M3/4/5P12 PROBLEM SHEET 4

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Exercise 1. Let G be a finite group, and $g \in G$ an element of order 2. Let V be a representation of G. Show that $\chi_V(g)$ is an integer and that

$$\chi_V(g) \equiv \dim V \pmod 2.$$

Hint: recall that $\chi_V(g)$ is a sum of eigenvalues of $\rho_V(g)$.

Exercise 2. Let $\chi : G \to \mathbb{C}$ be a function. Define $\ker \chi$ by

$$\ker \chi = \{g \in G : \chi(g) = \chi(e)\}.$$

Now suppose V is a representation of G, with $\rho_V : G \to \mathrm{GL}(V)$ the homomorphism giving the action of G on V, and χ_V the character of V.

Show that $\ker \chi_V = \ker \rho_V$.

Exercise 3. In this exercise we are going to work out the character table of $A_4 \subset S_4$, the group of even permutations of $\{1, 2, 3, 4\}$. There are 4 conjugacy classes in A_4, with representatives $e, (123), (132), (12)(34)$ and sizes $1, 4, 4, 3$ respectively.

(1) Show that A_4 has an irreducible representation U of dimension 3 with character given by

$$\chi_U(e) = 3, \chi_U(123) = \chi_U(132) = 0, \chi_U((12)(34)) = -1.$$

Hint: restrict a three-dimensional irrep of S_4 to the subgroup A_4

(2) Show that A_4 has three isomorphism classes of irreps of dimension 1, one isomorphism class of irreps of dimension 3 and these are all the irreps.

You've now shown that the character table of A_4 looks like:

	e	(123)	(132)	$(12)(34)$
χ_{triv}	1	1	1	1
χ_U	3	0	0	-1
χ_3	1	?	?	?
χ_4	1	?	?	?

(3) Show that $\chi_3((12)(34)) = \chi_4((12)(34)) = 1$. *Hint: use the fact that $\langle \chi, \chi' \rangle = 0$ if $\chi \neq \chi'$ are distinct irreducible characters.*

(4) Fill in the rest of the character table. *Hint: if χ is the character of a one-dimensional rep then $\chi(123)^3 = \chi(132)^3 = 1$. We also know that $\langle \chi_3, \chi_{triv} \rangle = \langle \chi_4, \chi_{triv} \rangle = 0$.*

(5) (More advanced question) Show that the representations with characters χ_3 and χ_4 are obtained by inflating representations of a quotient of A_4 which is isomorphic to the cyclic group C_3.

Exercise 4. (1) Let U be the three-dimensional irrep of A_4 found in the previous exercise. Find the decomposition of $U \otimes U$ into irreducibles.

(2) Let V be the two-dimensional irrep of S_4 found in lectures. Find the decomposition into irreducibles of the restriction of V to a representation of A_4.

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Exercise 5. Let G be a finite group such that every irrep of G is one-dimensional. Show that G is Abelian. *Hint: how many conjugacy classes does G have?*

Exercise 6. Let G be a finite group, with irreducible characters $\chi_1, \chi_2 \ldots, \chi_r$. Fix an element $g \in G$. Show that g is in the centre of G (i.e. $gh = hg$ for all $h \in G$) if and only if

$$\sum_{i=1}^{r} \chi_i(g)\overline{\chi_i(g)} = |G|.$$

Exercise 7.
(1) Write down the character table of S_3.
(2) Consider the class function $\phi : S_3 \to \mathbb{C}$ defined by $\phi(e) = 4, \phi(12) = 0, \phi(123) = -5$. Write ϕ as a linear combination of irreducible characters of S_3.
(3) Is ϕ the character of a representation of S_3?