M3/4/5P12 PROBLEM SHEET 5

Please send any corrections or queries to `j.newton@imperial.ac.uk`. The first exercise is left over from the chapter on character theory.

Exercise 1. Let G, H be two finite groups, let V be a representation of G and let W be a representation of H. Define a natural action of the product group $G \times H$ on the vector space $V \otimes W$ by

$$\rho_{V\otimes W}(g,h)(v \otimes w) = \rho_V(g)v \otimes \rho_W(h)w.$$

This defines a representation of $G \times H$.

(a) Find the character of $V \otimes W$ as a representation of $G \times H$, in terms of the characters χ_V of V and χ_W of W.
(b) Suppose V is an irrep of G and W is an irrep of H. Show that $V \otimes W$ is an irrep of $G \times H$.
(c) Supposes G has r distinct irreducible characters and H has s distinct irreducible characters. Show that $G \times H$ has at least rs distinct irreducible characters. By computing dimensions, show that $G \times H$ has exactly rs distinct irreducible characters and describe them in terms of the irreducible characters of G and of H.

Solution 1. (a) The character of $V \otimes W$ as a representation of $G \times H$ is given by

$$(g,h) \mapsto \chi_V(g)\chi_W(h).$$

To see this, we proceed as in lectures when we worked out the character of $V \otimes W$ when V and W are both representations of G.

If we fix bases A and B for V, W then the matrix for $\rho_V(g) \otimes \rho_W(h)$ with respect to the basis $A \otimes B$ is given by $M \otimes N$, where $M = [\rho_V(g)]_A$ and $N = [\rho_W(h)]_V$ — see Lemma 2.4 in the notes for the explicit description of the entries of this matrix. The trace of this matrix is equal to $\mathrm{Tr}(M)\mathrm{Tr}(N)$ — the proof of this is the same as the proof of Proposition 3.2 (2) in the notes. So we get that $\chi_{V\otimes W}(g,h) = \chi_V(g)\chi_W(h)$.

(b) We are going to use character theory to check that $V \otimes W$ is irreducible. We need to show that

$$\langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle = 1.$$

By definition we have

$$\langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle = \frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} \chi_{V\otimes W}(g,h)\overline{\chi_{V\otimes W}(g,h)}.$$

Applying part a), and noting that $|G \times H| = |G||H|$, we get

$$\langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle = \frac{1}{|G|}\frac{1}{|H|} \sum_{(g,h)\in G\times H} \chi_V(g)\chi_W(h)\overline{\chi_V(g)\chi_W(h)}$$

$$= \frac{1}{|G|}\left(\sum_{g\in G} \chi_V(g)\overline{\chi_V(g)}\right)\frac{1}{|H|}\left(\sum_{h\in H} \chi_W(h)\overline{\chi_W(h)}\right) = 1 \cdot 1 = 1.$$

The final equality holds by irreducibility of V and W.

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(c) Lets denote the r irreducible characters of G by $\chi_1, \ldots, \chi_r$ and the s irreducible characters of H by $\eta_1, \ldots, \eta_s$. Denote the dimensions by $d_1, \ldots, d_r$ and $e_1, \ldots, e_s$. By considering the tensor product representations, we get rs irreps of $G \times H$ with characters given by $\chi_i \eta_j$ for $1 \leq i \leq r, 1 \leq j \leq s$.

Let's show that these characters of $G \times H$ are all distinct. By a very similar calculation to what we did in the last part, we get

$$\langle \chi_i \eta_j, \chi_k \eta_l \rangle = \langle \chi_i, \chi_k \rangle \langle \eta_j, \eta_l \rangle = 0$$

unless $i = k$ and $j = l$. So we have produced rs distinct irreducible characters of $G \times H$.

Finally, the dimension of the rep with character $\chi_i \eta_j$ is equal to $d_i e_j$. So the sum of the squares of the dimensions gives $\sum_{i,j} d_i^2 e_j^2 = (\sum_i d_i^2)(\sum_j e_j^2) = |G||H| = |G \times H|$. So we know that we have found all of the irreducible characters.

The rest of the exercises are on algebras and modules.

Exercise 2. Find an isomorphism of algebras between $\mathbb{C}[C_3]$ and $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$.

Solution 2. Let's try to directly write down an isomorphism

$$f : \mathbb{C}[C_3] \to \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$$

We let $C_3 = \{e, g, g^2\}$. We need $f([e]) = (1,1,1)$, $f([g]) = (\lambda_1, \lambda_2, \lambda_3)$ and $f([g^2]) = (\lambda_1^2, \lambda_2^2, \lambda_3^2)$, where $\lambda_1, \lambda_2, \lambda_3$ are cube roots of unity. For f to be an isomorphism we need $(1,1,1), (\lambda_1, \lambda_2, \lambda_3), (\lambda_1^2, \lambda_2^2, \lambda_3^2)$ to be a basis for $\mathbb{C}^{\oplus 3}$, since $[e], [g], [g^2]$ are a basis for $\mathbb{C}[C_3]$. Conversely, any linear map f with these properties will be an algebra isomorphism. So we just need to choose $\lambda_1, \lambda_2, \lambda_3$.

Suppose $\lambda_1 = \lambda_2$. Then $(1,1,1), (\lambda_1, \lambda_2, \lambda_3), (\lambda_1^2, \lambda_2^2, \lambda_3^2)$ will all lie in the two-dimensional subspace $(x, x, z) \subset \mathbb{C}^{\oplus 3}$. So we need $\lambda_1, \lambda_2, \lambda_3$ to be three distinct cube roots of unity. This will then give a basis $(1,1,1), (\lambda_1, \lambda_2, \lambda_3), (\lambda_1^2, \lambda_2^2, \lambda_3^2)$ for $\mathbb{C}^{\oplus 3}$. One way to prove this is a basis is to use column orthogonality for the character table of C_3: the matrix

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}$$

is the character table of C_3, and its rows are linearly independent by row orthogonality, so its columns are also linearly independent.

We may as well choose $\lambda_1 = 1, \lambda_2 = e^{2\pi i/3}, \lambda_3 = e^{4\pi i/3}$. So we get an algebra isomorphism f defined by

$$f([e]) = (1,1,1), f([g]) = (1, \omega, \omega^2), f([g^2]) = (1, \omega^2, \omega)$$

where $\omega = e^{2\pi i/3}$.

Alternatively, you can use the Artin-Wedderburn theorem to write down the isomorphism.

Exercise 3. Let A and B be algebras. Show that the projection map $p : A \oplus B \to A$ defined by $p(a, b) = a$ is an algebra homomorphism, but that the inclusion map $i : A \to A \oplus B$ defined by $i(a) = (a, 0)$ is not.

Solution 3. The key point to remember is that algebra homomorphisms have to send the unit to the unit. We have $p(1_A, 1_B) = 1_A$, and we also have $p(a_1 a_2, b_1 b_2) = a_1 a_2 = p(a_1, b_1) p(a_2, b_2)$, so p is an algebra homomorphism. But the inclusion i sends 1_A to $(1_A, 0)$ which is not the unit in $A \oplus B$.

Exercise 4. Let A and B be algebras. Suppose M is an A-module and N is an A-module. The vector space $M \oplus N$ is naturally an $A \oplus B$-module, with action of $A \oplus B$ given by

$$(a, b) \cdot (m, n) = (a \cdot m, b \cdot n).$$

(a) Let X be an $A \oplus B$-module. Show that multiplication by $e_A := (1_A, 0)$ defines an $A \oplus B$-linear projection map

$$e_A : X \to X.$$

(b) Write $e_A X$ for the image of multiplication by e_A. Show that for $x \in e_A X$ we have $(a, b) \cdot x = (a, 0) \cdot x$ for all $a \in A, b \in B$.

(c) Show that there is an A-module M and a B-module N such that X is isomorphic to $M \oplus N$ as an $A \oplus B$-module.

(d) Describe the simple modules for $A \oplus B$ in terms of the simple modules for A and the simple modules for B.

Solution 4. (a) First we check that the map

$$e_A : X \to X$$

is $A \oplus B$ linear. If $(a, b) \in A \oplus B$ we have $e_A(a, b) = (a, 0) = (a, b)e_A$, so e_A is in the centre of $A \oplus B$. In particular, multiplication by e_A is an $A \oplus B$ linear map. Now we check that e_A is a projection: we have $e_A \circ e_A(a, b) = e_A(a, 0) = (a, 0) = e_A(a, b)$, so it is indeed a projection.

(b) If x is in the image of e_A then $x = e_A y$ for some $y \in A$. So $(a, b)x = (a, b)e_A y = (a, 0)y = (a, 0)e_A y = (a, 0)x$.

(c) Define e_B to be multiplication by $(0, 1_B)$. We claim that the image of e_B is equal to the kernel of e_A. Indeed, we have $e_A e_B = (0, 0)$, so the image of e_B is contained in the kernel of e_A. Conversely, if $e_A x = 0$, we have $x = (1_A, 1_B)x = e_A x + e_B x = e_B x$, so x is in the image of e_B. Since e_A is an $A \oplus B$-linear projection, we have an isomorphism of $A \oplus B$-modules $X = e_A X \oplus \ker(e_A) = e_A X \oplus e_B X$ (by Lemma 2.1 in the lecture notes). We let $M = e_A X$ with action of A given by $a \cdot x = (a, 0) \cdot x$. Similarly, we let $N = e_B X$ with action of B given by $b \cdot x = (0, b) \cdot x$. Then we have an isomorphism of $A \oplus B$-modules $X \cong M \oplus N$.

(d) Suppose X is a simple $A \oplus B$ module. $e_A X$ is a submodule of X, so it is either equal to X or $\{0\}$. If $e_A X = 0$ then $X = e_B X$. So we have either $X = e_A X$ or $X = e_B X$.

So X is isomorphic to either a simple A-module M, or a simple B-module N, where we think of them as $A \oplus B$ modules with action given by $(a, b)x = (a, 0)x$ or $(a, b)x = (0, b)x$ respectively. Note that if M or N was not simple, then X would not be simple, so this gives the simplicity of M or N.

Exercise 5. Show that the matrix algebra $M_n(\mathbb{C})$ is isomorphic to its own opposite algebra.

Solution 5. The transpose map gives an isomorphism $M_n(\mathbb{C}) \to M_n(\mathbb{C})^{op}$, since $(MN)^t = N^t M^t$.

Exercise 6. (a) What is the centre of $M_n(\mathbb{C})$?

Hint: $M_n(\mathbb{C})$ has a basis given by matrices E_{ij} with a 1 in the (i, j) entry and 0 everywhere else. Work out what it means for a matrix to commute with E_{ij}.

(b) If A and B are algebras, show that $Z(A \oplus B) = Z(A) \oplus Z(B)$.

(c) Let $n_1, \dots, n_r$ be positive integers. What is the centre of the algebra

$$\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})?$$

Solution 6. (a) Let's follow the hint. Let $M \in M_n(\mathbb{C})$. Then $E_{ij}M$ is the matrix whose ith row is the jth row of M, and the other entries are zero. On the other hand, ME_{ij} is the matrix whose jth column is the ith column of M.

Suppose $E_{ij}M = ME_{ij}$. Then comparing the (i,j) entries of these matrices we get $M_i i = M_j j$. The other entries in the matrices are all zero. We conclude that if $E_{ij}M = ME_{ij}$ for all i, j then M must be equal to λI_n for $\lambda \in \mathbb{C}$ (where I_n is the $n \times n$ identity matrix). So the centre of $M_n(\mathbb{C})$ is just given by the scalar matrices λI_n.

(b) Suppose $a \in Z(A)$ and $b \in Z(B)$. Then it's easy to check that $(a, b) \in Z(A \oplus B)$, so we have $Z(A) \oplus Z(B) \subset Z(A \oplus B)$. Conversely, if $(a, b) \in Z(A \oplus B)$ we have $(a, b)(x, 0) = (x, 0)(a, b)$ for all $x \in A$ which implies that $ax = xa$ for all $x \in A$. So $a \in Z(A)$. Similarly, we deduce that $b \in Z(B)$. So we get that $Z(A \oplus B) = Z(A) \oplus Z(B)$.

(c) By applying parts (a) and (b) we deduce that the centre of $\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})$ is given by $\bigoplus_{i=1}^{r} \mathbb{C} I_{n_i} \cong \mathbb{C}^{\oplus r}$.

Exercise 7. Let A be an algebra. Show that the map $f \mapsto f(1_A)$ gives an isomorphism of algebras between $\mathrm{Hom}_A(A, A)$ and A^{op}.

Solution 7. First we show that this map is an algebra homomorphism. The identity map gets sent to 1_A, and we have $f \circ g(1_A) = f(g(1_A)) = g(1_A)f(1_A)$.

Next we show that the map is injective. If $f(1_A) = 0$, then $f(a) = a \cdot f(1_A) = 0$ for all $a \in A$ so $f = 0$.

Finally, we show that the map is surjective. If $x \in A^{op}$ then we consider the map $f : A \to A$ given by $f(a) = a \cdot x$. This is an A-linear map, and $f(1_A) = x$.

Exercise 8. Let $A = \mathbb{C}[x]/(x^2)$ — recall that this has as a basis $\{1, x\}$, with 1 a unit and $x^2 = 0$. Show that A itself is not a semisimple A-module.

Solution 8. Consider the submodule $M = \mathbb{C}x \subset A$. This is a submodule because $1 \cdot x = x$ and $x \cdot x = 0$, so M is A-stable. We claim that M does not have a complementary submodule in A. Suppose $N \subset A$ is a submodule with $M + N = A$. In order for M and N to span A, we must have an element $\lambda 1 + \mu x \in N$ with $\lambda \in \mathbb{C}^\times$ (i.e. $\lambda \neq 0$) and $\mu \in \mathbb{C}$. Since N is a submodule, we have $\lambda^{-1}x(\lambda 1 + \mu x) = x \in N$ which implies that $x \in M \cap N$. We conclude that M does not have a complementary submodule in A.