Lecture 2

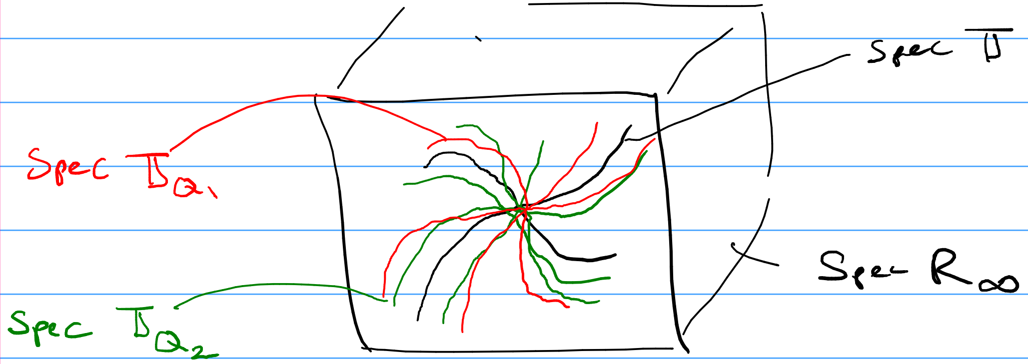
Patrick explained how to construct

$$R \twoheadrightarrow \mathbb{T} \quad \text{for modular forms.}$$

$\nearrow$ defo. ring $\qquad$ $\nwarrow$ Hecke alg.

Goal in the remaining lectures:
explain how to show $R \cong \mathbb{T}$ (or related statements) using the Taylor-Wiles method

<u>Idea</u>: embed both Spec R, Spec $\mathbb{T}$ in an ambient (maybe smooth) space, Spec R_∞, allow ramification/level at auxiliary primes



→ get more components
(more Hecke eigenforms)

→ take limit and show Hecke algebras fill out whole ambient space.

→ return to original level and get $R = \mathbb{T}$.

Following Calegari-Geraghty, we'll work in the context of CARs for GL_n / # field F.
Will present things conditional on some conjectures. With extra work + ideas can get unconditional results when F tot real or CM.

① loc sym spaces + their coh.

$G = \mathrm{Res}_{F/\mathbb{Q}} GL_n$ (conn. red / $\mathbb{Q}$)

$K \subset G(\mathbb{A}^\infty)$ cpt. open subgp. (<u>level</u>)

" $\prod K_v$ $K_v \subset GL_n(F_v)$

$K_\infty \subset G(\mathbb{R})$ max cpt.

K_∞^0 conn comp of identity

(product of $SO(n)$'s and $U(n)$'s)

$$X_K := G(\mathbb{Q}) \backslash \Big[\underbrace{G(\mathbb{R}) / \mathbb{R}_{>0} K_\infty^0}_{X_\infty \text{ symmetric space}} \times G(\mathbb{A}^\infty)/K \Big]$$

K suff small ⇝ real manifold

<u>Examples</u>

a) $n=1$ F signature (r,s)

$$X_K \twoheadrightarrow C_K = \pi_0\left(F^\times \backslash \mathbb{A}_F^\times / K \right)$$

finite ray class gp.

kernel $\cong (\mathbb{R}/\mathbb{Z})^{r+s-1}$ (Dir. unit thm.)

b) $G = GL_2/\mathbb{Q}$ $K = $ "$U_1(N)$"

$X_K = Y_1(N)$ modular curve

c) $G = GL_2/F$ F Im. quad field.

$X_\infty = \mathbb{H}^3 \cong SL_2(\mathbb{C})/SU(2)$

$X_K = \coprod \Gamma_i \backslash \mathbb{H}^3$ arith. subgps. of $SL_2(\mathbb{C})$.

(e.g. $SL_2(\mathcal{O}_F)$)

Cohomology: $H^*(X_K, \mathbb{C})$ can be computed with automorphic forms (Franke)

Thm 1. (Borel)

$$\bigoplus_{\pi} \left[(\pi^\infty)^K\right]^{\oplus m_i} \subset H^i(X_K, \mathbb{C})$$

Sum is over "weight 0" CARs of $G(\mathbb{A})$

e.g.: $GL_2/\mathbb{Q} \rightarrow$ π's generated by wt 2 cusp forms.

- (Clozel) Each π w/ $(\pi^\infty)^K \neq 0$ contributes in degrees $q_0, q_0+1, \ldots, q_0+l_0$.

$$l_0 = rk(G(\mathbb{R})) - rk(K_\infty) - 1$$

q_0: range is symmetric under Poincaré duality

Examples a) GL_1/F $l_0 = r+s-1$, $q_0 = 0$

b) $GL_2/\mathbb{Q}$ $l_0 = 0$, $q_0 = 1$ (Eichler-Shimura)

c) GL_2/IQF $l_0 = 1$, $q_0 = 1$

② Hecke actions

Fix S set of places w/ $K_v = GL_n(\mathcal{O}_v)$
$v \notin S$.

As in Patrick's talk: p-adic coeffs. $\mathcal{O} \subset E/\mathbb{Q}_p$,
$\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$.

$\mathbb{T}^S = \otimes$ of spherical Hecke algebras at $v \notin S$, $\mathcal{O}$-coeffs.

Have natural action

$$\mathbb{T}^S \longrightarrow \operatorname{End}_{D(\mathcal{O})}(R\Gamma(X_K, \mathcal{O}))$$

compatible w/ $\mathbb{T}^S \otimes_\iota \mathbb{C} \circlearrowright (\pi^\infty)^K$ in Thm 1.

Image $= \mathbb{T}^S(K)$, finite $\mathcal{O}$-algebra.

$\mathfrak{m} \triangleleft \mathbb{T}^S(K)$ max ideal w/ res. field k
($\leftrightarrow$ mod p Hecke evals appearing in coh of X_K)

$$R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}} \subset R\Gamma(X_K, \mathcal{O}) \text{ direct summand,}$$

$\circlearrowleft$

$\mathbb{T}^S(K)_{\mathfrak{m}} =: \mathbb{T}$

Conj A (Thm of Scholze if F CM or totally real)

$\exists$ abs semisimple rep $\bar{\rho}_{\mathfrak{m}} : G_F \longrightarrow GL_n(k)$

unram for $v \notin S$, $v \nmid p$,

$$\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v) \underset{\text{Satake}}{\sim} \mathbb{T} \xrightarrow[\text{Hecke eigensystem}]{} \mathbb{T}/\mathfrak{m} = k.$$

Conj B

mod p version of Thm 1

Suppose $\mathfrak{m}$ non-Eisenstein: $\bar{\rho}_{\mathfrak{m}}$ abs irred.

then $H^i(X_K, k)_{\mathfrak{m}} = 0$ for $i \neq [q_0, q_0 + l_0]$.

Consequence: $R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}}$ q.i. to complex of finite free $\mathcal{O}$-mods in degrees $[q_0, q_0 + l_0]$.

Examples

a) GL_1/F exercise!

b) $GL_2/\mathbb{Q}$: $H^1_{\mathfrak{m}}$ finite free $\mathcal{O}$-module

c) GL_2/IQF: $[C^1 \longrightarrow C^2]$ representing $(R\Gamma)_{\mathfrak{m}}$.

can (often will) have torsion in $H^2(X_K; \mathcal{O})_{\mathfrak{m}}$

can have $H^*(X_K, \mathcal{O})_{\mathfrak{m}}[\frac{1}{p}] = 0$

③ TW primes

We want to allow ramification so that $\mathbb{T}$ (or $R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}}$) grows in a controlled way.

Choose place $v \notin S$ with

$q_v \equiv 1 \mod p^m$ for $m \geq 1$. $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ distinct evals in k.

$$I_v \leq K_v \text{ Iwahori} \qquad \overset{\text{str. upper tri mod } \varpi_v}{I_v(1)} \subset I_v' \subset \overset{\text{upper tri. mod } \varpi_v}{I_v}$$

quotient
$$I_v / I_v' = \Delta_v = \underbrace{\left(I_v / I_v(1) \right)}_{\cong \begin{pmatrix} k^\times & & \\ & \ddots & \\ & & k^\times \end{pmatrix}} \otimes_{\mathbb{Z}} \mathbb{Z}/p^m$$

$$\Delta_v \cong \left(\mathbb{Z}/p^m\mathbb{Z} \right)^n.$$

$\mathcal{O}[\Delta_v]$ is a local ring (Δ_v p-group)

Proposition (see Khare-Thorne §5)

Assume $p > n$.

a) $R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}} \cong R\Gamma(X_{K^v I_v}, \mathcal{O})_{\tilde{\mathfrak{m}}}$

$\uparrow$ specify mod ϖ evals.

ordering of Frob_v evals $\longrightarrow$ $\alpha_1, \alpha_2, \dots, \alpha_n$ for "U_v" Hecke ops at v.

b) $C_{\{v\}} :=$ $\mathcal{O}$-linear dual of $R\Gamma(X_{K^v I_v'}, \mathcal{O})_{\tilde{\mathfrak{m}}}$, in $D(\mathcal{O}[\Delta_{Q_m}])$

$C_\emptyset :=$ ———————— $R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}}$

$C_{\{v\}} \simeq$ complex of finite free $\mathcal{O}[\Delta_v]$-modules in degrees $[q_0, q_0 + \ell_0]$

$$C_\emptyset = C_{\{v\}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}[\Delta_v]} \mathcal{O} .$$

& co-invariants.

$\mathbb{T}_{\{v\}} \curvearrowright C_{\{v\}}$, $\mathcal{O}[\Delta_v]$-Hecke alg, $\mathbb{T}_{\{v\}} \twoheadrightarrow \mathbb{T}$

Idea: a) $q = 1$ in k.

So Iwahori-Hecke algebra over k is very nice, $\cong k[X_*(T) \rtimes S_n]$

"no q-new forms at level $\Gamma_1(N) \cap \Gamma_0(q)$ with $\bar\rho = \bar\rho_{\mathfrak{m}}$".

b)
$$\begin{tikzcd} X_{K^v I_v'} \arrow[d, "\Delta_v\text{-torsor}"] \\ X_K \end{tikzcd}$$

⊛ $\mathcal{O}$-linear dual means we apply

$$R\mathrm{Hom}_{\mathcal{O}}(-,\mathcal{O}) : D^b(\mathcal{O}[\Delta_v]) \to D^b(\mathcal{O}[\Delta_v])$$

In particular, we have

$$H^{-i}\left(R\mathrm{Hom}_{\mathcal{O}}\left(R\Gamma(X_{K^v I_v^1}, \mathcal{O}), \mathcal{O} \right)\right) = H_i(X_{K^v I_v^1}, \mathcal{O})$$

example:

torsion subgp.

$$0 \to \mathrm{Ext}^1_{\mathcal{O}}\left(H^2(X_{K^v I_v^1}, \mathcal{O}), \mathcal{O} \right)$$

$$\to H_1(X_{K^v I_v^1}, \mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}\left(H^1(X_{K^v I_v^1}, \mathcal{O}), \mathcal{O} \right) \to 0$$

torsion-free

So we didn't throw away $\mathcal{O}$-torsion.

Lecture 4

Set-up $R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}}$ loc^d coh of locally symmetric space for GL_n/F.

$\circlearrowleft$

$\mathbb{T}$

$\uparrow$ Hecke algebra $\mathfrak{m}$ non-Eis.

$p > n$, unram in F. $K_v = GL_n(\mathcal{O}_v)$ $v|p$.

Assume we have $R \twoheadrightarrow \mathbb{T} \circlearrowright R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}}$

$\uparrow$

crys. HT wts $\{0, 1, \dots, n-1\}$ at $v|p$.

$\bar{\rho}_{\mathfrak{m}}(G_{F(\zeta_p)})$ enormous

Target theorem: $R \twoheadrightarrow \mathbb{T}$ has nilpotent kernel

Assuming: conjs on existence of Galois reps, local-global compatibility, and vanishing of mod p coh. outside $[q_0, q_0 + l_0]$.

(See 10 author paper for unconditional results)

Equiv: $\mathrm{Supp}_R\left(\bigoplus_i H^i(X_K, \mathcal{O})_{\mathfrak{m}}\right) = \mathrm{Spec}\, R$

Cor: Any Galois rep $\rho : G_F \to GL_n(E)$ corresponding to a pr. $R \to E$ is automorphic.

Pf Factors through $\mathbb{T} \to E$ and these Hecke eigenvalues appear in $H^*(X_K, \mathcal{O})[\frac{1}{p}]$ $\leftrightarrow$ come from CAR π.

① Allowing ram^n: $m \geq 1$, we choose sets of TW primes Q_m in F:

$|Q_m| = q$ indep^t of m, $q_v \equiv 1 \mod p^m$ for $v \in Q_m$

$$\begin{tikzcd} R_m \arrow[r, two heads] \arrow[d, two heads] & R \\ \mathbb{T}_m \arrow[r] & \mathbb{T} \end{tikzcd}$$

allow ram^n at Q_m

$\mathbb{T}_m$: image of $\left(\bigotimes \text{sph. Hecke alg for } v \notin S \cup Q_m \right) \otimes \mathcal{O}[\Delta_{Q_m}]$

$\Delta_{Q_m} = \prod \Delta_v \leftarrow$ lec 2

$\mathbb{T}_m \circlearrowright R\Gamma(X_{K^{Q_m}\prod_{v \in Q_m} \mathrm{Iw}_v'}, \mathcal{O})_{\tilde{m}}$

$\circlearrowright \quad C_m \nearrow \mathcal{O}$-linear dual.

This means: apply $R\mathrm{Hom}_{\mathcal{O}}(-,\mathcal{O}) : D^b(\mathcal{O}[\Delta_{Q_m}]) \to D^b(\mathcal{O}[\Delta_{Q_m}])$

$H^{-i}(C_m) = H_i(C_m)$ is a direct summand of $H_i(X_{\dots}, \mathcal{O})$

Local-global-compatibility at Q_m:

$R_m \longrightarrow \mathbb{T}_m$ is an $\mathcal{O}[\Delta_{Q_m}]$-alg. hom.

Can recover things at "TW level 0":

$$\begin{tikzcd} R_m \otimes_{\mathcal{O}[\Delta_{Q_m}]} \mathcal{O} \arrow[r, phantom, "\circlearrowright"] \arrow[d, phantom, "\cong" rotate=90] & C_m \otimes^{L}_{\mathcal{O}[\Delta_{Q_m}]} \mathcal{O} \arrow[d, phantom, "\cong" rotate=90] \\ R \arrow[r, phantom, "\circlearrowright"] & C \end{tikzcd}$$

Galois coh calc

$R_\infty = R_{loc}[[x_1, \dots, x_g]] \twoheadrightarrow R_m$

(equidim $1 + qn - l_0$

$= \left(\begin{array}{c} \text{dim of cotangent space} \\ m/m^2 \text{ in } \mathcal{O}[\Delta_{Q_m}] \end{array} \right) - l_0$

This is assuming $\mathrm{End}_{k[G_{F_v}]}(\bar{\rho}|_{G_{F_v}}) = k$

Otherwise, work with local lifting rings, not deformation rings.

② **Patching** Making some choices, get $S_\infty = \mathcal{O}[[t_1^{(i)}, \dots, t_n^{(i)} : 1 \le i \le q_r]]$

$$\begin{tikzcd} S_\infty \arrow[d, "t"] \\ \mathcal{O}[\Delta_{Q_m}] \end{tikzcd}$$

for each m.

$$\boxed{\dim S_\infty = \dim R_\infty - \ell_0} \leftarrow \text{key numerology}$$

Idea take limit over m.
(**Issue**: no transition maps!)

$$\bigcap_m \ker\left(S_\infty \longrightarrow \mathcal{O}[\Delta_{Q_m}]\right) = \{0\}$$

think

$$\mathbb{Z}_p[[t]] \longrightarrow \mathbb{Z}_p[[t]] \big/ \left((1+t)^{p^m} - 1\right)$$

So in the limit, our complexes of $\mathcal{O}[\Delta_{Q_m}]$-modules turn in to S_∞-modules.

In limit:

$$\begin{tikzcd} & S_\infty \arrow[d] & \\ R_\infty \arrow[r, two heads] & R^{\text{patch}} \arrow[r, phantom, "\circlearrowright"] & C_\infty \end{tikzcd}$$

$\simeq$ complex of fin free S_∞ modules in deg. $[q_0, q_0 + \ell_0]$

$$\mathbb{T}^{\text{patch}} = \overline{\text{image of } R^{\text{patch}} \text{ in } \mathrm{End}}_{D(S_\infty)}(C_\infty)$$

③ Commutative algebra

Prop C_∞ has non-zero coh. in only one degree $M_\infty = H_{q_0}(C_\infty)$ and M_∞ is a maximal CM R_∞-module

Impt: Support = union of irred components in $\operatorname{Spec} R_\infty$

Point is that amplitude l_0 of C_∞ gives lower bound on $\dim R^{patch}$, coincides with upper bound $\dim R_\infty$.

- $l_0 = 0$: $C_\infty = \underbrace{H_{q_0}(C_\infty)}_{M_\infty}[q_0]$ fin. free S_∞-mod.

$$\Rightarrow \quad M_\infty \text{ has } \operatorname{depth}_{R^{patch}} = \dim_{R^{patch}} = \dim S_\infty$$

$$\dim R_\infty = \dim S_\infty \geq \dim R^{patch}$$

- $l_0 > 0$: extra step $\dim R^{patch} \leq \dim S_\infty - l_0$

$$\Rightarrow \quad \operatorname{codim} \operatorname{supp}_{S_\infty}(H_*(C_\infty)) \geq l_0$$

[Calegari-Geraghty] $\Rightarrow$ $C_\infty \simeq \underbrace{H_{q_0}(C_\infty)}_{M_\infty}[q_0]$

and $\dim M_\infty = \dim S_\infty - l_0 = \dim R_\infty \geq \dim R^{patch}$

e.g. $l_0 = 1$, $C_\infty = [\mathbb{Z}_p[[t]] \xrightarrow{\times f} \mathbb{Z}_p[[t]]]$

$\dim_{S_\infty} H_*(C_\infty) \leq 1 \iff f \neq 0.$

Now same argument as $l_0 = 0 \Rightarrow$ Prop.

Thm: consequence of Prop

$\rho : G_F \to GL_n(E)$ } both from maps
$\rho_{\pi,\iota}$ aut Galois rep. } $R \to E$

$\rho|_{G_{F_v}}$, $\rho_{\pi,\iota}|_{G_{F_v}}$ on same

irred comp of R_v $\forall v \in S$, and $\rho_{\pi,\iota}|_{G_{F_v}}$ on unique irred component; $\Rightarrow$ ρ is automorphic.

Sample applications

- R_v irreducible $\forall v \Rightarrow R \to \mathbb{T}$ nilp kernel
- R_v smooth $\forall v \Rightarrow R \cong \mathbb{T}$, $H_{q_0}(X_K, \mathcal{O})_m$ free over $\mathbb{T}$.

More concrete:

$E/\mathbb{Q}$ ell curve, good red at 3 sst elsewhere, $\bar{\rho}_{E,3}$ abs irred and ramified at bad reduction primes.

$\Rightarrow$ E is modular.

R_v just one (Steinberg) component.

Taking the limit:

idea. $(M_m)_{m\geq 1}$ finite free

$\mathcal{O}[\Delta_{Q_m}]$-modules, all of rk r.

$\mathcal{F}$: non-prin. ultrafilter on $\mathbb{N}$: collection of subsets of $\mathbb{N}$. $\forall \Sigma \subset \mathbb{N}$, either Σ or $\Sigma^c \in \mathcal{F}$.

- upwards closed
- closed under finite intersect.

non-prin.: contains no finite sets.

$\mathcal{F}$ = "large" sets

$$M_\infty := \varprojlim M_{\infty,k}$$

$$M_{\infty,k} = \Big(\prod_m \big(M_m \otimes_{S_\infty} S_\infty/\mathfrak{m}^k_{S_\infty}\big)\Big)\Big/\sim$$

$M_{m,k}$: finite sets, bounded cardinality.

two elts are $\sim$ if they agree on a large set of m.

$M_{\infty,k}$ is finite free of rk r over $S_\infty/\mathfrak{m}^k_\infty$

Pf: $M_{m,k}$ iso to a fixed M_k for large set of m.

$\rightarrow$ M_∞ finite free over S_∞.