M3/4/5P12 PROBLEM SHEET 3

Please send any corrections or queries to `j.newton@imperial.ac.uk`.

Exercise 1. (1) Let V be a finite dimensional vector space. Consider the map

$$\alpha : V \to (V^*)^*$$

defined by letting $\alpha(v)$ be the linear map

$$\alpha(v) : V^* \to \mathbb{C}$$

given by $\alpha(v)(\delta) = \delta(v)$, for $\delta \in V^*$. Show that this map is an isomorphism of vector spaces.

(2) If V is a representation of G, show that α is a G-linear isomorphism.

Exercise 2. Let G be a finite group and consider the regular representation $\mathbb{C}G$. Show that the dual $(\mathbb{C}G)^*$ is isomorphic to $\mathbb{C}G$ as a representation of G.

The next two exercises explain a way to think about tensor products of vector spaces without fixing bases. They are not essential for the course.

Exercise 3. Let V, W and X be complex vector spaces. A map

$$f : V \times W \to X$$

is called bilinear if it is linear in each variable separately. That is, $f(av_1 + bv_2, w) = af(v_1, w) + bf(v_2, w)$ and $f(v, aw_1 + bw_2) = af(v, w_1) + bf(v, w_2)$ for $a, b \in \mathbb{C}$.

(1) Show that the map $\pi : V \times W \to V \otimes W$ which takes (v, w) to $v \otimes w$ is a bilinear map. *Note that we have implicitly fixed bases of V, W to define $V \otimes W$.*

(2) Show that for every bilinear map $f : V \times W \to X$ there is a unique linear map $h : V \otimes W \to X$ such that $f = h \circ \pi$.

(3) *This part is trickier* Suppose $\pi' : V \times W \to U$ is a bilinear map, and for every bilinear map $f : V \times W \to X$ there is a unique linear map $h : U \to X$ such that $f = h \circ \pi'$. Show that there is a unique isomorphism $i : U \to V \otimes W$ such that $i \circ \pi' = \pi$.

Remarks: Part (2) of the exercise says that tensor products are a way to turn bilinear maps into linear maps.

We can also use this exercise to give an alternative (basis-independent) definition of the tensor product. We say that a vector space U, together with a bilinear map $\pi : V \times W \to U$ 'is a tensor product' of V and W if for every bilinear map $f : V \times W \to X$ there is a unique linear map $h : V \otimes W \to X$ such that $f = h \circ \pi$. Part (2) says that a tensor product of V and W exists (it's the tensor product $V \otimes W$ we have already defined with a chosen basis of V and W).

Part (3) says that a tensor product of V and W is unique up to unique isomorphism, so to all intents and purposes any two tensor products of V and W are the same mathematical object.

Date: Sunday 14$^{\text{th}}$ February, 2016.

Exercise 4. Let V, W be two vectors spaces, with bases A and B. Let $\mathbb{C}[V \times W]$ be the (infinite dimensional) complex vector space with basis given by symbols $\{v * w : v \in V, w \in W\}$. Define a linear map $\mathbb{C}[V \times W] \to V \otimes W$ by taking $v * w$ to $v \otimes w$.

(1) Let $E \subset \mathbb{C}[V \times W]$ be the subspace spanned by the elements

$$(av_1 + bv_2) * w - a(v_1 * w) - b(v_2 * w), \quad v * (aw_1 + bw_2) - a(v * w_1) - b(v * w_2)$$

with $a, b \in \mathbb{C}$, $v_i, v \in V$ and $w_i, w \in W$.

Note that a map of sets $f : V \times W \to X$ gives a linear map

$$F : \mathbb{C}[V \times W] \to X$$

defined by $F(v*w) = f(v, w)$. Show that f is bilinear if and only if $F(u) = 0$ for all $u \in E$.

(2) Show that the map $(v, w) \mapsto v * w$ defines a bilinear map from $V \times W$ to the quotient vector space $\mathbb{C}[V \times W]/E$.

(3) Using Exercise 3, show that there is a unique isomorphism

$$i : \mathbb{C}[V \times W]/E \to V \otimes W$$

satisfying $i(v * w + E) = v \otimes w$.

Exercise 5. (1) Let V and W be representations of G and suppose W has dimension one. Show that $V \otimes W$ is irreducible if and only if V is irreducible.

(2) Let V and W be representations of G. Show that $V \otimes W$ is isomorphic as a representation of G to $W \otimes V$.

Exercise 6. (1) Let V be a representation of G and consider the map $f : V \otimes V \to V \otimes V$ given by $f(v_1 \otimes v_2) = v_2 \otimes v_1$. Show that f is a G-linear map.

(2) Define S^2V to be the subspace of $x \in V \otimes V$ such that $f(x) = x$. Define $\wedge^2 V$ to be the subspace of $x \in V \otimes V$ such that $f(x) = -x$. Show that S^2V and $\wedge^2 V$ are subrepresentations of $V \otimes V$ and $V \otimes V \cong S^2V \oplus \wedge^2 V$.

(3) Show that $(1/2)(f + \mathrm{id}_V)$ is a projection with image S^2V and $(1/2)(f - \mathrm{id}_V)$ is a projection with image $\wedge^2 V$.

(4) Show that if $A = \{a_1, \dots, a_n\}$ is a basis of V, then $\{a_i \otimes a_j - a_j \otimes a_i : i < j\}$ is a basis of $\wedge^2 V$. What are the dimensions of S^2V and $\wedge^2 V$ in terms of $\dim(V) = n$? Can you find a basis for S^2V?

(5) Suppose $g \in G$ and $\lambda_1, \dots, \lambda_n$ are the eigenvalues (with multiplicity) of $\rho_V(g)$. Show that the eigenvalues of $\rho_{\wedge^2 V}(g)$ are $\{\lambda_i \lambda_j : i < j\}$.

(6) Show that the characters $\chi_{\wedge^2 V}$ and $\chi_{S^2 V}$ are given by

$$\chi_{\wedge^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2} \qquad \chi_{S^2 V}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}.$$

Exercise 7. Let G be a group acting on a finite set Ω. Recall that we have defined a representation $\mathbb{C}\Omega$ of G. Show that the character $\chi_{\mathbb{C}\Omega}$ satisfies: $\chi_{\mathbb{C}\Omega}(g)$ is equal to the number of fixed points for g in Ω.

Exercise 8. (1) Let G be a finite group. Show that if G is simple (i.e. G is non-trivial and the only normal subgroups of G are $\{e\}$ and G) then a representation of G is either trivial or faithful.

(2) Suppose every non-trivial irreducible representation of a finite group G is faithful. Show that G is a simple group. *Hint: if G is not simple then there is a normal subgroup N of G such that G/N is simple.*