M3/4/5P12 PROGRESS TEST 2

Question 1. Let G be a finite group. Let V be a representation of G, with character χ_V.

(a) What is the definition of the *dual representation* V^*?

(b) What is the character χ_{V^*} of the dual representation V^*, in terms of χ_V? Justify your answer.

(c) Let W be another representation of G, with character χ_W. What is the character $\chi_{V\otimes W}$ of the tensor product representation $V \otimes W$, in terms of χ_V and χ_W? *You just need to state the answer.*

(d) Let V_{triv} be the one-dimensional trivial representation of G, with character χ_{triv}. Show that

$$\langle \chi_{V\otimes W}, \chi_{triv}\rangle = \langle \chi_V, \chi_{W^*}\rangle.$$

(e) Suppose V and W are irreducible representations. If W^* is not isomorphic to V, how many copies of V_{triv} appear in the decomposition of $V \otimes W$ into irreducibles? How many copies of V_{triv} appear in the decomposition of $V \otimes V^*$ into irreducibles? Justify your answers.

Solution 1. (a) The vector space V^* is $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. The action of G is defined by $\rho_{V^*}(g)f = f \circ \rho_V(g^{-1})$. **1 mark**

(b) If we fix a basis B for V and consider the dual basis B^* for V^*, then the matrix $[\rho_{V^*}(g)]_{B^*} = [\rho_V(g)]_B^{-t}$, the inverse transpose matrix. So we have $\chi_{V^*}(g) = \chi_V(g^{-1})$, since a matrix and its transpose have the same trace. Since $\chi_V(g) = \zeta_1 + \zeta_2 + \cdots \zeta_n$ where the ζ_i are the eigenvalues of $\rho_V(g)$ (which are roots of unity), we have $\chi_V(g^{-1}) = \zeta_1^{-1} + \zeta_2^{-1} + \cdots \zeta_n^{-1} = \overline{\chi_V(g)}$. So $\chi_{V^*} = \overline{\chi_V}$. 1 mark for $\chi_{V^*} = \overline{\chi_V}$, 2 more marks for justification. Total: **3 marks**.

(c) $\chi_{V\otimes W} = \chi_V \chi_W$. **1 mark**

(d) We have $\langle \chi_{V\otimes W}, \chi_{triv}\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\chi_W(g)$. On the other hand,

$$\langle \chi_V, \chi_{W^*}\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\overline{\chi_{W^*}(g)} = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)\chi_W(g)$$

where the last equality is by part (b). So we get the same answer. **3 marks**

(e) The number of times V_{triv} appears in the decomposition of $V \otimes W$ is equal to $\langle \chi_{V\otimes W}, \chi_{triv}\rangle$. By the previous part, this is equal to $\langle \chi_V, \chi_{W^*}\rangle$. Since W is irreducible, W^* is irreducible. Since χ_V and χ_{W^*} are not isomorphic, $\langle \chi_V, \chi_{W^*}\rangle = 0$, so the first answer is 0.

The number of times V_{triv} appears in the decomposition of $V \otimes V^*$ is equal to $\langle \chi_{V\otimes V^*}, \chi_{triv}\rangle$. By the previous part, this is equal to $\langle \chi_V, \chi_{V^{**}}\rangle$. Since V is irreducible and $V^{**} \cong V$, this number is equal to 1. **2 marks**, partial credit if you do something sensible but don't get all the way to the correct answers!

Question 2. (a) Let G be a finite group. State the column orthogonality relations for the irreducible characters of G.

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(b) Here is an incomplete character table for a group of order 24, with 7 conjugacy classes.

	$g_1 = e$	g_2	g_3	g_4	g_5	g_6	g_7
Size of conj. class	1	1	6	4	4	4	4
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2	ω^2	ω
χ_4	2	-2	0	-1	-1	1	1

In the table, $\omega = e^{2\pi i/3}$.

(i) Find another irreducible character χ_3 of dimension 1 (i.e. with $\chi_3(e) = 1$).
(ii) Find two more distinct irreducible characters χ_5, χ_6 of dimension two.
(iii) Work out the complete character table for this group of order 24, justifying your answer.

Solution 2. (a) Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G, and let $g, h \in G$. The column orthogonality relations are

$$\sum_{i=1}^{r} \overline{\chi_i(g)}\chi_i(h) = 0$$

if h is not conjugate to G, and

$$\sum_{i=1}^{r} \overline{\chi_i(g)}\chi_i(h) = \frac{|G|}{|C(g)|}$$

if h is conjugate to g, where $C(g)$ is the conjugacy class of g. **2 marks**

(b) We write V_i for an irrep with character χ_i.

(i) We let $\chi_3 = \overline{\chi_2}$. This is an irreducible character because V_2^* is an irrep. It is distinct from χ_1 and χ_2, since its value on g_4 is different. **2 marks** *Some of you tried to find χ_3 by using the row orthogonality relations. But this gives you something like 4 equations in 6 unknowns, so it seems impossible to determine χ_3 this way.*

(ii) We let $\chi_5 = \chi_2\chi_4$ and $\chi_6 = \chi_3\chi_4$. These are irreducible characters because V_2, V_3 have dimension 1, so the tensor product representations $V_2 \otimes V_4$ and $V_3 \otimes V_4$ are irreducible. They are distinct from each other and χ_4 because they take different values on g_4. **2 marks**

(iii) So far we have:

	$g_1 = e$	g_2	g_3	g_4	g_5	g_6	g_7
Size of conj. class	1	1	6	4	4	4	4
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2	ω^2	ω
χ_3	1	1	1	ω^2	ω	ω	ω^2
χ_4	2	-2	0	-1	-1	1	1
χ_5	2	-2	0	$-\omega$	$-\omega^2$	ω^2	ω
χ_6	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2
χ_7							

We know there are 7 irreducible characters because there are 7 conjugacy classes. We can work out χ_7 using column orthogonality. There are a few different ways to do this, but here's an example.
We have $1^2 + 1^2 + 1^2 + 2^2 + 2^2 + 2^2 + \chi_7(e)^2 = 24$, so $\chi_7(e) = 3$. Using column orthogonality for the first and second columns, we get $3 - 12 + 3\chi_7(g_2) = 0$, so $\chi_7(g_2) = 3$. For the first and third columns, we get $3 + 3\chi_7(g_3) = 0$ so $\chi_7(g_3) = -1$. Finally, for the last 4 columns, applying the column orthogonality relation to each column with itself

gives

$$6 \times 1 + |\chi_7(g_i)|^2 = \frac{|G|}{|C(g_i)|} = \frac{24}{4} = 6$$

which implies that $\chi_7(g_i) = 0$ for $i \geq 4$. This completes the character table. To summarise, we have

	$g_1 = e$	g_2	g_3	g_4	g_5	g_6	g_7
χ_7	3	3	-1	0	0	0	0

4 marks