R \rangle$ given by

$$R\mathbb{Z}G \rightarrow X\mathbb{Z}G \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z}$$

Suppose $x_1, x_2 \in X$ are both positive under χ . Then in $\widehat{\mathbb{Z}G_\chi}$, $x_i - 1$ has an inverse (given by the obvious formal series), so $\frac{x_1}{1-x_1} - \frac{x_2}{1-x_2}$ maps to 0 under the augmentation map. In the Cayley graph with respect to X , the infinite path λ represents is an element of $H_1(G, \widehat{\mathbb{Z}G_\chi})$. If we can form a path between two vertices on this path lying fully in $\Gamma_{\chi,r}$, then we have an infinite path in $\Gamma_{\chi,r}$ homologous to the original. If we can repeat this for increasing r , we have a sequence of homologous infinite paths tending to 0, and so our infinite path is trivial in homology.

If we can't find such a path, then we get a non-zero element of the homology (waving our hands)

Our formal proof of the above theorem is 'No.'

Our proof sketch finishes by noting that the hypotheses of the above theorem are intuitive. $\square$

Definition 5.5. The *deficiency* of a finite presentation $G = \langle X \mid R \rangle$ is $|X| - |R|$, and for finitely presented G , the *deficiency* of G is the maximum of the deficiencies of all presentations, denoted $d(G)$.

Theorem 5.4. (Bi-N-St, 1987) Let G be a finitely presented group of deficiency at least 2. Then $\Sigma(G) = \emptyset$.

Theorem 5.5. (Bieri, 2007) If G is finitely presented then

1. $\Sigma(G, \mathbb{Z}) \neq \{0\}$ implies the deficiency is at most 1.
2. $\Sigma^1(G, \mathbb{Z}) = \Sigma^2(G, \mathbb{Z})$ if $d(G) = 1$.
3. If $\Sigma(G, \mathbb{Z}) \neq \{0\}$ then G is of cohomological dimension 2.

Bieri conjectured that if $d(G) = 1$ and $\Sigma^1(G, \mathbb{Z})$ is non-zero, then G is an ascending HNN extension over a finitely generated free base group, and Bieri showed this in the case that $\Sigma(G)$ is symmetric.

Kielak discusses G with $d(G) = 1$ and shows that if $\mathbb{Z}G$ embeds in a division ring, then $\Sigma(G)$ is the dual of an integral polytope.

Proof. (Of 5.4) Suppose χ is discrete and $N = \ker \chi$. Suppose $\chi(t) = 1$ and $Q = G/N$ is generated by $Nt = Q$. Take a deficiency d presentation for G where $d \geq 2$. We modify it to be of the form

$$G = \langle t, a, b_1, \dots \mid b_i = v_i \rangle$$

Then $A = N/N'N^p$ is the quotient of a free $\mathbb{F}_p Q$ module of rank $m - 1$, where m is the number of generators, quotient by a submodule generated by at most $m - 2$ elements.

But $\mathbb{F}_p Q$ is a PID, so A contains a copy of $\mathbb{F}_p Q$. However, $\mathbb{F}_p Q$ is not finitely generated over $\mathbb{F}_p[q]$ or $\mathbb{F}_p[q^{-1}]$. So $\chi, -\chi \notin \Sigma(G/NN^p)$, and so neither is in $\Sigma(G)$, so it is empty as it is open and contains no discrete character. $\square$

Proof. (Of 5.5) Take χ and suppose

$$F_2 \rightarrow F_1 \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0$$

is a partial resolution coming from a presentation where F_1 is rank m and F_2 is rank r . We tensor with $\widehat{\mathbb{Z}G_\chi}$ and call the result $(*)$. If $\chi \in \Sigma^1(G, \mathbb{Z})$ then by 5.1, $H_i(G, \widehat{\mathbb{Z}G_\chi}) = 0$ for $i = 0, 1$, so the result is exact. Since $\widehat{\mathbb{Z}G_\chi}$ is free, the sequence $(*)$ splits, so $\hat{F}_1 = \partial(\hat{F}_2) \oplus \widehat{\mathbb{Z}G_\chi}$ where ∂ is the differential, so $\dagger: \hat{F}_2 \oplus \widehat{\mathbb{Z}G_\chi} \rightarrow \hat{F}_1$ is an epimorphism, so $r + 1 \geq m$, and we are done with (a).

For (b) and (c), we assume the deficiency is 1. We continue with the resolutions and maps above. Since $d(G) = 1$, the domain and codomain of $\dagger$ have the same rank, so we may regard it as an endomorphism of $(\widehat{\mathbb{Z}G_\chi})^n$. By von Neumann finiteness 5.2, so $\ker \partial_2 = 0$. If we extend our partial resolution, then $H_2(G, \widehat{\mathbb{Z}G_\chi})$ is 0, so $\chi \in \Sigma^2(G, \mathbb{Z})$ using 5.1, and the converse is trivial so we are done with (b).

For (c), $F_i \rightarrow \hat{F}_i$ is injective, so the injectivity of $\hat{\partial}_2$ implies ∂_2 is injective, so G has cohomological dimension at most 2. $\square$

Definition 5.6. An *integral polytope* P is a finite subset of a lattice $Q \subseteq \mathbb{R}^n$. A *marking* on P is a map $\phi: P \rightarrow \{0, 1\}$, and the *marked vertices* are the preimages of 1. χ is *marked* with respect to P if the minimum of χ over P is achieved at a single point, which we require to be marked.

Theorem 5.6. (Kielak, 2020). Let G be of deficiency 1 and Q the torsion free part of $G/G' \subseteq H_1(G, \mathbb{R}) = \mathbb{R}^n$. Then there are open subsets U_i covering $\text{Hom}(Q, \mathbb{R}) \setminus \{0\}$ such that for $\chi \in U_i$, we have $\chi \in \Sigma(G)$ if and only if $\chi \in \Sigma(A_i)$ for some matrix A_i over $\mathbb{Z}G$.

Definition 5.7. G is *agrarian* if $\mathbb{Z}G$ embeds in a division ring.

Theorem 5.7. (Kielak) Suppose G is agrarian and A is a square matrix over $\mathbb{Z}G$. Then there is a marked polytope $P(A)$ such that $\chi \in \Sigma(A)$ if and only if χ is marked with respect to $P(A)$.

Remark 5.2. For deficiency 1 agrarian groups, the marked polytopes $P(A)$ can be put together to give one marked polytope, and $\chi \in \Sigma(G)$ if and only if χ is marked with respect to this polytope.

Proof. (of 5.6) We again have a partial resolution

$$F_2 \rightarrow X\mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0$$

where X generates G and the rank of F_2 is $|X| - 1$. We again have the hatted resolution. We take $\chi \neq 0$, so we know $\chi \in \Sigma(G)$ if and only if $\chi \in \Sigma^1(G, \mathbb{Z})$ if and only if $H_1(G, \widehat{\mathbb{Z}G_\chi}) = 0$.

We let $U_i = \{\chi \neq 0 \mid \chi(x_i) \neq 0\}$ for $x_i \in X$. If $\chi(x_i) > 0$, then $x_i - 1$ is invertible in $\widehat{\mathbb{Z}G_\chi}$, otherwise if $\chi(x_i) < 0$ then $x_i - 1 = x_i(1 - x_i^{-1})$ is invertible. So if $\chi \in U_i$ then $\hat{\partial}_i$ is an isomorphism $x_i \widehat{\mathbb{Z}G_\chi} \rightarrow \widehat{\mathbb{Z}G_\chi}$, so $H_0(G, \widehat{\mathbb{Z}G_\chi}) = 0$. We may split $\hat{F}_{1,\chi} = x_i \widehat{\mathbb{Z}G_\chi} \oplus \hat{F}'_{1,\chi}$ and read off that $H_1(G, \widehat{\mathbb{Z}G_\chi}) = 0$ if and only if $\hat{F}_{2,\chi} \rightarrow \hat{F}'_{1,\chi}$ is surjective.

If ∂_2 is represented by a matrix A which is $m \times (m-1)$, then this composition is represented by A_i by removing the i th row. So for $\chi \in U_i$, we have $\chi \in \Sigma(G)$ if and only if $\chi \in \Sigma(A_i)$ $\square$

Lemma 5.8. $\lambda \in \mathbb{Z}Q$ is invertible in $\widehat{\mathbb{Z}Q_\chi}$ if and only if $\min\{\chi(q) \mid q \in \text{supp } \lambda\}$ is achieved at only one point with invertible coefficient (ie, if and only if $\chi \in \Sigma(\mathbb{Z}Q/\lambda\mathbb{Z}Q)$).

Proof. Cf 2.2. $\square$

Proof. (of 5.7) Suppose we were working over a commutative ring $\mathbb{Z}Q$. Then $A \in M_n(\mathbb{Z}Q)$ is invertible over $\widehat{\mathbb{Z}Q_\chi}$ if and only if $\det A$ is invertible in $\widehat{\mathbb{Z}Q_\chi}$.

We set P to be the convex hull of the support of the determinant, a finite subset of Q . We mark P so that A is invertible in $\widehat{\mathbb{Z}Q_\chi}$ if and only if χ is marked in P .

In the non-commutative case, for division rings we have an analogue of determinant, which maps to $D^\times/[D^\times, D^\times] \cup \{0\}$, and we can do the same thing, by embedding $\mathbb{Z}[G, G]/(G/[G, G]) \rightarrow D(G/[G, G])$, and embedding $D^\times$ into the group $P(DQ)$ defined below. There is a non-trivial point that the image is a single polytope, and not a formal sum, but that is true. How to mark it is also non-trivial and refer to Kielak for that. $\square$

Definition 5.8. We define the Grothendieck group on formal symbols for polytopes (via the obvious addition), and denote it $P(DQ)$.

Definition 5.9. A *Poincaré duality group* G of dimension n is a group with a G module $\mathcal{O}$ (the orientation module) which is isomorphic to $\mathbb{Z}$ as an abelian group and a class $f \in H_n(G, \mathcal{O})$, the *fundamental class* such that for any G module M , the cup product with f gives an isomorphism

$$H^i(G, M) \rightarrow H_{n-i}(G, M \otimes \mathcal{O})$$

When $\mathcal{O}$ is trivial, G is *orientable*.

Theorem 5.9. (Kielak, 2020) If G is a PD^3 agrarian group of type F (fundamental group of a finite dimensional CW complex), then $\Sigma(G) \cup \{0\} = \Sigma^i(G, \mathbb{Z})$ for all $i \geq 1$, and there is a marked polytope P such that $\chi \in \Sigma(G)$ if and only if χ is marked with respect to P .

Remark 5.3. 3-manifold groups satisfy these conditions (Kielak-Linton, 2023)

Proof. There is a free resolution (from Kielak's paper) $\mathcal{F}$ of $\mathbb{Z}$ of length 3 which we write

$$0 \rightarrow F_2 \rightarrow F_2 \rightarrow X\mathbb{Z}G \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0$$

where X generates G . Then for $\chi \neq 0$, there is $x \in X$ with $\chi(x) \neq 0$. Let U_χ be a closed neighbourhood of χ such that $\phi(x) \neq 0$ for $\phi \in U_\chi$. We assume it is the convex hull of finitely many ϕ_i and let $\widehat{\mathbb{Z}G_{U_\chi}}$, the intersection of $\widehat{\mathbb{Z}G_{\phi_i}}$.

Tensoring with this, we get an exact sequence as above but with hats, and the $H_0(G, \widehat{\mathbb{Z}G_{U_\chi}})$ is 0 via this complex, as $x - 1$ is invertible in $\widehat{\mathbb{Z}G_{U_\chi}}$. Poincaré duality implies the transpose of $\hat{\partial}_3$ is also onto. Since $\hat{F}_3$ is a free module, $\hat{F}_2 = \hat{F}_3 + \hat{F}_2'$ with the latter stably free.

We modify our resolution by taking the direct sum with $F_3 \rightarrow F_3 \rightarrow 0$, the identity map with the 0 in degree 0. The new chain complex $(\mathcal{E}, \delta)$ is a free resolution of $\mathbb{Z}$ by finitely generated $\mathbb{Z}G$ modules.

We use this to calculate homology, and set $k_i = \text{rk } E_i$. $k_0 = 1$, and we assume $\mathbb{Z}G$ embeds in a division ring D . Considering the homology of $\mathcal{E} \otimes_{\mathbb{Z}G} D$, the Euler characteristic is $k_0 - k_1 + k_2 - k_3$, so as the Euler characteristic is 0 by Poincaré duality, we have $k_1 + k_3 = k_2 + 1$.

We have

$$\hat{E}_2 = \hat{F}_2 \oplus \hat{F}_3 = (\hat{F}_3 \oplus \hat{F}_2') \oplus \hat{F}_3 = \hat{F}_3 \oplus F_2$$

and if M is the (invertible) matrix such that $M\delta_3$ is the $k_3 \times k_3$ identity with some 0 rows appended. By truncating, we get an M_0 over $\mathbb{Z}G$. We truncate as follows: pick $c \in \mathbb{R}$ such that $c + \phi_j(g) > 0$ for all g in the support of entries of δ_3 , and truncate at c .

By the choice of c , $M_0\delta_3 = I + M_1$ where if $g \in \text{supp } M_1$, then $\phi_j(g) > 0$ for all j , and hence for all $\phi \in U_\chi$. This implies that the span of the last basis vectors (past the identity part) is a complement to the image of the injective map $M_0\delta_3$, and therefore when computing the homology of $\hat{\mathcal{E}}$, we can disregard $\hat{E}_3$ and replace $\hat{E}_2$ by its complement $\hat{E}_2'$, so $H_i(G, \widehat{\mathbb{Z}G_\phi}) = 0$ for $i \geq 2, \phi \in U_\chi$.

Importantly, the differential $\hat{E}_2' \rightarrow \hat{E}_1$ is represented by a $k_1 \times (k_2 - k_3)$ matrix over $\mathbb{Z}G$, and hence has one more row than column. Finish as in the deficiency 1 argument. $\square$

In 1993, Linnell introduced a class $\mathcal{C}$, the smallest class of groups containing free groups and closed under extensions by elementary amenable groups and direct unions. He showed:

Theorem 5.10. *If $G \in \mathcal{C}$ is torsion free, then G satisfies the strong Atiyah conjecture, and such a group embeds in a division ring.*

Theorem 5.11. *(Kielak-Linton, 2023) $\mathcal{C}$ is closed under free products.*

Theorem 5.12. *(Friedl-Lück 2019, Kielak-Linton 2023) Fundamental groups of 3-manifolds are in $\mathcal{C}$.*

We prove none of the above.