M3/4/5P12 PROGRESS TEST 1 (ALTERNATIVE VERSION)

Note: all representations are assumed to be on finite dimensional complex vector spaces.

Question 1. Let G be a finite group and let $\chi : G \to \mathbb{C}^\times$ be a group homomorphism. Let V be a representation of G. We define a map

$$e_\chi : V \to V$$

by

$$e_\chi(v) = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\rho_V(g)v.$$

We also define a subspace V^χ of V by

$$V^\chi = \{v \in V : \rho_V(g)v = \chi(g)v \text{ for all } g \in G\}.$$

(a) Show that V^χ is a subrepresentation of V.

(b) Show that e_χ is a G-linear map, that $e_\chi \circ e_\chi = e_\chi$, and that the image of e_χ is equal to V^χ (i.e. e_χ is a *G-linear projection* with image V^χ).

(c) Now suppose we have another group homomorphism $\chi' : G \to \mathbb{C}^\times$. Show that if $\chi \neq \chi'$ then $e_{\chi'} \circ e_\chi = 0$.

(d) Consider the linear map $f : V \to V$ given by $\sum_\chi e_\chi$, where the sum runs over all the homomorphisms $\chi : G \to \mathbb{C}^\times$. Show that f is a G-linear projection, and that the kernel of f has no one-dimensional subrepresentations.

Solution 1. (a) We need to show that V^χ is G-stable. Suppose $v \in V^\chi$and $h \in G$. We need to show that $\rho_V(h)v \in V^\chi$. But we have $\rho_V(h)v = \chi(h)v$ so for any $g \in G$ we have

$$\rho_V(g)(\rho_V(h)v) = \chi(g)\chi(h)v = \chi(g)(\rho_V(h)v).$$

Therefore we have $\rho_V(h)v \in V^\chi$. **2 marks**

(b) First we check that e_χ is G-linear. We have

$$e_\chi(\rho_V(h)v) = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\rho_V(gh)v = \rho_V(h)\frac{1}{|G|}\sum_{g\in G}\chi(h^{-1}gh)^{-1}\rho_V(h^{-1}gh)v$$

where in the last equality we use that $\chi(h^{-1}gh) = \chi(g)$. Since conjugation by h gives a permutation of G, this is equal to $\rho_V(h)e_\chi(v)$, so e_χ is G-linear.

Next we check that the image of e_χ is contained in V^χ. We have

$$\rho_V(h)e_\chi(v) = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\rho_V(hg)v = \chi(h)\frac{1}{|G|}\sum_{g\in G}\chi(hg)^{-1}\rho_V(hg)v = \chi(h)e_\chi(v)$$

since multiplication by h gives a permutation of G.

Finally, it remains to check that e_χ is equal to the identity on V^χ. This shows that e_χ is a G-linear projection with image V^χ. If $v \in V^\chi$ we have

$$e_\chi v = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\rho_V(g)v = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\chi(g)v = v$$

so we are done. **4 marks**

(c) The G-linear map e_χ has image equal to V^χ. The restriction of $e_{\chi'}$ to V^χ gives a map from V^χ to $(V^\chi)^{\chi'}$ but if $\chi' \neq \chi$ we have $(V^\chi)^{\chi'} = 0$. So the composition $e_{\chi'} \circ e_\chi$ is equal to 0. **2 marks**

(d) Using parts b) and c) we can check that

$$(\sum_\chi e_\chi) \circ (\sum_\chi e_\chi) = \sum_\chi e_\chi \circ e_\chi = \sum_\chi e_\chi$$

so $\sum_\chi e_\chi$ is a G-linear projection. Suppose we have a one-dimensional subrepresentation U of V. The action of G on U is given by $\rho_U(g)u = \chi(g)u$ for some homomorphism $\chi : G \to \mathbb{C}^\times$. So we conclude that e_χ is equal to the identity on U and the other maps $e_{\chi'}$ are equal to 0. This implies that U is not in the kernel of f. **2 marks**

Question 2. Consider the symmetric group S_4 of permutations of $\{1, 2, 3, 4\}$. Write Ω for the subset $\{(12)(34), (13)(24), (14)(23)\} \subset S_4$.

Define an action of S_4 on Ω by $g \cdot \omega = g\omega g^{-1}$. Consider the representation of S_4 on the vector space $\mathbb{C}\Omega$ with basis $\{[\omega] : \omega \in \Omega\}$ and group action defined by

$$\rho_{\mathbb{C}\Omega}(g)[\omega] = [g \cdot \omega].$$

(a) By computing eigenspaces for $\rho_{\mathbb{C}\Omega}(12)$ and $\rho_{\mathbb{C}\Omega}(13)$, or otherwise, show that $\mathbb{C}\Omega$ has a unique one-dimensional subrepresentation U_1, which is spanned by $\sum_{\omega\in\Omega}[\omega]$.

(b) Deduce that $\mathbb{C}\Omega$ is isomorphic as a representation of S_4 to $U_1 \oplus U_2$ where U_2 is an irreducible two-dimensional representation of S_4. *You don't need to find U_2 explicitly.*

(c) Show that S_4 has an irreducible representation of dimension 3. *You may assume without proof that S_4 has exactly two isomorphism classes of one-dimensional representations. Again, you don't need to find this representation explicitly.*

Solution 2. (a) Let's write down the matrices for $\rho_{\mathbb{C}\Omega}(12)$ and $\rho_{\mathbb{C}\Omega}(13)$ with respect to the basis $\{[(12)(34)], [(13)(24)], [(14)(23)]\}$ of $\mathbb{C}\Omega$. We have

$$\rho_{\mathbb{C}\Omega}(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\rho_{\mathbb{C}\Omega}(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The eigenspaces for (13) are given by: a $+1$ eigenspace of vectors (a, b, a) and a -1 eigenspace of vectors $(a, 0, -a)$.

The eigenspaces for (12) are given by: a $+1$ eigenspace of vectors (a, b, b) and a -1 eigenspace of vectors $(0, b, -b)$.

Suppose $v = (a, b, c)$ is a non-zero simultaneous eigenvector. If it is a -1-eigenvector for (13) then we have $b = 0$. Looking at the possibilities for (12) then tells us that $v = 0$. The same argument applies if v is a -1-eigenvector for (12). So v must have eigenvalue $+1$ for both (12) and (13). We deduce that v is in the span of $(1, 1, 1)$. This shows that the only one-dimensional subrepresentation is the one spanned by $(1, 1, 1) = \sum_{\omega\in\Omega}[\omega]$. **4 marks**

(b) By Maschke's theorem, we know that $\mathbb{C}\Omega$ is isomorphic to $U_1 \oplus U_2$ where U_2 is two-dimensional. Since U_1 is the unique one-dimensional subrepresentation of $\mathbb{C}\Omega$, we deduce that U_2 is irreducible.**2 marks**

(c) We have $24 = \sum_{i=1}^{r} d_i^2$, where the d_i are the dimensions of the irreps (up to isomorphism). We know that S_4 has two one-dimensional and one two-dimensional irrep. So we get

$$24 = 1 + 1 + 4 + \cdots$$

so we have $18 = \sum_{i=4}^{r} d_i^2$. The only possibility is then that we have two more irreps of dimension 3, since 4 does not divide 18 and there are no more representations of dimension 1. **4 marks**