M3/4/5P12 PROGRESS TEST 1 (ALTERNATIVE VERSION)

PLEASE WRITE YOUR NAME AND CID NUMBER ON EVERY SCRIPT THAT YOU HAND IN. FAILURE TO DO THIS MAY RESULT IN YOU NOT RECEIVING MARKS FOR QUESTIONS THAT YOU ANSWER.

Note: all representations are assumed to be on finite dimensional complex vector spaces.

Question 1. Let G be a finite group and let $\chi : G \to \mathbb{C}^\times$ be a group homomorphism. Let V be a representation of G. We define a map

$$e_\chi : V \to V$$

by

$$e_\chi(v) = \frac{1}{|G|} \sum_{g \in G} \chi(g)^{-1} \rho_V(g) v.$$

We also define a subspace V^χ of V by

$$V^\chi = \{v \in V : \rho_V(g)v = \chi(g)v \text{ for all } g \in G\}.$$

(a) Show that V^χ is a subrepresentation of V.
(b) Show that e_χ is a G-linear map, that $e_\chi \circ e_\chi = e_\chi$, and that the image of e_χ is equal to V^χ (i.e. e_χ is a *G-linear projection* with image V^χ).
(c) Now suppose we have another group homomorphism $\chi' : G \to \mathbb{C}^\times$. Show that if $\chi \neq \chi'$ then $e_{\chi'} \circ e_\chi = 0$.
(d) Consider the linear map $f : V \to V$ given by $\sum_\chi e_\chi$, where the sum runs over all the homomorphisms $\chi : G \to \mathbb{C}^\times$. Show that f is a G-linear projection, and that the kernel of f has no one-dimensional subrepresentations.

Question 2. Consider the symmetric group S_4 of permutations of $\{1, 2, 3, 4\}$. Write Ω for the subset $\{(12)(34), (13)(24), (14)(23)\} \subset S_4$.

Define an action of S_4 on Ω by $g \cdot \omega = g\omega g^{-1}$. Consider the representation of S_4 on the vector space $\mathbb{C}\Omega$ with basis $\{[\omega] : \omega \in \Omega\}$ and group action defined by

$$\rho_{\mathbb{C}\Omega}(g)[\omega] = [g \cdot \omega].$$

(a) By computing eigenspaces for $\rho_{\mathbb{C}\Omega}(12)$ and $\rho_{\mathbb{C}\Omega}(13)$, or otherwise, show that $\mathbb{C}\Omega$ has a unique one-dimensional subrepresentation U_1, which is spanned by $\sum_{\omega \in \Omega} [\omega]$.
(b) Deduce that $\mathbb{C}\Omega$ is isomorphic as a representation of S_4 to $U_1 \oplus U_2$ where U_2 is an irreducible two-dimensional representation of S_4. *You don't need to find U_2 explicitly.*
(c) Show that S_4 has an irreducible representation of dimension 3. *You may assume without proof that S_4 has exactly two isomorphism classes of one-dimensional representations. Again, you don't need to find this representation explicitly.*

Date: Thursday February 18, 2016.