M4/5P12 GROUP REPRESENTATION THEORY MASTERY MATERIAL: INDUCED REPRESENTATIONS

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1. Induced representations

1.1. **Definition and basic properties.**

Definition 1.1. Let H be a subgroup of a finite group G. Let V be a representation of H. We define a representation

$$\mathrm{Ind}_H^G(V)$$

of G with underlying vector space given by H-equivariant fuctions from G to V:

$$\mathrm{Ind}_H^G(V) = \{f : G \to V | f(hg) = \rho_V(h)f(g) \text{ for } h \in H\}.$$

The action of G is given by $(g \cdot f)(g') = f(g'g)$.

Recall that we write $H\backslash G$ for the set of *right cosets* of H in G. In other words, we have

$$G = \coprod_{i=1}^{d} Hg_i$$

for a finite set $\{g_1, \ldots, g_d\}$ of coset representatives. We then write $C_i = Hg_i$, and the set of cosets $H\backslash G$ is equal to $\{C_1, \ldots, C_d\}$. We now fix the set of coset representatives $\{g_1, \ldots, g_d\}$.

Lemma 1.1. *Let $f : G \to V \in \mathrm{Ind}_H^G(V)$. Then f is determined by its values on the elements $g_1, \ldots, g_d$.*

Proof. Let $f \in \mathrm{Ind}_H^G(V)$ and let g be an arbitrary element of G. There is a unique $i \in \{1, \ldots, d\}$ such that $g = hg_i$ for some $h \in H$. We have $f(g) = f(hg_i) = \rho_V(h)f(g_i)$. So f is determined by the values $f(g_1), \ldots, f(g_d)$. □

Definition 1.2. For $f : G \to V$ a function, the *support* of f, denoted $\mathrm{Supp}(f)$ is the set $\{g \in G : f(g) \neq 0\}$.

Remark 1.1. If $f \in \mathrm{Ind}_H^G(V)$ then $f(hg) = \rho_V(h)f(g)$ for $h \in H$. So if $g \in \mathrm{Supp}(f)$ then $hg \in \mathrm{Supp}(f)$ for all $h \in H$. We deduce that $\mathrm{Supp}(f)$ is a union of right cosets of H.

Definition 1.3. If C is a right coset of H in G we write V_C for the subspace of $\mathrm{Ind}_H^G(V)$ comprising functions f such that $\mathrm{Supp}(f) = C$, together with the zero function. Equivalently, we can define V_C to be the subspace of $\mathrm{Ind}_H^G(V)$ comprising functions f which are zero outside C.

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Lemma 1.2. (1) *The linear map*

$$\bigoplus_{C\in H\backslash G} V_C \to \mathrm{Ind}_H^G(V)$$
$$(f_C)_{C\in H\backslash G} \mapsto \sum_{C\in H\backslash G} f_C$$

is an isomorphism of vector spaces.

(2) *Let C be a right coset and fix a coset representative $g_C \in C$. The linear map*

$$ev_{g_C} : V_C \to V$$
$$f \mapsto f(g_C)$$

is an isomorphism of vector spaces.

(3) *If $g \in G$ then the map*

$$\rho_{\mathrm{Ind}_H^G(V)}(g) : \mathrm{Ind}_H^G(V) \to \mathrm{Ind}_H^G(V)$$

restricts to a map

$$\rho_{\mathrm{Ind}_H^G(V)}(g)|_{V_C} : V_C \to V_{Cg^{-1}}$$

Moreover, if we fix coset representatives g_C of C and g_D of $D = Cg^{-1}$ we have a commutative diagram

$$\begin{tikzcd} V_C \arrow[r, "\rho_{\mathrm{Ind}_H^G(V)}(g)"] \arrow[d, "ev_{g_C}"'] & V_{Cg^{-1}} \arrow[d, "ev_{g_D}"] \\ V \arrow[r, "\rho_V(g_D g g_C^{-1})"] & V \end{tikzcd}$$

Proof. (1) We give an inverse to this map. If $f \in \mathrm{Ind}_H^G(V)$ then we define f_C to be the function which is equal to f on the coset C, but equal to zero everywhere else. Then $f_C \in V_C$ and the map $f \mapsto (f_C)_{C\in H\backslash G}$ gives the desired inverse. Note that we have $\sum_{C\in H\backslash G} f_C = f$.

(2) Again we write down an inverse map: if $v \in V$ we define $f \in V_C$ by $f(hg_C) = \rho_V(h)v$.

(3) Suppose $f \in V_C$. Then f vanishes outside C. Since $\left(\rho_{\mathrm{Ind}_H^G(V)}(g)f\right)(g') = f(g'g)$, the function $\rho_{\mathrm{Ind}_H^G(V)}(g)f$ vanishes when $g'g \notin C$. In other words, it vanishes when $g' \notin Cg^{-1}$. This says that $\rho_{\mathrm{Ind}_H^G(V)}(g)f$ is an element of $V_{Cg^{-1}}$, as desired.

Now we need to check the diagram commutes. The first important thing to note is that $g_D g g_C^{-1} \in H$, so the lower horizontal map actually makes sense. Now we need to check that the composition $ev_{g_D} \circ \rho_{\mathrm{Ind}_H^G(V)}(g)$ is equal to $\rho_V(g_D g g_C^{-1}) \circ ev_{g_C}$. Let's start with $f \in V_C$. We have

$$ev_{g_D} \circ \rho_{\mathrm{Ind}_H^G(V)}(g)f = (\rho_{\mathrm{Ind}_H^G(V)}(g)f)(g_D) = f(g_D g).$$

On the other hand,

$$\rho_V(g_D g g_C^{-1}) \circ ev_{g_C} f = \rho_V(g_D g g_C^{-1})(f(g_C)) = f(g_D g g_C^{-1} g_C) = f(g_D g),$$

where the penultimate equality is a special case of the property $f(hg) = \rho_V(h)f(g)$. ☐

Remark 1.2. The above Lemma gives us a fairly explicit description of $\mathrm{Ind}_H^G(V)$. We can think of it as a direct sum of copies of V, indexed by the set of cosets $H\backslash G$, and with the action of G given by the commutative diagram appearing in part (3) of the above.

Lemma 1.3. *The dimension of* $\mathrm{Ind}_H^G(V)$ *is* $[G:H]\dim(V)$ *where* $[G:H] = |G|/|H|$.

Proof. Note that $[G:H]$ is the number of right cosets for H in G. So this follows from parts 1 and 2 of Lemma 1.2. ☐

Example 1.1. (1) Let H be any subgroup of G. Consider the induction $\mathrm{Ind}_H^G\mathbb{C}H$ of the regular representation. We claim that this is isomorphic to the regular representation of G. An isomorphism

$$\alpha : \mathbb{C}G \to \mathrm{Ind}_H^G\mathbb{C}H$$

is given by letting $\alpha([g])$ be the map from G to $\mathbb{C}H$ which is equal to zero outside the coset Hg^{-1} and takes an element hg^{-1} to $[h]$.

(2) Let H be any subgroup of G. Consider the trivial one-dimensional representation V_{triv} of H, and the induction $\mathrm{Ind}_H^G V_{triv}$. Now the elements of $\mathrm{Ind}_H^G V_{triv}$ are functions $f : G \to \mathbb{C}$ such that $f(hg) = f(g)$ for all $h \in H, g \in G$. In other words, they are functions from the set of right cosets $H\backslash G$ to $\mathbb{C}$. We claim that this is isomorphic to the representation $\mathbb{C}\Omega$, where Ω is the set of *left* cosets G/H with action of G given by left multiplication.

An isomorphism

$$\alpha : \mathbb{C}\Omega \to \mathrm{Ind}_H^G V_{triv}$$

is given by letting $\alpha([gH])$ be the map which takes the right coset Hg^{-1} to 1, and takes every other coset to 0.

(3) Let $G = D_{2n} = \langle s, t : s^n = t^2 = e, tst = s^{-1}\rangle$ be a dihedral group of order $2n$. We let $H \subset G$ be the (cyclic order n) subgroup generated by s. Let $\zeta \in \mathbb{C}$ with $\zeta^n = 1$ and let V_ζ be the one-dimensional rep with of H with basis vector $[b]$ on which s acts as multiplication by ζ. Let's work out what the representation $\mathrm{Ind}_H^G(V_\zeta)$ is. It has dimension two, with a basis given by f_1, f_2 where $f_1(s^i) = \zeta^i b$, $f_1(s^i t) = 0$, and $f_2(s^i) = 0$, $f_2(s^i t) = \zeta^i b$. We can write down the matrices giving the action of s and t with respect to this basis.

We have $(sf_1)(s^i) = f_1(s^{i+1}) = \zeta f_1(s^i)$ and $(sf_1)(s^i t) = f_1(s^i ts) = f_1(s^{i-1}t) = 0$ so $sf_1 = \zeta f_1$. We have $(sf_2)(s^i) = f_2(s^{i+1}) = 0$ and $(sf_2)(s^i t) = f_2(s^i ts) = f_2(s^{i-1}t) = \zeta^{-1} f_2(s^i t)$ so $sf_2 = \zeta^{-1} f_1$. So we conclude that s acts with matrix

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}.$$

Now for t: we have $(tf_1)(s^i) = f_1(s^i t) = 0$ and $(tf_1)(s^i t) = f_1(s^i t^2) = f_1(s^i) = \zeta^i b$ so $tf_1 = f_2$. On the other hand we have $(tf_2)(s^i) = f_2(s^i t) = \zeta^i b$ and $(tf_2)(s^i t) =$

$f_2(s^it^2) = f_2(s^i) = 0$ so $tf_2 = f_1$. We conclude that s acts with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Compare with Problem Sheet 2, Exercise 6, where we wrote down these matrices to describe some two-dimensional representations of G. Now you know a sensible way to construct these representations 'by pure thought'.

1.2. **An alternative description of the induced representation.**

Definition 1.4. Let H be a subgroup of a finite group G. There is a natural algebra homomorphism $\mathbb{C}H \to \mathbb{C}G$, and we can think of $\mathbb{C}G$ as a $\mathbb{C}H$-module (an element of $\mathbb{C}H$ acts on $\mathbb{C}G$ by multiplication on the left).

If V is a representation of H (and therefore a $\mathbb{C}H$-module) we can then consider the vector space of $\mathbb{C}H$-linear maps $\mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V)$.

Finally, we make $\mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V)$ into a $\mathbb{C}G$-module by defining $a \cdot x$ to be the element of $\mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V)$ given by

$$(a \cdot x)(b) = x(ba)$$

for $x \in \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V)$ and $a, b \in \mathbb{C}G$.

Lemma 1.4. *The map*

$$\begin{aligned} \alpha : \mathrm{Ind}_H^G(V) &\to \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V) \\ f &\mapsto \left(\sum \lambda_g[g] \mapsto \sum \lambda_g f(g)\right) \end{aligned}$$

is an isomorphism of $\mathbb{C}G$*-modules.*

Proof. First we'll check that we've defined a $\mathbb{C}G$-linear map. It suffices to check that $\alpha([h]f) = [h]\alpha(f)$ for every $f \in \mathrm{Ind}_H^G(V)$ and $h \in G$. The map $\alpha([h]f)$ takes $\sum \lambda_g[g]$ to $\sum \lambda_g f(gh)$. On the other hand $[h]\alpha(f)$ takes $\sum \lambda_g[g]$ to $\alpha(f)(\sum \lambda_g[g][h]) = \sum \lambda_g f(gh)$. We get the same result, so we have shown that α is G-linear.

To show α is an isomorphism, we'll write down the inverse α^{-1}. We define

$$\begin{aligned} \alpha^{-1} : \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V) &\to \mathrm{Ind}_H^G(V) \\ f &\mapsto (g \mapsto f([g])) \end{aligned}$$

□

1.3. **Frobenius reciprocity.**

Recall that if V is a representation of G and H is a subgroup of G, then we get a representation of H on the vector space V by restricting the action of G. We write $\mathrm{Res}_H^G V$ for this representation of H.

Theorem 1.1 (Frobenius reciprocity). *Let V be a representation of G and let W be a representation of H. There is an isomorphism*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, W).$$

Proof. We can write down an isomorphism explicitly. If $f \in \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ we take it to the H-linear map $v \mapsto (f(v))(e)$. Conversely, if $f \in \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ we take it to the G-linear map

$$\begin{aligned} V &\to \mathrm{Ind}_H^G W \\ v &\mapsto (g \mapsto f(\rho_V(g)v)) \end{aligned}$$

□

Remark. Switching to $\mathbb{C}G$-modules, the above proposition says that there is an isomorphism

$$\mathrm{Hom}_{\mathbb{C}G}(V, \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, W)) \cong \mathrm{Hom}_{\mathbb{C}H}(V, W).$$

Corollary 1.1 (Frobenius reciprocity formula)**.** *Let V be a representation of G and let W be a representation of H. Then we have*

$$\langle \chi_{\mathrm{Ind}_H^G W}, \chi_V \rangle = \langle \chi_W, \chi_{\mathrm{Res}_H^G V} \rangle.$$

Proof. Recall that if U, V are two representations of G, then $\langle \chi_U, \chi_V \rangle = \dim_{\mathbb{C}} \mathrm{Hom}_G(V, U)$. □

1.4. **The character of an induced representation.** Given H a subgroup of G and a representation V of H, we want to work out the character $\chi_{\mathrm{Ind}_H^G(V)}$ of the induced representation in terms of the character χ_V of V.

Recall that we've fixed a set of representatives $g_1, \ldots, g_d$ for the right cosets $H\backslash G = \{C_1, \ldots, C_d\}$.

Proposition 1.1. *We have*

$$\chi_{\mathrm{Ind}_H^G(V)}(g) = \sum_{i : g_i g g_i^{-1} \in H} \chi_V(g_i g g_i^{-1}) = \frac{1}{|H|} \sum_{g' \in G : g' g (g')^{-1} \in H} \chi_V(g' g (g')^{-1}).$$

Proof. We apply Lemma 1.2. We have $\mathrm{Ind}_H^G(V) = \bigoplus_{i=1}^d V_{C_i}$ and $g \in G$ gives maps

$$\rho_{\mathrm{Ind}_H^G(V)}(g) : V_C \to V_{Cg^{-1}}.$$

To compute the trace $\chi_{\mathrm{Ind}_H^G(V)}(g)$, we only need to consider cosets C with $C = Cg^{-1}$. In terms of representatives we consider the g_i such that $Hg_i = Hg_i g^{-1}$, or in other words, such that $g_i g g_i^{-1} \in H$. Assuming that $g_i g g_i^{-1} \in H$, Part (3) of Lemma 1.2 tells us that the map

$$\rho_{\mathrm{Ind}_H^G(V)}(g) : V_C \to V_C$$

has trace $\chi_V(g_i g g_i^{-1})$. Summing over all the cosets C, we get the desired formula for $\chi_{\mathrm{Ind}_H^G(V)}(g)$.

To get the final equality, we just note that if we sum over all elements of G instead of coset representatives we count the contribution of each coset $|H|$ times, so dividing by $|H|$ gives the same result as summing over coset representatives. □

1.5. **Restriction of induced representations.** Suppose we take a representation V of H, with H a subgroup of a finite group G. We can consider the restriction $\mathrm{Res}_H^G \mathrm{Ind}_H^G V$ of the induced representation. For simplicity we only consider the case when H is a normal subgroup of G. See sections 7.3 and 7.4 of Serre 'Linear representations of finite groups' for the whole story, or google 'Mackey theory'.

Definition 1.5. Suppose $H \triangleleft G$ is a normal subgroup and that V is a representation of H. For $g \in G$ define a representation V^g of H with underlying vector space the same as V by

$$\rho_{V^g}(h) = \rho_V(ghg^{-1})$$

for $h \in H$.

Lemma 1.5. *Suppose $H \triangleleft G$ is a normal subgroup and that V is a representation of H.*

(1) *Let $g, g' \in G$ such that g, g' belong to the same right coset C of H in G. Then there is a H-linear isomorphism $V^g \cong V^{g'}$.*
(2) *If V is an irreducible representation of H, then V^g is irreducible for all $g \in G$.*

Proof. (1) Since $g, g' \in C$ there is an $h \in H$ such that $g' = hg$. Now consider the map $\rho_V(h) : V^g \to V^{g'}$. We claim that this map is H-linear. Indeed, we have

$$\rho_V(h)\rho_V(ghg^{-1}) = \rho_V(hghg^{-1}) = \rho_V(g'h(g')^{-1}h) = \rho_V(g'h(g')^{-1})\rho_V(h).$$

This says that $\rho_V(h)$ is an H-linear map from V^g to $V^{g'}$. Since $\rho_V(h)$ is an invertible linear map, we conclude that we have the desired H-linear isomorphism.

(2) Suppose $U \subset V^g$ is an H-stable subspace. Then, by definition of V^g we have that for $u \in U$ and $h \in H$, $\rho_V(ghg^{-1})u \in U$. Since $gHg^{-1} = H$, we conclude that $\rho_V(h)u \in U$ for all $h \in H$, so U is also an H-stable subspace of V. By irreducibility of V, U is either 0 or all of V, so we deduce that V^g is irreducible.

□

Lemma 1.6. *Suppose $H \triangleleft G$ is a normal subgroup and that V is a representation of H. Let $g_1, \ldots, g_d$ be representatives for the right cosets $H\backslash G = \{C_1, \ldots, C_d\}$. There is an H-linear isomorphism*

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G V \cong \bigoplus_{i=1}^{d} V^{g_i}.$$

Proof. To show that there is such an isomorphism we just need to show that the characters of both sides are equal. Proposition 1.1 says that the character of the left hand side is

$$h \mapsto \sum_{i : g_i h g_i^{-1} \in H} \chi_V(g_i h g_i^{-1}).$$

Since H is a normal subgroup of G the sum appearing here is just

$$\sum_{i=1}^{d} \chi_V(g_i h g_i^{-1}) = \sum_{i=1}^{d} \chi_{V^{g_i}}(h).$$

□

Proposition 1.2. *Suppose $H \lhd G$ is a normal subgroup and that V is an irreducible representation of H. Let $g_1, \dots, g_d$ be representatives for the right cosets $H \backslash G = \{C_1, \dots, C_d\}$. Then $\mathrm{Ind}_H^G V$ is an irreducible representation of G if and only if V^g is not isomorphic to V for every $g \in G - H$.*

Proof. We know that $\mathrm{Ind}_H^G V$ is an irreducible representation of G if and only if

$$\langle \chi_{\mathrm{Ind}_H^G V}, \chi_{\mathrm{Ind}_H^G V} \rangle = 1.$$

By Corollary 1.1 and Lemma 1.6 we have

$$\langle \chi_{\mathrm{Ind}_H^G V}, \chi_{\mathrm{Ind}_H^G V} \rangle = \langle \chi_V, \chi_{\mathrm{Res}_H^G \mathrm{Ind}_H^G V} \rangle = \sum_{i=1}^{d} \langle \chi_V, \chi_{V^{g_i}} \rangle.$$

We may as well assume that $g_1 = e$, so we have

$$\langle \chi_{\mathrm{Ind}_H^G V}, \chi_{\mathrm{Ind}_H^G V} \rangle = 1 + \sum_{i=2}^{d} \langle \chi_V, \chi_{V^{g_i}} \rangle.$$

So $\mathrm{Ind}_H^G V$ is irreducible if and only if $\langle \chi_V, \chi_{V^{g_i}} \rangle = 0$ for all $i \geq 2$. Since V and V^{g_i} are irreducible, this holds if and only if V^{g_i} is not isomorphic to V for every $i \geq 2$. By Lemma 1.5, this holds if and only if V^g is not isomorphic to V for every $g \in G - H$. $\square$

1.6. **Some motivation: a theorem of Artin.** This section is non-examinable! We've seen that induction gives examples of representations of G, constructed out of reps of subgroups H. This is interesting in its own right. Another reason to be interested in induction is the following theorem of Artin:

Theorem 1.2. *Let V be a representation of a finite group G. Then there exist cyclic subgroups $H_1, H_2, \dots H_n$ of G, representations U_i of H_i and rational numbers $\lambda_i \in \mathbb{Q}$ such that*

$$\chi_V = \sum_{i=1}^{n} \lambda_i \chi_{\mathrm{Ind}_{H_i}^G U_i}.$$

See Chapter 9 of Serre 'Linear representations of finite groups' for the proof of this. This tells us that using induction (and allowing linear combinations of characters with rational coefficients) we can generate all the characters of a finite group from the representations of cyclic subgroups.

Artin was actually motivated to prove this theorem by number theory. He had defined L-functions (generalisations of the Riemann zeta function) for Galois extensions of number fields L/K and representations ρ of $\mathrm{Gal}(L/K)$. More precisely, Artin L-functions are functions in a complex variable s $L(L/K, \rho, s)$ defined when the real part of s is sufficiently large, and holomorphic in this region of the complex plane. See (for example) `en.wikipedia.org/wiki/Artin_L-function`. Specialising to $L = K = \mathbb{Q}$ (and ρ the trivial representation of the trivial group) gives the Riemann zeta function. The case when the Galois group $\mathrm{Gal}(L/K)$ is Abelian was well understood (using class field theory), and the representation theoretic result of Artin allowed him to establish some properties of his more general L-functions in the case where $\mathrm{Gal}(L/K)$ is non-Abelian, by reducing to the

Abelian case. For example, he could show that $L(L/K, \rho, s)$ extends to a meromorphic function over the whole complex plane.

If ρ is a non-trivial representation, it is expected that $L(L/K, \rho, s)$ extends to a *holomorphic* function over the whole complex plane. This expectation is called the Artin conjecture, and is one of the most important open problems in number theory.

See `http://math.osu.edu/~cogdell/artin-www.pdf` for a discussion of Artin's work, and more recent developments.