SHIMURA CURVES, THE DRINFELD CURVE AND SERRE WEIGHTS

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1. Introduction

We let F be a totally real number field, and fix a place $\mathfrak{p}|p$ of F, with p an odd prime. Denote the local field $F_{\mathfrak{p}}$ by K_0, and denote its residue field by k_0. We have $k_0 \cong \mathbb{F}_q$ with $q = p^f$ for some $f \geq 1$. We denote by $X(\mathfrak{p})$ a Shimura curve associated to a suitable indefinite quaternion algebra over F, with some fixed level outside $\mathfrak{p}$ and level at $\mathfrak{p}$ corresponding to the principal congruence subgroup

$$U(\mathfrak{p}) = \ker(\mathrm{GL}_2(\mathcal{O}_{K_0}) \to \mathrm{GL}_2(k_0)).$$

Then $X(\mathfrak{p})$ has semistable reduction over a finite extension K/K_0 with ramification degree $q^2 - 1$ and residue field k some (explicit) extension of k_0. An explicit semistable model $\widetilde{X}$, in the modular curve case, is described in unpublished work of Edixhoven [Edi01]. The description of this model is also a special case of the recent work of Weinstein [Wei], which describes semistable models for modular curves of any level.

The genesis of this work was the authors' attempts to investigate the consequences for the mod p Langlands programme of the geometry of $\widetilde{X}$. The étale cohomology group

$$H^1_{et}(X(\mathfrak{p})_{\overline{F}}, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$$

is of importance in the mod p Langlands programme — for example, the weight part of Serre's conjecture is a statement about its structure as a $\mathrm{GL}_2(k_0) \times \mathrm{Gal}(\overline{F}/F)$-representation. This étale cohomology group compares with the log-crystalline cohomology of $\widetilde{X}$ over the base S appearing in Breuil's integral p-adic Hodge theory, but in the present paper we only consider the de Rham cohomology of the normalisation of $\widetilde{X}_k$. The structure of this de Rham cohomology group already has some consequences for one direction (weight elimination) of the weight part of Serre's conjecture ([BDJ10],[Sch08a]).

Our first goal in this paper is to describe the analogous semistable model in the Shimura curve case. This essentially follows from a special case of the generalised semistable models for Lubin–Tate spaces with level $\mathfrak{p}$ structure which are described in [Yos10]. The irreducible components of the special fibre $\widetilde{X}_k$ are of two kinds. The first kind have a purely local description, and are isomorphic to compactifications of the Deligne-Lusztig curves for the group $\mathrm{GL}_2(k_0)$ (with the natural action of $\mathrm{GL}_2(k_0)$ on $X(\mathfrak{p})$ corresponding to the natural action of $\mathrm{GL}_2(k_0)$ on the Deligne-Lusztig curves) — these curves are forms of the Drinfeld curve with affine equation $xy^q - yx^q = 1$. The second kind of irreducible components are Igusa curves, which already appear as irreducible components of the regular integral model for $X(\mathfrak{p})$ defined by Carayol [Car86]. We denote the union of the 'Drinfeld' components by $\overline{Y}_k$ and the union of the 'Igusa' components

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by $\overline{Z}_k$. For $l \neq p$, the l-adic cohomology of $\overline{Y}_k$ is comprised of cuspidal representations of $\mathrm{GL}_2(k_0)$, whilst the l-adic cohomology of $\overline{Z}_k$ is comprised of induced representations.

Another goal of this paper is to indicate the way in which the 'cuspidal' and 'induced' pieces of cohomology blend together when one considers a cohomology theory with mod p coefficients (namely, de Rham cohomology of the special fibre or mod p étale cohomology of the generic fibre). This blending can be observed in the structure of finite flat group schemes which are naturally associated to Galois representations appearing in the mod p étale cohomology of $X(\mathfrak{p})$. In what remains of the introduction, we describe the contents of this paper in a little more detail.

1.1. **Consequences for the weight part of Serre's conjecture.** We need to set up a little more notation in order to state our results precisely.

We denote by $I(K/K_0)$ the inertia group of K/K_0 and define a fundamental character of niveau $2f$

$$\omega_{2f} : I(K/K_0) \to k^\times$$
$$g \mapsto \frac{g\varpi}{\varpi} \bmod \varpi,$$

where ϖ is any uniformiser of K. Set $\omega_f = \omega_{2f}^{1+q}$. Denote the ramification degree of $K_0/\mathbb{Q}_p$ by e_0. We let L be the unramified quadratic extension of K_0 inside K, with residue field k_L, and write G_F and G_{K_0} for the absolute Galois groups of F and K_0 respectively. Finally, we write I for the inertia subgroup of G_{K_0}. We prove the following in Corollaries 6.17 6.19, 6.24 and 6.26:

Theorem 1.1. *Let W be an irreducible representation of $\mathrm{GL}_2(k_0)$*

$$W = \det{}^a \bigotimes_{\tau : k_0 \hookrightarrow \overline{\mathbb{F}}_p} (\mathrm{Symm}^{y_\tau} k_0^2) \otimes_\tau \overline{\mathbb{F}}_p$$

with $1 \leq y_\tau < p$ for every τ and not all the y_τ equal to $p-1$ (we say the weight W is weakly regular when these conditions hold). If $f = 1$ we modify the definition of weakly regular slightly (see Definition 6.7).

If moreover we have $e_0 \leq y_\tau \leq p - 1 - e_0$ for all τ, we say the weight W is strongly e_0-regular.

Suppose r is an irreducible two dimensional $\overline{\mathbb{F}}_p$-representation of G_F, such that r appears as an $\overline{\mathbb{F}}_p[G_F]$-submodule of $\mathrm{Hom}_{\mathrm{GL}_2(k_0)}(W, H^1_{et}(X(\mathfrak{p})_{\overline{F}}, \overline{\mathbb{F}}_p))$.

(i) *Suppose that $r|_{G_{K_0}}$ is irreducible (the supersingular case). Then there exists a subset $J \subset \mathrm{Hom}(k_L, \overline{\mathbb{F}}_p)$, which bijects with $\mathrm{Hom}(k_0, \overline{\mathbb{F}}_p)$ on restriction to k_0, together with integers $-1 \leq \epsilon_\tau \leq e_0 - 1$ for $\tau \in \mathrm{Hom}(k_0, \overline{\mathbb{F}}_p)$, such that $r|_I = \rho \oplus \rho^q$ where*

$$\rho = \omega_f^{-a} \prod_{\tau \in J} \tau \circ \omega_{2f}^{-(1+y_\tau+\epsilon_\tau)} \prod_{\tau \notin J} \tau \circ \omega_{2f}^{-(e_0-1-\epsilon_\tau)}.$$

The integers ϵ_τ are ≥ 0 unless τ and $\tau \circ \sigma^{-1}$ are both in J, or both in the complement of J (here σ^{-1} is the inverse of the q-power Frobenius morphism). If W is strongly e_0-regular then all of the ϵ_τ are ≥ 0.

(ii) *Suppose that $r|_{G_{K_0}}$ is reducible (the ordinary case). Then there exists a subset $J \subset \mathrm{Hom}(k_0, \overline{\mathbb{F}}_p)$ together with integers $-1 \le \epsilon_\tau \le e_0$ for $\tau \in \mathrm{Hom}(k_0, \overline{\mathbb{F}}_p)$, such that $r|_I = \begin{pmatrix} \rho' & * \\ 0 & \rho'' \end{pmatrix}$ where*

$$\rho' = \omega_f^{-a} \prod_{\tau \in J} \tau \circ \omega_f^{-(1+y_\tau+\epsilon_\tau)} \prod_{\tau \notin J} \tau \circ \omega_f^{-(e_0-1-\epsilon_\tau)}$$

and

$$\rho'' = \omega_f^{-a} \prod_{\tau \notin J} \tau \circ \omega_f^{-(1+y_\tau+\epsilon_\tau)} \prod_{\tau \in J} \tau \circ \omega_f^{-(e_0-1-\epsilon_\tau)}.$$

The integers ϵ_τ are ≥ 0 unless τ and $\tau \circ \sigma^{-1}$ are both in J. The integers ϵ_τ are $\le e_0 - 1$ unless τ and $\tau \circ \sigma^{-1}$ are both in the complement of J. If W is strongly e_0-regular and we exclude the cases where $y_\tau = e_0$ for every τ or $y_\tau = p-1-e_0$ for every τ, then all of the ϵ_τ are in the interval $[0, e_0 - 1]$.

This result was already known in many cases [Gee11], [GS11a], [Sch08b], [Sch08a]. Whilst this work was in preparation, Gee, Liu and Savitt [GLS] have established it in general, in the stronger form with the ϵ_τ in the interval $[0, e_0 - 1]$ for all τ. However, our methods are completely different to theirs, and in many situations we find a natural geometric interpretation for the sets J appearing in the above theorem. These results are outlined in Section 1.2.

Since our argument is geometric, amongst the above references it is closest to the work of Schein in [Sch08b], [Sch08a]. We work entirely in characteristic p, avoiding any combinatorics related to the irreducible constituents of the reductions of characteristic 0 $\mathrm{GL}_2(k_0)$-representations. It is plausible that analogous arguments to Schein's, using the semistable model $\widetilde{X}$, would be able to prove stronger results in the ordinary case than those we present here. We avoid any computations with reductions of crystalline representations, instead working only with finite flat group schemes killed by p and their associated Breuil–Kisin modules.

1.2. **Geometric interpretation of the weight part of Serre's conjecture.** In many cases, we are able to give a geometric interpretation for the subset J appearing in Theorem 1.1 (and in the conjectures of [BDJ10], [Sch08a]). For simplicity we here just state our result in the supersingular case, so we make all the assumptions of Theorem 1.1, including the strongly e_0-regular assumption on the weight W and suppose $r|_{G_{K_0}}$ is irreducible. We wish to understand the subset J appearing in the conclusion of the Theorem.

The inclusion of r in $\mathrm{Hom}_{\mathrm{GL}_2(k_0)}(W, H^1_{et}(X(\mathfrak{p})_{\overline{F}}, \overline{\mathbb{F}}_p))$ allows us to construct a finite flat $\overline{\mathbb{F}}_p$-module scheme (in fact, we work over a finite coefficient field) $\mathcal{H}$ over $\mathcal{O}_K$ with generic fibre descending to K_0, such that $\mathcal{H}(\overline{K}) \cong \rho(1)$ and such that the (contravariant) crystalline Dieudonné module $\mathbb{D}$ of $\mathcal{H}_k^\vee$ naturally embeds in

$$[H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p.$$

The free $k \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$-module $\mathbb{D}$ breaks up as a direct sum $\oplus_\tau \mathbb{D}_\tau$ of one-dimensional $\overline{\mathbb{F}}_p$-vector spaces indexed by embeddings $\tau : k \hookrightarrow \overline{\mathbb{F}}_p$, and we obtain natural embeddings

$$\iota_\tau : \mathbb{D}_\tau \hookrightarrow [H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)] \otimes_{k,\tau} \overline{\mathbb{F}}_p.$$

For each τ, the image of ι_τ is contained in either $H^1_{dR}(\overline{Y}_k/k) \otimes_{k,\tau} \overline{\mathbb{F}}_p$ or $H^1_{dR}(\overline{Z}_k/k) \otimes_{k,\tau} \overline{\mathbb{F}}_p$, we say that τ is a 'Drinfeld embedding' for ρ in the first case, and an 'Igusa embedding' for ρ in the second case. We prove that whether τ is Drinfeld or Igusa depends only on the restriction of τ to k_L, that τ is a Drinfeld embedding for ρ if and only if it is an Igusa embedding for ρ^q (this requires the strongly e_0-regular assumption), and give (in Theorem 6.18) the following description of the subset $J \subset \mathrm{Hom}(k_L, \overline{\mathbb{F}}_p)$

Theorem 1.2. *An embedding $\tau : k \hookrightarrow \overline{\mathbb{F}}_p$ is a Drinfeld embedding for ρ if and only if*

$$\tau|_{k_L} \in J.$$

1.3. **Finite flat models and the mod p Langlands programme.** Finally, our methods produce some information about the structure of some natural finite flat models over $\mathrm{Spec}(\mathcal{O}_K)$ for certain $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p}) \times \mathrm{GL}_2(k_0)$-subrepresentations of $H^1_{et}(X(\mathfrak{p})_{\overline{F}}, \overline{\mathbb{F}}_p)$. These finite flat models arise from the quasi-finite flat group scheme $\mathrm{Pic}^0(\widetilde{X}/\mathcal{O}_K)[p]$ and we believe they are natural objects to study in the mod p Langlands programme.

We sketch an outline of our motivation: given a maximal ideal $\mathfrak{m}$ of a suitable Hecke algebra, one obtains a (log) finite flat group scheme $\mathcal{G} = \mathrm{Pic}^0(\widetilde{X}/\mathcal{O}_K)[\mathfrak{m}]$ with descent datum to $F_\mathfrak{p}$ on the generic fibre. The generic fibre of $\mathcal{G}$ gives representation of $G_{K_0} \times \mathrm{GL}_2(k_0)$ which is conjecturally described (up to multiplicities) by [BP12] in many cases when p is unramified in K_0. The $\mathrm{GL}_2(k_0)$-socle of this representation is the object of study in the weight part of Serre's conjecture. In any case, one expects this representation to look like $(\rho \otimes \pi_\mathfrak{m})^{\oplus m}$, where m is some multiplicity (which will depend the global situation), ρ is the restriction to $G_{F_\mathfrak{p}}$ of the mod p Galois representation attached to $\mathfrak{m}$ and $\pi_\mathfrak{m}$ is a representation of $\mathrm{GL}_2(k_0)$. Moreover, $\pi_\mathfrak{m}$ should depend only on the local Galois representation ρ. The results of [EGS], when p is unramified in K_0, prove that this is indeed the case.

One can also ask if the whole of $\mathcal{G}$, not just its generic fibre, is determined by ρ. More precisely, do we have $\mathcal{G} = (\mathcal{G}_\mathfrak{m})^{\oplus m}$, with $\mathcal{G}_\mathfrak{m}$ a (log) finite flat model for $\rho \otimes \pi_\mathfrak{m}$, which depends only on the local Galois representation ρ? We believe that the answer may be 'Yes', and that there should be some version of a mod p Langlands philosophy incorporating these finite flat models. One of our goals is to provide some evidence for this belief, and our results in this direction are collected in Section 7. Briefly, we are able to understand the piece of $\mathcal{G}$ contributing to the strongly e_0-regular Serre weights in $\pi_\mathfrak{m}$. In future work we hope to go 'beyond the socle'.

1.4. **Outline of this paper.** The structure of our paper is as follows: In Section 2 we describe the semistable models of modular and Shimura curves which we will be studying over the course of the paper. These models were first described by Edixhoven in the modular curve case [Edi01] and the necessary models for the first covering of Lubin–Tate spaces appear in [Yos10] (in more generality).

In Section 3 we discuss some of the characteristic p representation theory of $\mathrm{GL}_2(\mathbb{F}_q)$ which we will need to describe the structure of the de Rham cohomology of the irreducible components of the special fibre of our semistable curves. In Section 4 we give this description.

We then use the theory of Breuil–Kisin modules to relate (pieces of) the mod p étale cohomology of $X(p)$ to the de Rham cohomology of the (normalisation of the) curve $\widetilde{X}_k$. For our applications to the weight part of Serre's conjecture we could probably have used Raynaud's theory of vector space schemes, as in [Sch08b], [Sch08a], but we found working with Breuil–Kisin modules more convenient. Also, the results of Section 7 are most easily phrased using Breuil–Kisin modules. Section 5 summarises the results we need about Breuil–Kisin modules with coefficients and descent data. Everything we use is contained in [Kis09] and [Sav08], except for some elementary facts about Breuil–Kisin modules with actions of $\mathrm{GL}_2(\mathbb{F}_q)$.

In Section 6 we explain the proof of Theorems 1.1 and 1.2. In Section 7 we discuss our results on the finite flat models appearing in $\mathrm{Pic}^0(\widetilde{X}/\mathcal{O}_K)$. All the work for this final section is done in Section 6, we just extract the statements from the proofs of that section.

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2. Geometry of Shimura curves

In this section we describe semistable models of Shimura curves with full level at a prime dividing p. The models are due to Edixhoven [Edi01] (unpublished) in the case of modular curves, and a generalisation to unitary Shimura varieties of 'Harris–Taylor' type follows from the results of [Yos10].

Let F be a totally real field with $[F:\mathbb{Q}] = d$. Let B be a central simple algebra over F of degree 4, and G be the algebraic group over $\mathbb{Q}$ defined by $G(R) = (B \otimes_{\mathbb{Q}} R)^\times$ for any $\mathbb{Q}$-algebra R. We look at the Riemann surfaces $G(\mathbb{Q})\backslash G(\mathbb{A})/(U \times U_\infty)$, with a sufficiently small open compact subgroup U of $G(\mathbb{A}^\infty)$, where $\mathbb{A}^\infty = \mathbb{Q} \otimes \widehat{\mathbb{Z}}$, and the maximal connected compact modulo centre subgroup U_∞ of $G(\mathbb{R})$. We consider the cases where $G(\mathbb{R})/U_\infty \cong \mathbb{C} \setminus \mathbb{R}$, i.e. Shimura curves over F.

We let B be a quaternion algebra over F which is split at one infinite place $\tau : F \to \mathbb{R}$ and non-split at every other infinite place. In the case $B = M_2(\mathbb{Q})$, $G = \mathrm{GL}_2$ we have $U_\infty = \mathbb{R}^\times \mathrm{SO}(2)$, and we let $X(\mathbb{C})$ be the compactification of $G(\mathbb{Q})\backslash G(\mathbb{A})/(U \times U_\infty)$. Otherwise, we have $U_\infty = \mathbb{R}^\times \mathrm{SO}_2(\mathbb{R}) \times (\mathbb{H}^\times)^{d-1}$ in $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H}^\times)^{d-1}$, where $\mathbb{H}$ is the Hamiltonian quaternions over $\mathbb{R}$, and we set $X(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/(U \times U_\infty)$ (which is compact).

The proper smooth curve $X/\mathbb{C}$ has a canonical model over F, where we choose the Shimura data as in [Car86]. Let $\det = N_{B/F} : B^\times \to F^\times$ be the reduced norm on B. Extended to $(B \otimes_{\mathbb{Q}} R)^\times$ for any $\mathbb{Q}$-algebra R, it gives a homomorphism $\det : G \to \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ of algebraic groups over $\mathbb{Q}$. Note that $\det U_\infty = (\mathbb{R}^\times_{>0})^d$ and $\pi_0(X(\mathbb{C})) \cong F^\times \backslash \mathbb{A}_F^\times / \det(U \times U_\infty)$, the narrow ideal class group corresponding to $\det U$, where $\mathbb{A}_F = F \otimes_{\mathbb{Q}} \mathbb{A}$. If $F(U)$ is the abelian extension of F such that $F^\times \backslash \mathbb{A}_F^\times / \det(U \times U_\infty) \cong \mathrm{Gal}\big(F(U)/F\big)$ by the global Artin map, then $X \to \mathrm{Spec}\, F$ factors through $\mathrm{Spec}\, F(U)$, with $X/F(U)$ geometrically connected.

2.1. **No level at p.** Let p be a prime and $\mathfrak{p} = \mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the primes of F above p. We assume that B is split at $\mathfrak{p}$, i.e. $B \otimes_F F_{\mathfrak{p}} \cong M_2(F_{\mathfrak{p}})$. Let $F_{\mathfrak{p}_i}$ be the completion of F at $\mathfrak{p}_i$, a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_{\mathfrak{p}_i}$ be the ring of integers of $F_{\mathfrak{p}_i}$, and $B_{\mathfrak{p}_i} = B \otimes_F F_{\mathfrak{p}_i}$.

Note that $G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p)$ where $\mathbb{A}^{\infty,p} = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell$, and $G(\mathbb{Q}_p) = \prod_{i=1}^t B_{\mathfrak{p}_i}^\times$, so we write $G(\mathbb{A}^\infty) = G^{\infty,\mathfrak{p}} \times \mathrm{GL}_2(F_{\mathfrak{p}})$ with $G^{\infty,\mathfrak{p}} = G(\mathbb{A}^{\infty,p}) \times \prod_{i=2}^t B_{\mathfrak{p}_i}^\times$. Now we choose $U \subset G(\mathbb{A}^\infty)$ to be of the form $U = U^{\mathfrak{p}} \times \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ with a sufficiently small $U^{\mathfrak{p}} \subset G^{\infty,\mathfrak{p}}$. The base change of X/F to

$F_\mathfrak{p}$ extends to a proper smooth curve over $\mathcal{O}_\mathfrak{p}$ ([KM85], [Car86]). Also $\mathfrak{p}$ is unramified in $F(U)/F$, and letting $\mathcal{O}_F$, $\mathcal{O}_U$ be the ring of integers of F, $F(U)$ and $\mathcal{O}_{U,\mathfrak{p}} := \mathcal{O}_U \otimes_{\mathcal{O}_F} \mathcal{O}_\mathfrak{p}$, the morphism $X \to \mathrm{Spec}\,\mathcal{O}_\mathfrak{p}$ factors through $\mathrm{Spec}\,\mathcal{O}_{U,\mathfrak{p}}$ with $X/\mathcal{O}_{U,\mathfrak{p}}$ geometrically connected.

In the modular curve case X is obtained as a moduli of elliptic curves with level U^p-structure ([KM85]), and the Barsotti-Tate group $\mathcal{G} := \mathcal{A}[p^\infty]$ for the universal elliptic curve $\mathcal{A}$ over X is of height 2 and dimension 1. We denote by $\mathcal{G}_1 = \mathcal{G}[p]$ the finite flat group scheme over X obtained as the p-torsion of $\mathcal{G}$. In the Shimura curve case, except when $F = \mathbb{Q}$, the canonical models are obtained only by relating them to unitary Shimura curves of PEL-type over a CM extension of F ([Car86]). We recall this construction.

Let E_0 be an imaginary quadratic field in which p splits and $E = FE_0$, which is a CM field. Let $z \mapsto \overline{z}$ be the complex conjugation of E/F, so that $N_{E/F}(z) = z\overline{z}$. Extended to $(E \otimes_\mathbb{Q} R)^\times$ for any $\mathbb{Q}$-algebra R, it gives $N_{E/F} : \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) \to \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$. Now we define the algebraic group G' over $\mathbb{Q}$ by

$$G'(R) = \{(g, z) \in G(R) \times (E \times_\mathbb{Q} R)^\times \mid \det(g) = N_{E/F}(z),\ \det(g) \cdot N_{E/F}(z) \in R^\times\}.$$

Then it has a homomorphism $\nu : G' \to \mathbb{G}_m$ defined by $\nu(g, z) = \det(g) \cdot N_{E/F}(z)$. Then G and G' have the same derived subgroup $G_1 = \mathrm{Ker}\,(\det)$, and $\mathrm{Ker}\,\nu = G_1 \times \mathrm{Ker}\,(N_{E/F})$.

We interpret G' as a unitary similitude group. Let $B' := B \otimes_F E$, which is a quaternion algebra over E. Let $\mathrm{tr}_{B'/E}$ be its reduced trace, and $\mathrm{tr} := \mathrm{tr}_{E/\mathbb{Q}} \circ \mathrm{tr}_{B'/E} : B' \to \mathbb{Q}$. Define an involution on B' by $\overline{b \otimes z} = \overline{b} \otimes \overline{z}$, where $b \mapsto \overline{b}$ is the canonical involution on B. We can choose $\delta \in B'$ with $\overline{\delta} = \delta$ so that the involution $* : B' \to B'^{\mathrm{op}}$ defined as $b^* := \delta^{-1}\overline{b}\delta$ is of second kind (i.e. $*|_E$ is $z \mapsto \overline{z}$) and positive ($\mathrm{tr}(bb^*) > 0$ for every $b \in B \setminus \{0\}$). Let $V = B'$, considered as a left B'-module of rank 1. Let $(\ ,\ ) : V \times V \to \mathbb{Q}$ be an alternating pairing defined by $(x, y) = \mathrm{tr}(x\beta y^*)$ for some $\beta \in B'$ with $\beta^* = -\beta$. Then $(bx, y) = (x, b^*y)$ for all $b \in B'$. Then for any choice of δ and β we have

$$G'(R) = \{g \in \mathrm{Aut}_{B' \otimes_\mathbb{Q} R}(V \otimes_\mathbb{Q} R) \mid (gx, gy) = \nu_g(x, y) \text{ for } \nu_g \in R^\times\},$$

and the character $G' \ni g \mapsto \nu_g \in \mathbb{G}_m$ is our previous ν.

For an E-algebra R such that $B' \otimes_E R \cong M_2(R)$ (e.g. $R = E \otimes_\mathbb{Q} \mathbb{R}$), once we fix such an isomorphism we denote by ε the element corresponding to the diagonal matrix $(1, 0)$ in $\mathrm{End}_{B' \otimes_E R}(V \otimes_E R) \cong M_2(R)$ (since $B'^{\mathrm{op}} \cong B'$). We fix $\tau_0 : E_0 \to \mathbb{C}$, which gives $E \to \mathbb{C}$ above each $F \to \mathbb{R}$. Then the Hermitian pairing on $\varepsilon(V \otimes_\mathbb{Q} \mathbb{R}) \cong (E \otimes_\mathbb{Q} \mathbb{R})^2 \cong (\mathbb{C}^d)^2$ (by τ_0) associated to $(\ ,\ )$ has invariants $(1, 1)$ at $\tau : F \hookrightarrow \mathbb{R}$ and $(0, 2)$ at all the other. Thus $G \times_\mathbb{Q} \mathbb{R} \cong \mathrm{G}\big(\mathrm{U}(1, 1) \times \mathrm{U}(0, 2)^{d-1}\big)$, where $\mathrm{G}(\cdot)$ denotes the unitary similitude group. The associated PEL-type Shimura varities X' with $X'(\mathbb{C}) = G'(\mathbb{Q})\backslash G'(\mathbb{Q})/U' \cdot U'_\infty$, where U' is a sufficiently small open compact subgroup of $G'(\mathbb{A}^\infty)$ and U'_∞ is the maximal compact modulo centre subgroup of $G'(\mathbb{R})$, are the special cases of the ones treated in [HT01].

Now fix a prime $\mathfrak{p}_0$ of E_0 above p, and let $\mathfrak{q} = \mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the primes of E above $\mathfrak{p}_0$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, so that $E_{\mathfrak{q}_i} = F_{\mathfrak{p}_i}$ and $B_{\mathfrak{p}_i} = B' \otimes_E E_{\mathfrak{q}_i}$. Let $\mathcal{O}_E$ be the ring of integers of E and $\mathcal{O}_{E,(p)} = \mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$. We choose a maximal $\mathcal{O}_{E,(p)}$-order $\mathcal{O}_{B'}$ of B' with $\mathcal{O}_{B'}^* = \mathcal{O}_{B'}$, and let $\mathcal{O}_{B,\mathfrak{p}_i} = \mathcal{O}_{B'} \otimes_{\mathcal{O}_{E,(p)}} \mathcal{O}_{\mathfrak{p}_i}$. Note that $G'(\mathbb{A}^\infty) = G'(\mathbb{A}^{\infty,p}) \times G'(\mathbb{Q}_p)$ with $G'(\mathbb{Q}_p) = \left(\prod_{i=1}^t B_{\mathfrak{p}_i}^\times\right) \times \mathbb{Q}_p^\times$. Take $U' \subset G'(\mathbb{A}^\infty)$ to be of the form $U' = U'^p \times U'^{\mathfrak{p}}_p \times GL_2(\mathcal{O}_\mathfrak{p}) \times \mathbb{Z}_p^\times$, with a sufficiently small $U'^p \subset G'(\mathbb{A}^{\infty,p})$ and $U'^{\mathfrak{p}}_p := \prod_{i=2}^t \mathrm{Ker}\left(\mathcal{O}_{B,\mathfrak{p}_i}^\times \to (\mathcal{O}_{B,\mathfrak{p}_i}/\mathfrak{p}_i^{m_i})^\times\right) \subset \prod_{i=2}^t B_{\mathfrak{p}_i}^\times$ for $m_i \in \mathbb{N}$. Then the corresponding curve X'

has a canonical model over E, whose base change to $E_{\mathfrak{q}} = F_{\mathfrak{p}}$ extends to a proper smooth curve over $\mathcal{O}_{\mathfrak{p}}$ ([HT01], III.4).

We describe this integral model of X' over $\mathcal{O}_{\mathfrak{p}}$. Let $\mathcal{O}_{\mathfrak{p}_0}$ be the ring of integers for $E_{0,\mathfrak{p}_0}$. Choose an isomorphism $\mathcal{O}_{B,\mathfrak{p}} \cong M_2(\mathcal{O}_{\mathfrak{p}})$, which gives $B_{\mathfrak{p}}^\times \cong \mathrm{GL}_2(F_{\mathfrak{p}})$, and let $\varepsilon \in \mathcal{O}_{B,\mathfrak{p}}$ be the element corresponding to the diagonal matrix $(1,0)$. We write $\Lambda = \varepsilon\mathcal{O}_{B,\mathfrak{p}}$, a free $\mathcal{O}_{\mathfrak{p}}$-module of rank 2 with a fixed isomorphism $\Lambda \cong \mathcal{O}_{\mathfrak{p}}^2$. The $\mathcal{O}_{\mathfrak{p}}$-scheme X' represents a functor $\mathfrak{X}$ from the category of pairs (S,s), where S is a connected locally Noetherian $\mathcal{O}_{\mathfrak{p}}$-scheme and s is a geometric point of S, to the category of sets. It sends (S,s) to the set of equivalence classes of quadruples $(A,\lambda,i,\eta^{\mathfrak{p}})$, where

- A/S is an abelian scheme of dimension $4d$;
- $i : \mathcal{O}_{B'} \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $\mathrm{Lie}\, A \otimes_{(\mathcal{O}_{\mathfrak{p}_0} \otimes_{\mathbb{Z}_p} \mathcal{O}_S), 1\otimes 1} \mathcal{O}_S$ is locally free of rank 2 and the two actions of $\mathcal{O}_F$ (from $\mathcal{O}_{B'}$ and $\mathcal{O}_S$) coincide;
- $\lambda : A \to A^\vee$ is a prime-to-p polarisation with $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for all $b \in \mathcal{O}_{B'}$;
- $\eta^{\mathfrak{p}} := \left(\overline{\eta}^p,\ (\eta_i)_{2\le i\le t}\right)$ is a level structure outside $\mathfrak{p}$, i.e.:
- $\overline{\eta}^p$ is a $\pi_1(S,s)$-invariant U'^p-orbit of isomorphisms $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A_s$ of $B' \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules, which take the pairing $(\ ,\ )$ to an $(\mathbb{A}^{\infty,p})^\times$-multiple of the λ-Weil pairing on $V^p A_s$, where $V^p A_s := \left(\varprojlim_{p \nmid N} A[N](k(s))\right) \otimes_{\mathbb{Z}} \mathbb{Q}$;
- $\eta_i : \left(\mathfrak{p}_i^{-m_i}\mathcal{O}_{B,\mathfrak{p}_i}/\mathcal{O}_{B,\mathfrak{p}_i}\right)_S \xrightarrow{\sim} A[\mathfrak{p}_i^{m_i}]$ is an isomorphism of S-schemes with $\mathcal{O}_{B,\mathfrak{p}_i}$-actions,

and two quadruples $(A,\lambda,i,\eta^{\mathfrak{p}}), (A',\lambda',i',\eta'^{\mathfrak{p}})$ are equivalent if there is a prime-to-p isogeny $A \to A'$ which sends $(i,\eta^{\mathfrak{p}})$ to $(i',\eta'^{\mathfrak{p}})$ and λ to $\nu\lambda'$ for some $\nu \in \mathbb{Z}_{(p)}^\times$. For two geomteric points s, s' of S the sets $\mathfrak{X}(S,s)$ and $\mathfrak{X}(S,s')$ are in canonical bijection, hence we think of $\mathfrak{X}$ as a functor from connected locally Noetherian $\mathcal{O}_{\mathfrak{p}}$-schemes, and extend it to a functor from all locally Noetherian $\mathcal{O}_{\mathfrak{p}}$-schemes by setting $\mathfrak{X}(\coprod_i S_i) = \prod_i \mathfrak{X}(S_i)$.

Given $(A,\lambda,i,\eta^{\mathfrak{p}})$ as above, let $\mathcal{G}_A = \varepsilon A[\mathfrak{p}^\infty]$, a Barsotti-Tate $\mathcal{O}_{\mathfrak{p}}$-module; it has $\mathcal{O}_{\mathfrak{p}}$-height 2, i.e. the multiplication by a uniformiser ϖ_0 of $\mathcal{O}_{\mathfrak{p}}$ has degree q^2, where q is the order of the residue field $k_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$ of $F_{\mathfrak{p}}$. Over a base in which p is nilpotent it is 1-dimensional and *compatible*, i.e. the two actions of $\mathcal{O}_{\mathfrak{p}}$ on $\mathrm{Lie}\,\mathcal{G}_A$ coincide ([HT01], II.1). Let $\mathcal{A}$ be the universal abelian scheme over X', and $\mathcal{G} = \mathcal{G}_{\mathcal{A}}$.

To relate X to X', let $F_{\mathfrak{p}}^{\mathrm{ur}}$ be the maximal unramified extension of $F_{\mathfrak{p}}$. For each connected component of $X \otimes_F F_{\mathfrak{p}}^{\mathrm{ur}}$, we can choose U' so that it is canonically isomorphic to a connected component of $X' \otimes_E F_{\mathfrak{p}}^{\mathrm{ur}}$ as $F_{\mathfrak{p}}^{\mathrm{ur}}$-schemes ([Car86], 4.5.4). By descent from $X \otimes_F F_{\mathfrak{p}}^{\mathrm{ur}}$, the above integral model of X' gives an integral model of X, a proper smooth scheme over $\mathcal{O}_{\mathfrak{p}}$ ([Car86], 6.1). Let $X_{\mathfrak{p}} = X \otimes_{\mathcal{O}_{\mathfrak{p}}} k_{\mathfrak{p}}$. At each geometric point s of $X_{\mathfrak{p}}$, the completion $\widehat{\mathcal{O}}_{X,s}$ of the strict henselisation of X at s is isomorphic to $\widehat{\mathcal{O}}_{X',s'}$ for the corresponding point $s' \in X'_{\mathfrak{p}}(\overline{k}_{\mathfrak{p}})$, hence is isomorphic to the universal deformation ring of $\mathcal{G}_{s'}$ ([Car86], 6.6). We write $\mathcal{G}_s = \mathcal{G}_{s'}$. For each n, if we choose a sufficiently small U, there exists a finite flat $\mathcal{O}_{\mathfrak{p}}$-module scheme $\mathcal{G}_n$ on X which pulls back over $\widehat{\mathcal{O}}_{X,s} \cong \widehat{\mathcal{O}}_{X',s'}$ to $\mathcal{G}[\mathfrak{p}^n]$, the $\mathfrak{p}^n$-torsion of $\mathcal{G}$ ([Car86], 6.5). Since we can vary $U^{\mathfrak{p}}$, the Hecke algebra $\mathbb{T}_U = \mathbb{Z}[U^{\mathfrak{p}} \backslash G^{\infty,\mathfrak{p}} / U^{\mathfrak{p}}]$ away from $\mathfrak{p}$ acts on X from the right by algebraic correspondences.

Now we describe the supersingular points on X. For any field κ of characteristic p, a point $s \in X_{\mathfrak{p}}(\kappa)$ is called *supersingular* if $\mathcal{G}_{\overline{s}}$ is a formal (i.e. connected) Barsotti-Tate $\mathcal{O}_{\mathfrak{p}}$-module for $\overline{s} \in X_{\mathfrak{p}}(\overline{\kappa})$ above s. Let

X_{ss} be the reduced closed subscheme of $X_{\mathfrak{p}}$, whose underlying set is (the closure of) the set of all closed points of $X_{\mathfrak{p}}$ which are supersingular. Then it is a finite étale $k_{\mathfrak{p}}$-scheme, stable under the Hecke correspondences in $\mathbb{T}_U$. Hence the q-th power Frobenius morphism $\mathrm{Fr} : X \to X$ restricts to an automorphism of X_{ss} as a $k_{\mathfrak{p}}$-scheme. In particular Fr acts on the set $X_{\mathrm{ss}}(\overline{k}_{\mathfrak{p}})$. Also, recall that $X_{\mathfrak{p}} \to \mathrm{Spec}\, k_{\mathfrak{p}}$ factors through $\mathrm{Spec}(\mathcal{O}_U/\mathfrak{p})$ with geometrically connected fibres, hence X_{ss} is a finite étale $\mathcal{O}_U/\mathfrak{p}$-scheme and we have a map

$$X_{\mathrm{ss}}(\overline{k}_{\mathfrak{p}}) \to \mathrm{Spec}(\mathcal{O}_U/\mathfrak{p})(\overline{k}_{\mathfrak{p}}) \cong F^\times \backslash \mathbb{A}_F^\times / \det(U \times U_\infty), \tag{2.1}$$

equivariant with the Fr-action and the action of the inverse of a uniformiser of $F_{\mathfrak{p}}$.

To describe the Fr-set $X_{\mathrm{ss}}(\overline{k}_{\mathfrak{p}})$, let $\overline{B}$ be a totally definite quaternion algebra over F whose non-split places are exactly the places where B is non-split together with $\mathfrak{p}$ and τ, and define an algebraic group $\overline{G}$ over $\mathbb{Q}$ by $\overline{G}(R) = (\overline{B} \otimes_{\mathbb{Q}} R)^\times$. Note $\overline{G}(\mathbb{A}^\infty) = G^{\infty,\mathfrak{p}} \times \overline{B}_{\mathfrak{p}}^\times$, where $\overline{B}_{\mathfrak{p}} = \overline{B} \otimes_F F_{\mathfrak{p}}$ is the quaternion algebra over $F_{\mathfrak{p}}$. Let $\mathcal{O}_{\overline{B},\mathfrak{p}}$ be the maximal order of $\overline{B}_{\mathfrak{p}}$, and $v : \overline{B}_{\mathfrak{p}}^\times / \mathcal{O}_{\overline{B},\mathfrak{p}}^\times \xrightarrow{\cong} \mathbb{Z}$ be the normalised valuation. Fix a supersingular point $s_0 \in X_{\mathrm{ss}}(\overline{k}_{\mathfrak{p}})$. Then there is a bijection of finite sets with right $\mathbb{T}_U$-actions ([Car86], 11.2)

$$X_{\mathrm{ss}}(\overline{k}_{\mathfrak{p}}) \xrightarrow{\cong} \overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}^\infty)/(U^{\mathfrak{p}} \times \mathcal{O}_{\overline{B},\mathfrak{p}}^\times) \xrightarrow{\cong} \overline{G}(\mathbb{Q}) \backslash \Big((G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}) \times \mathbb{Z} \Big), \tag{2.2}$$

which sends s_0 to 1. Here the second bijection is defined by sending the class of $\delta \in \overline{G}(\mathbb{A}^\infty)$ to the class of $(\delta^{\mathfrak{p}}, -v(\delta_{\mathfrak{p}}))$. The action of Fr on the left hand side coincides with the action of $\Pi^{-1} \in \overline{B}_{\mathfrak{p}}^\times$, on the middle term, where $v(\Pi) = 1$. The action on the right hand side is therefore given by $h \mapsto h+1$ on $\mathbb{Z}$. The map (2.1) is given by the reduced norm $\det = N_{\overline{B}/F} : \overline{B}^\times \to F^\times$.

The right hand side is concretely described as follows. Fix a set of representatives $\Sigma_0 \subset G^{\infty,\mathfrak{p}}$ for the double cosets in $\overline{G}(\mathbb{Q}) \backslash G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}$, with $1 \in \Sigma_0$. Then the image of $g \times \mathbb{Z}$ for $g \in \Sigma_0$ is given by $\Gamma_g \backslash \mathbb{Z}$, where the subgroup $\Gamma_g \subset \overline{G}(\mathbb{Q})$ is the intersection $\overline{G}(\mathbb{Q}) \cap gU^{\mathfrak{p}}g^{-1}$ inside $G^{\infty,\mathfrak{p}}$, considered as a discrete subgroup of $\overline{B}_{\mathfrak{p}}^\times$ and acting on $\mathbb{Z}$ via $-v$. We have $\Gamma_g = \delta_g^{\mathbb{Z}}$ for a unique $\delta_g \in \overline{B}_{\mathfrak{p}}^\times$ with $v(\delta_g) > 0$, hence

$$\overline{G}(\mathbb{Q}) \backslash \Big((G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}) \times \mathbb{Z} \Big) \xrightarrow{\cong} \coprod_{g \in \Sigma_0} \Gamma_g \backslash \mathbb{Z} \xrightarrow{\cong} \coprod_{g \in \Sigma_0} \mathbb{Z}/v(\delta_g)\mathbb{Z}, \tag{2.3}$$

and Fr acts by $h \mapsto h+1$ in each $\mathbb{Z}/v(\delta_g)\mathbb{Z}$. The order of the automorphism Fr on $X_{\mathrm{ss}}(\overline{k}_{\mathfrak{p}})$ is the least common multiple of $v(\delta_g)$ for $g \in \Sigma_0$.

We make (2.2) into an isomorphism of schemes with a descent datum to $k_{\mathfrak{p}}$. Let n be the least common multiple of two and the order of Fr on $X_{\mathrm{ss}}(\overline{k}_{\mathfrak{p}})$, and let k be a finite extension of $k_{\mathfrak{p}}$ with $[k : k_{\mathfrak{p}}] = n$ (we ensure that n is even so that for $x \in X_{\mathrm{ss}}(\overline{k}_{\mathfrak{p}})$ the endomorphisms of $\mathbb{G}_x$ descend to k). Let $X_{\mathrm{ss}/k} = X_{\mathrm{ss}} \otimes_{k_{\mathfrak{p}}} k$. We have $X_{\mathrm{ss}}(\overline{k}) = X_{\mathrm{ss}}(k) = X_{\mathrm{ss}/k}(k)$ by definition of n.

We define a k-scheme $\underline{\mathbb{Z}}_k = \coprod_{\mathbb{Z}} \mathrm{Spec}\, k$, where we write $\underline{\mathbb{Z}}_k^h$ for the copy of $\mathrm{Spec}\, k$ at $h \in \mathbb{Z}$. Let $\mathrm{Fr} : \underline{\mathbb{Z}}_k \xrightarrow{\cong} \underline{\mathbb{Z}}_k$ be an automorphism as k-schemes, defined by $\underline{\mathbb{Z}}_k^h \xrightarrow{\cong} \underline{\mathbb{Z}}_k^{h+1}$ for every $h \in \mathbb{Z}$. Let $\overline{B}_{\mathfrak{p}}^\times$ act on $\underline{\mathbb{Z}}_k$ from the left via $\delta \mapsto \mathrm{Fr}^{-v(\delta)}$. Using this left $\overline{B}_{\mathfrak{p}}^\times$-action, which clearly commutes with Fr, we replace $\mathbb{Z}$ in (2.5) by $\underline{\mathbb{Z}}_k$, and obtain an isomorphism of k-schemes with Fr-action

$$X_{\mathrm{ss}/k} \xrightarrow{\cong} \overline{G}(\mathbb{Q}) \backslash \Big((G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}) \times \underline{\mathbb{Z}}_k \Big). \tag{2.4}$$

By taking the base change to $\overline{k}_{\mathfrak{p}}$ over k, we obtain an isomorphism

$$\overline{X}_{\text{ss}} \xrightarrow{\cong} \overline{G}(\mathbb{Q})\backslash\Big(\big(G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}\big) \times \underline{\mathbb{Z}}\Big) \tag{2.5}$$

of $\overline{k}_{\mathfrak{p}}$-schemes with Fr-action, where $\overline{X}_{\text{ss}} = X_{\text{ss}} \otimes_{k_{\mathfrak{p}}} \overline{k}_{\mathfrak{p}}$ and $\underline{\mathbb{Z}} = \coprod_{\mathbb{Z}} \operatorname{Spec} \overline{k}_{\mathfrak{p}}$.

2.2. **The Lubin-Tate spaces, and the full level at p.** Let us relieve some of our notational burden by writing $K_0 = F_{\mathfrak{p}}$, $\mathcal{O} = \mathcal{O}_{\mathfrak{p}}$, and $k_0 = k_{\mathfrak{p}} \cong \mathbb{F}_q$ with $q = p^f$. Let $k \cong \mathbb{F}_{q^n}$ be as above, and we write $\overline{k}$ for $\overline{k}_{\mathfrak{p}} \cong \overline{\mathbb{F}}_p$. We reserve K for a finite extension of K_0 appearing later. Let $D = \overline{B}_{\mathfrak{p}}$ and $\mathcal{O}_D = \mathcal{O}_{\overline{B},\mathfrak{p}}$, with its maximal ideal $\mathfrak{p}_D$ and residue field $\kappa = \mathcal{O}_D/\mathfrak{p}_D \cong \mathbb{F}_{q^2}$. Let W be the completion of the ring of integers of K_0^{ur}, and $X_W = X \otimes_{\mathcal{O}} W$. A formal spectrum $\operatorname{Spf} R$ for a complete Noetherian local ring R will have its maximal ideal as the ideal of definition if not otherwise mentioned.

We desribe the p-adic uniformisation for the formal neighbourhood of $\overline{X}_{\text{ss}}$ in X_W. Recall that we fixed a supersingular point $s_0 \in X_{\text{ss}}(\overline{k})$. Let $\mathbb{G} = \mathcal{G}_{s_0}$, the Barsotti-Tate $\mathcal{O}$-module at s_0. It is a (strict) formal $\mathcal{O}$-module of $\mathcal{O}$-height 2 and dimension 1 over $\overline{k}$, which is unique up to isomorphism. We denote by LT the Lubin-Tate space for $\mathbb{G}$ (the Rapoport-Zink space [RZ96]); it is a formal scheme over $\operatorname{Spf} W$, whose reduced subscheme is $\mathrm{LT}^{\text{red}} = \underline{\mathbb{Z}} = \coprod_{\mathbb{Z}} \operatorname{Spec} \overline{k}$. On the category $\mathcal{C}$ of Artinian local W-algebras $\iota : W \to R$ with the maximal ideal $\mathfrak{m} \subset R$ such that ι induces $\overline{k} \cong R/\mathfrak{m}$, the set $\mathrm{LT}(R)$ is the set of all isomorphism classes of pairs $(\mathcal{G}, \rho)$, where $\mathcal{G}$ is a Barsotti-Tate $\mathcal{O}$-module over R and a quasi-isogeny $\rho : \mathbb{G} \otimes_{\overline{k}} R/\mathfrak{m} \longrightarrow \mathcal{G} \otimes_R R/\mathfrak{m}$. For $h \in \mathbb{Z}$, we denote the locus where the $\mathcal{O}$-height of ρ is h by LT^h, hence $\mathrm{LT} = \coprod_{h\in\mathbb{Z}} \mathrm{LT}^h$. Then $\mathrm{LT}^0 = \operatorname{Spf} A_0$, where we let A_0 be the universal deformation ring of $\mathbb{G}$, non-canonically isomorphic to $W[[T]]$.

Note that $\mathcal{O}_D$ (resp. $D^\times$) is the endomorphism ring (resp. the self-quasi-isogeny group) of $\mathbb{G}$, and $v : D^\times/\mathcal{O}_D^\times \xrightarrow{\cong} \mathbb{Z}$ is given by the $\mathcal{O}$-height. Hence $D^\times$ acts on LT from the right as $\rho \mapsto \rho \circ \delta$ for $\delta \in D^\times$, and we consider this as a left action by making δ act as δ^{-1} from the right. In particular, a uniformiser $\Pi \in D^\times$, a self-isogeny of $\mathbb{G}$ with $\mathcal{O}$-height 1 (degree q), defines an isomorphism $\Pi : \mathrm{LT}^h \xrightarrow{\cong} \mathrm{LT}^{h-1}$ for every h by $\rho \mapsto \rho \circ \Pi^{-1}$, hence an isomorphism $\mathrm{LT} \cong \coprod_{\mathbb{Z}} \operatorname{Spf} A_0$. This left $D^\times$-action extends the action on $\underline{\mathbb{Z}}$ defined previously, since $\mathcal{O}_D^\times$ acts trivially on $\underline{\mathbb{Z}}$.

For any $\mathbb{F}_p$-algebra R, let $\sigma_q : R \to R$ be the q-th power ring endomomorphism. Let $\mathbb{G}^{(q)}$ be the base change of $\mathbb{G}$ with respect to $\sigma_q^* : \operatorname{Spec} \overline{k} \to \operatorname{Spec} \overline{k}$, which is (non-canonically) isomorphic to $\mathbb{G}$. Let $\mathrm{LT}^{(q)}$ be the Lubin-Tate space defined for $\mathbb{G}^{(q)}$. Then:

(i) Let $\sigma_q : W \to W$ be the arithmetic Frobenius, the continuous $\mathcal{O}$-automorphism lifting σ_q on $\overline{k}$. Then $\big(\mathrm{LT} \otimes_{W,\sigma_q} W\big)(R) = \mathrm{LT}\big(W \xrightarrow{\sigma_q} W \xrightarrow{\iota} R\big)$ is the set of isomorphism classes of $(\mathcal{G}, \rho)$, where $\mathcal{G}/R$ is as before (since R is unchanged as an $\mathcal{O}$-algebra), and $\rho : \mathbb{G} \otimes_{\overline{k},\,\iota\circ\sigma_q} R/\mathfrak{m} = \mathbb{G}^{(q)} \otimes R/\mathfrak{m} \longrightarrow \mathcal{G} \otimes R/\mathfrak{m}$. Therefore $\mathrm{LT}^{(q)} = \mathrm{LT} \otimes_{W,\sigma_q} W$.

(ii) Letting $\mathrm{LT}_{\mathfrak{p}} = \mathrm{LT} \otimes_W \overline{k}$, the formal scheme over $\overline{k}$ obtained by reduction $\operatorname{mod} \mathfrak{p}$, its base change $\mathrm{LT}_{\mathfrak{p}}^{(q)}$ with respect to σ_q on $\overline{k}$ is just $\mathrm{LT}_{\mathfrak{p}}^{(q)} = \mathrm{LT}^{(q)} \otimes_W \overline{k}$ by (i). The q-th power Frobenius morphism $\mathrm{Fr} : \mathrm{LT}_{\mathfrak{p}} \to \mathrm{LT}_{\mathfrak{p}}^{(q)}$ is given by $(\mathcal{G}, \rho) \mapsto (\mathcal{G}^{(q)}, \rho^{(q)})$ on the R-valued points, where $(\cdot)^{(q)}$ denotes the base change with respect to σ_q on R.

(iii) The q-th power Frobenius morphism $\mathrm{Fr}_{\mathbb{G}} : \mathbb{G} \to \mathbb{G}^{(q)}$ is an isogeny of $\mathcal{O}$-height 1, hence $\rho \mapsto \rho \circ \mathrm{Fr}_{\mathbb{G}}$ defines an isomorphism $w : \mathrm{LT}^{(q)} \xrightarrow{\cong} \mathrm{LT}$ of formal schemes over $\mathrm{Spf}\, W$ (the Weil descent), with $w : \mathrm{LT}^{(q),h} \xrightarrow{\cong} \mathrm{LT}^{h+1}$ for every h. The action of $w \circ \mathrm{Fr} : \mathrm{LT}^h_{\mathfrak{p}} \longrightarrow \mathrm{LT}^{h+1}_{\mathfrak{p}}$ on $\mathrm{LT}^{\mathrm{red}} = \underline{\mathbb{Z}}$ is what we denoted by Fr at the end of §2.2.

Using the left action of $D^\times$ on LT, we replace the $\underline{\mathbb{Z}}$ in the right hand side of (2.5) by LT, and claim that the formal completion $\widehat{X}_W|_{\overline{X}_{\mathrm{ss}}}$ of X_W along its closed subscheme $\overline{X}_{\mathrm{ss}}$ is given by

$$\widehat{X}_W|_{\overline{X}_{\mathrm{ss}}} = \coprod_{s \in X_{\mathrm{ss}}(\overline{k})} \mathrm{Spf}\, \widehat{\mathcal{O}}_{X,s} \xrightarrow{\cong} \overline{G}(\mathbb{Q}) \backslash \Big((G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}) \times \mathrm{LT} \Big), \tag{2.6}$$

as formal schemes over $\mathrm{Spf}\, W$ with the $\mathbb{T}_U$-actions from the right. This is seen as follows. The right hand side is a finite disjoint union of formal schemes of the form $\Gamma_g \backslash \mathrm{LT}$, as in (2.3), and $\Gamma_g \backslash \mathrm{LT} \cong \coprod_{\mathbb{Z}/v(\delta)\mathbb{Z}} \mathrm{Spf}\, A_0$ non-canonically. Since (2.5) says that the $\overline{k}$-points of (2.6) are in $\mathbb{T}_U$-equivariant bijection, and also $\widehat{\mathcal{O}}_{X,s} \cong A_0$ for each $s \in X_{\mathrm{ss}}(\overline{k})$ (since $\mathcal{G}_s \cong \mathbb{G}$), we see that (2.6) is an isomorphism of formal schemes over $\mathrm{Spf}\, W$.

For every $s \in X_{\mathrm{ss}}(k)$, canonically $\mathcal{G}_{\mathrm{Fr}(s)} \cong \mathcal{G}_s^{(q)}$ (by the moduli interpretation of X'), hence we have $\widehat{\mathcal{O}}_{X,\mathrm{Fr}(s)} \cong \widehat{\mathcal{O}}_{X,s} \otimes_{W,\sigma_q} W$ as in (i) above.

This picture descends to $X_{\mathrm{ss}/k}$ as follows. Let K_0'/K_0 be the unramified extension with the residue field extension k/k_0 of degree n, and let $|k| = q^n = q'$. Let $\mathcal{O}'$ be its ring of integers, and $X_{\mathcal{O}'} = X \otimes_{\mathcal{O}} \mathcal{O}'$. Since $\mathrm{Fr}^n = \mathrm{id}$ on X_{ss} defines a descent data $\mathbb{G} \xrightarrow{\cong} \mathbb{G}^{(q')}$, let $\mathbb{G}_k$ be the corresponding formal Barsotti-Tate $\mathcal{O}$-module over k with $\mathbb{G} = \mathbb{G}_k \otimes_k \overline{k}$. Let LT_k be the Lubin-Tate space defined for $\mathbb{G}_k$, which is a formal scheme over $\mathrm{Spf}\, \mathcal{O}'$, non-canonically isomorphic to $\coprod_{\mathbb{Z}} \mathrm{Spf}\, \mathcal{O}'[[T]]$, with $D^\times$-action and the Weil descent (note that k contains the quadratic extension of k_0, hence all endomorphisms in $\mathrm{End}(\mathbb{G}) = \mathcal{O}_D$ are defined on $\mathbb{G}_k$). Then $\mathrm{LT}_k \otimes_{\mathcal{O}'} W \cong \mathrm{LT}$ with the $D^\times$-action and Weil descent. Then the formal completion of $X_{\mathcal{O}'}$ along its closed subscheme $\overline{X}_{\mathrm{ss}/k}$ is given by

$$\widehat{X}_{\mathcal{O}'}|_{\overline{X}_{\mathrm{ss}/k}} = \coprod_{s \in X_{\mathrm{ss}}(k)} \mathrm{Spf}\, \widehat{\mathcal{O}}_{X,s} = \overline{G}(\mathbb{Q}) \backslash \Big((G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}) \times \mathrm{LT}_k \Big), \tag{2.7}$$

as formal schemes over $\mathrm{Spf}\, \mathcal{O}'$ with the $\mathbb{T}_U$-actions from the right, where $\widehat{\mathcal{O}}_{X,s}$ is the complete local ring of $X_{\mathcal{O}'}$ at s.

Next we choose an open compact subgroup of $G(\mathbb{A}^\infty)$ of the form $U = U^{\mathfrak{p}} \times U_1$, with $U_1 = \mathrm{Ker}\big(\mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(k_0)\big)$, and let $X(\mathfrak{p})$ be the corresponding canonical model over F, which is an étale $\mathrm{GL}_2(k_0)$-covering over X. Here we recall an integral model of $X(\mathfrak{p})$ over $\mathcal{O}$, which is a regular scheme, proper flat over $\mathcal{O}$ ([KM85], [Car86]). Choosing $U^{\mathfrak{p}}$ small enough so that we have the finite flat $\mathcal{O}$-module scheme $\mathcal{G}_1$ over X, the integral model for $X(\mathfrak{p})$ over $\mathcal{O}$ is defined as the relative moduli over X of the Drinfeld level $\mathfrak{p}$-structures on $\mathcal{G}_1$ ([KM85] 1.8, [HT01] II.2):

$$\eta : \mathfrak{p}^{-1}\Lambda/\Lambda \longrightarrow \mathcal{G}_1,$$

where $\Lambda = \mathcal{O}^2$ (for Shimura curves $\Lambda = \varepsilon\mathcal{O}_{B,\mathfrak{p}} \cong \mathcal{O}^2$, and the corresponding model for X' is as appears in [HT01], III.4). For each geometric point $s \in X(\mathfrak{p})(\overline{k})$, the completed strict henselisation $\widehat{\mathcal{O}}_{X(\mathfrak{p}),s}$ is isomorphic to the universal deformation ring of $\mathcal{G}_s$ with Drinfeld level $\mathfrak{p}$-structures ([Car86], 7.4), hence

$X(\mathfrak{p})$ is regular and finite over X ([Dri74], §4). As $X(\mathfrak{p}) \to X$ is a finite morphism between regular schemes of the same dimension, it is flat ([AK70] V, 3.6).

Now we look at the formal neighbourhood of $X_{\rm ss}$ for $X(\mathfrak{p})$. For any field κ' of characteristic p and $t \in X_{\rm ss}(\kappa')$, there is a unique Drinfeld level $\mathfrak{p}$-structure of $(\mathcal{G}_1)_t$, hence a unique point $s \in X(\mathfrak{p})(\kappa')$ above t. Thus $X_{\rm ss}$ can be considered as a reduced subscheme of $X(\mathfrak{p})$. We denote by $\mathrm{LT}(\mathfrak{p})$ the first covering of the Lubin-Tate space for $\mathbb{G}$, non-canonically isomorphic to $\coprod_{\mathbb{Z}} \mathrm{Spf}\, A$, where A is the universal deformation ring of $\mathbb{G}$ with Drinfeld level $\mathfrak{p}$-structures. Then $\mathrm{LT}(\mathfrak{p})$ has a $\mathrm{GL}_2(k_0)$-action from the right, and $\mathrm{LT}(\mathfrak{p}) \to \mathrm{LT}$ is a $D^\times$-equivariant finite flat covering, which is an étale $\mathrm{GL}_2(k_0)$-torsor on the associated rigid spaces. Since $\widehat{\mathcal{O}}_{X(\mathfrak{p}),s} \cong A$, the formal completion of $X(\mathfrak{p})_W = X(\mathfrak{p}) \otimes_{\mathcal{O}} W$ along its closed subscheme $\overline{X}_{\rm ss}$ is given by

$$\widehat{X(\mathfrak{p})}_W|_{\overline{X}_{\rm ss}} = \coprod_{s \in X_{\rm ss}(\overline{k})} \mathrm{Spf}\, \widehat{\mathcal{O}}_{X(\mathfrak{p}),s} = \overline{G}(\mathbb{Q})\backslash\Big((G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}) \times \mathrm{LT}(\mathfrak{p}) \Big), \tag{2.8}$$

as formal schemes over $\mathrm{Spf}\, W$ with actions of $\mathbb{T}_U$ and $\mathrm{GL}_2(k_0)$ from the right. This is a finite flat covering of (2.6). It also descends to $X_{{\rm ss}/k}$ as

$$\widehat{X(\mathfrak{p})}_{\mathcal{O}'}|_{X_{{\rm ss}/k}} = \coprod_{s \in X_{\rm ss}(k)} \mathrm{Spf}\, \widehat{\mathcal{O}}_{X(\mathfrak{p}),s} = \overline{G}(\mathbb{Q})\backslash\Big((G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}) \times \mathrm{LT}(\mathfrak{p})_k \Big), \tag{2.9}$$

as formal schemes over $\mathrm{Spf}\, \mathcal{O}'$ with actions of $\mathbb{T}_U$ and $\mathrm{GL}_2(k_0)$ from the right, where $\widehat{\mathcal{O}}_{X(\mathfrak{p}),s}$ is the complete local ring of $X(\mathfrak{p})_{\mathcal{O}'}$ at s and $\mathrm{LT}(\mathfrak{p})_k$ is the first covering of LT_k. This will be our starting point in constructing the Hecke-equivariant semistable models of $X(\mathfrak{p})$ in the next subsection.

Here we collect some facts on LT and $\mathrm{LT}(\mathfrak{p})$ we will need, or rather its connected components $\mathrm{Spf}\, A_0$ and $\mathrm{Spf}\, A$, all essentially due to Drinfeld [Dri74].

Let ϖ_0 be a uniformiser of K_0. Recall that a (formal) coordinate for a 1-dimensional formal $\mathcal{O}$-module $\mathcal{G}$ over an $\mathcal{O}$-algebra R is a choice of an isomorphism $\mathrm{Spf}\, R[[X]] \xrightarrow{\cong} \mathcal{G}$, with the ideal of definition (X), as formal schemes over R. It gives the formal $\mathcal{O}$-module law $X +_{\mathcal{G}} Y \in R[[X,Y]]$ and $[\cdot] = [\cdot]_{\mathcal{G}} : \mathcal{O} \longrightarrow R[[X]]$. The formal additive group law $\widehat{\mathbb{G}}_a$ is given by $X + Y$ and $[a](X) = aX$ for all $a \in \mathcal{O}$, and every $\mathcal{G}$ is congruent to $\widehat{\mathbb{G}}_a \mod \deg 2$ by definition. A formal $\mathcal{O}$-module over a k_0-algebra R has *$\mathcal{O}$-height* $h \in \mathbb{Z}_{>0}$ if $[\varpi_0](X)$ is a power series in X^{q^h} but not in $X^{q^{h+1}}$. The $\mathcal{O}$-height is independent of the choice of coordinates, but a consequence of Lazard's lemma is that if $\mathcal{G}$ has $\mathcal{O}$-height h, then we can choose a coordinate so that $\mathcal{G} \equiv \widehat{\mathbb{G}}_a \mod \deg q^h$ ([Dri74], 1.5).

For $\mathbb{G}/\overline{k}$ of $\mathcal{O}$-height 2 and a uniformiser ϖ_0 of K_0, we can choose a coordinate so that (1) $[\varpi_0](X) = X^{q^2}$, (2) $\mathbb{G} \equiv \widehat{\mathbb{G}}_a \mod \deg q^2$ and (3) the formal $\mathcal{O}$-module law is defined over a quadratic extension of k_0 ([Dri74], Proof of 1.7). Then an element $\delta \bmod \mathfrak{p} \in \mathcal{O}_D/\mathfrak{p}$ is determined by $[\delta](X) \equiv a_\delta X + b_\delta X^q \pmod{\deg q^2}$ with $a_\delta, b_\delta \in \mathbb{F}_{q^2} \subset \overline{k}$ ([Dri74], Proof of 1.7(c)). Since a uniformiser Π of $\mathcal{O}_D = \mathrm{End}(\mathbb{G})$ has $\mathcal{O}$-height 1, an element $\delta \bmod \Pi \in \mathcal{O}_D/\mathfrak{p}_D$ is determined by the leading coefficient $a_\delta \in \mathbb{F}_{q^2}$ of $[\delta](X)$. Therefore $\mathbb{G}/\overline{k}$ gives a k_0-homomorphism $\kappa = \mathcal{O}_D/\mathfrak{p}_D \to \overline{k}$ by $\delta \mapsto a_\delta/a_1$. Also for $\mathbb{G}_k/k$, since all endomorphisms are defined over k, looking at the leading coefficients gives $\kappa \to k$.

The deformation theory of $\mathbb{G}_k/k$ or $\mathbb{G}/\overline{k}$, height 2 and dimension 1 ([Dri74], 4.2), shows that the universal deformation $\mathcal{G}$ on $A_0 = \mathcal{O}'[[T]]$ or $W[[T]]$ has height 1 on $A_0/\mathfrak{p}$ and height 2 on $A_0/(\mathfrak{p}, T)$. More precisely, by changing T by its unit multiple if needed, its formal $\mathcal{O}$-module law satisfies

$$[\varpi_0](X) \equiv \varpi_0 X + TX^q + uX^{q^2} \pmod{\deg q^2+1}, \tag{2.10}$$

with $u \in A_0^\times$, and since A_0 is Henselian and $u \bmod (\mathfrak{p}, T) \in k$ or $\overline{k}$ has a (q^2-1)-th root so does u, hence we can make $u = 1$ by replacing X by $X' = \sqrt[q^2-1]{u}X$. On the universal deformation ring A with Drinfeld level $\mathfrak{p}$-structures, we have $\mathcal{G} \otimes_{A_0} A$ with the universal object $\eta : \mathfrak{p}^{-1}\Lambda/\Lambda \longrightarrow \mathcal{G}[\mathfrak{p}]$, a morphism of finite flat $\mathcal{O}$-module schemes. Once we fix a coordinate of $\mathcal{G}$ and take the A-valued points, it gives a map $\eta : \mathfrak{p}^{-1}\Lambda/\Lambda \cong k_0^2 \longrightarrow \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of A. Then A is regular with $\mathfrak{m} = (X_1, X_2)$, where $X_1 = \eta(1,0)$ and $X_2 = \eta(0,1)$ ([Dri74], 4.3). Only the case of $\mathbb{G}/\overline{k}$ is treated in [Dri74], but the proofs are valid for $\mathbb{G}_k/k$.

2.3. **The Drinfeld curve and a semistable model.** The goal of this subsection is to describe a semistable model of $X(\mathfrak{p})$, obtained by blowing up in $X_{\mathrm{ss}/k}$. First we define the Drinfeld curve, which will appear in the exceptional divisors. Let Dr^0 be the affine curve over $k_0 \cong \mathbb{F}_q$ defined as

$$\mathrm{Dr}^0 = \operatorname{Spec} k_0[X_1, X_2]/\big((X_1X_2^q - X_1^qX_2)^{q-1} - 1\big) \tag{2.11}$$

$$= \operatorname{Spec} k_0[X_1, X_2]/\Big(\prod_{(a_1,a_2)\in k_0^2\setminus\{(0,0)\}} (a_1X_1 + a_2X_2) - 1\Big), \tag{2.12}$$

a closed subscheme of $\mathbb{A}^2_{k_0}$ (in fact defined over $\mathbb{F}_p$), with an action of $\mathrm{GL}_2(k_0)$ from the right, by its linear action on (X_1, X_2). It has connected components labelled by $\boldsymbol{\mu}_{q-1}(k_0) = k_0^\times$, and each component is smoothly compactified by taking its closure in $\mathbb{P}^2_{k_0}$. We denote the resulting smooth compactification of Dr^0 by $\overline{\mathrm{Dr}}^0$.

Let k be as in the end of §2.1, and $\Gamma = \boldsymbol{\mu}_{q^2-1}(k) \cong \mathbb{F}_{q^2}^\times$. The curve $\mathrm{Dr}_k^0 = \mathrm{Dr}^0 \otimes_{k_0} k$ over k has an action of Γ from the right, where $\zeta \in \Gamma$ acts as $(X_1, X_2) \mapsto (\zeta^{-1}X_1, \zeta^{-1}X_2)$. This action commutes with the right $\mathrm{GL}_2(k_0)$-action.

Now recall $\kappa = \mathcal{O}_D/\mathfrak{p}_D \cong \mathbb{F}_{q^2}$, and let $\iota^\pm : \kappa \to k$ be the two k_0-homomorphisms. We have two left actions of $\kappa^\times$ on Dr_k^0, where $u \in \kappa^\times$ acts via $\iota^\pm(u^{-1}) \in \Gamma$. We denote by $\mathrm{Dr}_k^\pm$ this curve over k with the corresponding left $\kappa^\times$-action and the right $\mathrm{GL}_2(k_0) \times \Gamma$-action defined above, which commute with each other. Now we define a k-scheme Dr_k

$$\mathrm{Dr}_k = \coprod_{h\in\mathbb{Z}} \mathrm{Dr}_k^h, \quad \text{where} \quad \mathrm{Dr}_k^h = \begin{cases} \mathrm{Dr}_k^+ & (h : \text{even}), \\ \mathrm{Dr}_k^- & (h : \text{odd}), \end{cases}$$

with the right $\mathrm{GL}_2(k_0) \times \Gamma$-action on each Dr_k^h.

We extend the left action of $\kappa^\times$ to an action of $D^\times$ as follows. We define $\Pi : \mathrm{Dr}_k^h \xrightarrow{\cong} \mathrm{Dr}_k^{h-1}$ to be the identity morphisms, decreasing the index h by 1. Then since

$$1 \longrightarrow \kappa^\times = (\mathcal{O}_D/\mathfrak{p}_D)^\times \longrightarrow D^\times/1 + \mathfrak{p}_D \xrightarrow{v} \mathbb{Z} \longrightarrow 1$$

is exact and $D^\times/1 + \mathfrak{p}_D \cong \kappa^\times \rtimes \Pi^{\mathbb{Z}}$ with $\Pi u \Pi^{-1} = u^q$ for $u \in \kappa^\times$, the actions of $\kappa^\times$ and Π combine to give a left action of $D^\times/1 + \mathfrak{p}_D$ on Dr_k. The right $\mathrm{GL}_2(k_0) \times \Gamma$-action on Dr_k commutes with this left $D^\times$-action, since it commutes with the actions of Π and $\kappa^\times$.

Now let $\mathrm{Dr}_k^{(q)}$ be the base change of Dr_k with respect to σ_q on k. We have $\mathrm{Dr}_k \cong \mathrm{Dr}_k^{(q)}$ as k-schemes, and the right Γ-action is twisted by $\zeta \mapsto \zeta^q$. Since the $\kappa^\times$-actions on Dr_k^+ and Dr_k^- are exchanged, we have $\mathrm{Dr}_k^{(q),h} \xrightarrow{\cong} \mathrm{Dr}_k^{h+1}$ equivariant with the left $D^\times$-action and the right $\mathrm{GL}_2(k_0)$-action. If we forget the Γ-action, we can consider the q-th power Frobenius morphism $\mathrm{Fr} : \mathrm{Dr}_k \to \mathrm{Dr}_k^{(q)}$ as $\mathrm{Fr} : \mathrm{Dr}_k^h \to \mathrm{Dr}_k^{h+1}$.

Using the above $D^\times$-action on Dr_k, we replace $\mathbb{Z}$ in (2.5) by Dr_k to obtain a k-scheme

$$\coprod_{s \in X_{\mathrm{ss}}(k)} \mathrm{Dr}_k^0 \;=\; \overline{G}(\mathbb{Q}) \backslash \Big(\big(G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}} \big) \times \mathrm{Dr}_k \Big), \tag{2.13}$$

with the actions of $\mathbb{T}_U$ and $\mathrm{GL}_2(k_0) \times \Gamma$ from the right.

In section 2.4 we show that this scheme is a quotient by the $\kappa^\times$-action on

$$\Big(\coprod_{s \in X_{\mathrm{ss}}(k)} \kappa^\times \Big) \times \mathrm{Dr}_k^0 \;=\; \Big(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}^\infty) / \big(U^{\mathfrak{p}} \times (1 + \mathfrak{p}_D) \big) \Big) \times \mathrm{Dr}_k^0, \tag{2.14}$$

acting diagonally (from the right on the first and from the left on the second factor).

Recall that K_0'/K_0 is the unramified extension with ring of integers $\mathcal{O}'$ and residue field k. Let ϖ_0 be a suitable uniformiser of K_0. Let $K = K_0'(\varpi)$ with $\varpi = \sqrt[q^2-1]{-\varpi_0}$, a totally tamely ramified abelian extension of K_0', with the ring of integers $\mathcal{O}_K$ and the residue field k. Then ϖ is a uniformiser of K, and we identify its Galois group $\mathrm{Gal}(K/K_0')$ with Γ by the canonical isomorphism

$$\mathrm{Gal}(K/K_0') \ni \sigma \xrightarrow{\cong} \frac{\sigma(\varpi)}{\varpi} \bmod \varpi \in \boldsymbol{\mu}_{q^2-1}(k) = \Gamma \subset k^\times.$$

Now we can state our main geometric input.

Theorem 2.1. *Let F be a totally real field and $\mathfrak{p}$ its finite place. Let $X(\mathfrak{p})$ be the regular integral model over $\mathcal{O} = \mathcal{O}_{\mathfrak{p}}$ of the modular or Shimura curve with full level $\mathfrak{p}$ and sufficiently small level $U^{\mathfrak{p}}$ outside $\mathfrak{p}$. Let $X(\mathfrak{p})_{\mathcal{O}'} = X(\mathfrak{p}) \otimes_{\mathcal{O}} \mathcal{O}'$, where $\mathcal{O}'/\mathcal{O}$ be the unramified extension with the residue field k, the extension of $k_0 = k_{\mathfrak{p}}$ given in the end of §2.1.*

Let $\tilde{X}$ be an $\mathcal{O}_K$-scheme obtained from $X(\mathfrak{p})_{\mathcal{O}'}$ by (1) blowing up all $s \in X_{\mathrm{ss}}(k)$, (2) taking the base change from $\mathcal{O}'$ to $\mathcal{O}_K$, and (3) normalising. Then:

(i) *The curve $\tilde{X}$ has semistable reduction, and retains the actions of $\mathbb{T}_U$ and $\mathrm{GL}_2(k_0)$ from the right. It is a disjoint union of connected components X_ξ labelled by $\xi \in k_0^\times$, and the special fibre of each X_ξ is a grid made of $q+1$ Igusa components (horizontal) and an irreducible component of Dr_k^0 above each $s \in X_{\mathrm{ss}}(k)$ (vertical).*

(ii) *Let Y_k be the complement of the Igusa components in $\tilde{X}_k := \tilde{X} \otimes_{\mathcal{O}_K} k$ and let Z_k be the complement of the Drinfeld components. Then the normalisation of $\tilde{X}_k$ is a disjoint union $\overline{Y}_k \coprod \overline{Z}_k$, where $\overline{Y}_k$ and $\overline{Z}_k$ are the smooth compactifications of the (disjoint union of) smooth affine curves Y_k and Z_k respectively.*

Moreover, there is an isomorphism

$$Y_k \xrightarrow{\cong} \coprod_{s \in X_{\mathrm{ss}}(k)} \mathrm{Dr}_k^0 \;=\; \overline{G}(\mathbb{Q}) \backslash \Big(\big(G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}} \big) \times \mathrm{Dr}_k \Big) \quad \textit{as } k\textit{-schemes}, \tag{2.15}$$

equivariant for the actions of $\mathbb{T}_U$ and $\mathrm{GL}_2(k_0)$ from the right. Moreover in (2.15), the inertial action of Γ on Y_k, i.e. the action induced by the $\Gamma \subset \mathrm{Gal}(K/K_0')$-action on the base of $\tilde{X}/\mathcal{O}_K$, is induced by the right Γ-action on Dr_k.

Proof. (i): Since the procedure (1),(2),(3) is invariant under the actions of $\mathbb{T}_U$ and $\mathrm{GL}_2(k_0)$, the morphism $\tilde{X} \to X$ is equivariant under their actions. For the semistability and the configuration of components, the question is étale local on $X(\mathfrak{p})$, and since the local picture around s depends only on $\mathcal{G}_s$, everything is seen near X_{ss} (Deligne's homogeneity principle, [KM85] 5.2). Hence we can perform (1),(2),(3) on (2.9) or even (2.8), which is written out in detail in [Yos10], apart from proving that we have normal crossings at the crossing points.

Here we sketch the argument in [Yos10]. Recall the remarks from the end of the last subsection. The condition that the universal object $\eta : k_0^2 \to \mathfrak{m} \subset A$ on $\mathcal{G}[\mathfrak{p}]/A$ is a Drinfeld level structure says

$$([\varpi_0](X)) = \Big(\prod_{\underline{a}\in k_0^2}(X - \eta(\underline{a}))\Big) = \Bigg(\prod_{(a_1,a_2)\in k_0^2}\Big(X - ([a_1](X_1) +_{\mathcal{G}} [a_2][X_2])\Big)\Bigg) \tag{2.16}$$

as principal ideals in $A[[X]]$. Since $[\varpi_0](X) \equiv X^{q^2} \pmod{\mathfrak{m}}$, the quotient $U(X) = [\varpi_0](X)/\Big(\prod(X - \eta(\underline{a}))\Big)$ lies in $1 + \mathfrak{m}\cdot A[[X]]$. Hence comparing the leading terms gives

$$(-1)^{q^2}\varpi_0 \;=\; u \cdot \prod_{(a_1,a_2)\in k_0^2\setminus\{(0,0)\}}\Big([a_1](X_1) +_{\mathcal{G}} [a_2][X_2]\Big), \tag{2.17}$$

where u is the constant term of $U(X)$, hence in $1+\mathfrak{m}$. Blowing up and normalisation over $\mathcal{O}_K$ amounts to the change of variables

$$X_1 = \varpi X_1', \;\; X_2 = \varpi X_2' \tag{2.18}$$

and dividing out by $-\varpi_0 = \varpi^{q^2-1}$ on the exceptional divisor away from the crossings, i.e. after inverting the Igusa components ([Yos10], §5.1). As $[a_1](X_1) +_{\mathcal{G}} [a_2][X_2] \equiv a_1X_1 + a_2X_2 \pmod{\deg 2}$, by reducing $\bmod\ \varpi$ we get the equation (2.12) of the Drinfeld curve.

Proving that the complete local rings at the crossing points are of the form $\mathcal{O}_K[[V_1, V_2]]/(\varpi - V_1V_2)$ is done as in [Edi01]. Indeed, after blowing up, the exceptional divisor has multiplicity q^2-1, whilst the proper transforms of the Igusa components have multiplicity $q-1$. The normalisation of $\mathrm{Spec}(\mathcal{O}_K[[U,V]]/(U^{q^2-1}V^{q-1} - \varpi^{q^2-1}))$ is a disjoint union of copies of $\mathrm{Spec}(\mathcal{O}_K[[V_1,V_2]]/(\varpi - V_1V_2))$, indexed by the $q-1$ roots of unity ξ, as can be seen by writing $U^{q^2-1}V^{q-1} - \varpi^{q^2-1} = \prod_{\xi\in k_0^\times}(U^{q+1}V - \xi\varpi^{q+1})$ and then computing the normalisation of $\mathrm{Spec}(\mathcal{O}_K[[U,V]]/(U^{q+1}V - \xi\varpi^{q+1}))$ as in [Edi90, 2.2.2.4].

(ii): Now (2.15) is obtained by performing (1),(2),(3) on (2.9) as seen in (i), and the action of $\zeta \in \Gamma$ on Y_k induced from its action on the base of $\tilde{X}/\mathcal{O}_K$ is given by $\varpi \mapsto \zeta\varpi$, which has the same effect as $(X_1', X_2') \mapsto (\zeta^{-1}X_1', \zeta^{-1}X_2')$ on Dr_k, in view of (2.18). □

2.4. **Compactification of Dr_k and a $\kappa^\times$ torsor.** Recall that we denote by $\overline{\mathrm{Dr}}^0$ the smooth proper curve over k_0 obtained from Dr^0 by taking the disjoint union of the closures of each of its connected components in $\mathbb{P}^2_{k_0}$. The connected components of Dr^0 are labelled by $\mu_{q-1}(k_0)$, and the component labelled by $\xi \in$

$\mu_{q-1}(k_0)$ is given by

$$\mathrm{Spec}\, k_0[X_1, X_2]/(X_1X_2^q - X_1^qX_2 - \xi).$$

The closure of this connected component in $\mathbb{P}^2_{k_0}$ is given by

$$\mathrm{Proj}\, k_0[X, Y, Z]/(XY^q - X^qY - \xi Z^{q+1}).$$

The right action of G extends naturally to $\overline{\mathrm{Dr}}^0$, and the right action of Γ on Dr^0_k extends to $\overline{\mathrm{Dr}}^0_k$. We will explicitly describe these actions on the boundary divisor $\Delta := \overline{\mathrm{Dr}}^0_k \backslash \mathrm{Dr}^0_k$. We see that Δ consists of $q + 1$ points $[x : y : 0]$ in each component, with $[x : y] \in \mathbb{P}^1(k_0)$.

The right action of $g \in G$ on Δ (which is a map of k-schemes) is given by the usual right action of G on $\mathbb{P}^1(k_0)$, together with changing the connected component from ξ to $\det(g)\xi$. The action of $\zeta \in \Gamma = \mu_{q^2-1}(k)$ is given by the identity, together with changing the connected component from ξ to $\zeta^{-(1+q)}\xi$.

We now have the obvious smooth compactification of Dr_k:

$$\overline{\mathrm{Dr}}_k = \coprod_{h\in\mathbb{Z}} \overline{\mathrm{Dr}}^h_k$$

where each $\overline{\mathrm{Dr}}^h_k$ is equal to $\overline{\mathrm{Dr}}^0_k$ as a k-scheme but has the left $\kappa^\times$ action extending that on Dr^h_k. The k-scheme $\overline{\mathrm{Dr}}_k$ has a right action of $G \times \Gamma$ and a left action of $D^\times$, extending the actions on Dr_k.

When $U^{\mathfrak{p}}$ is sufficiently small, the action of $\overline{G}(\mathbb{Q})$ on $\left(G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}\right) \times \overline{\mathrm{Dr}}_k$ remains free, so the quotient

$$\overline{G}(\mathbb{Q})\backslash\Big(\left(G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}\right) \times \overline{\mathrm{Dr}}_k\Big)$$

is a smooth proper k-scheme. Therefore, the isomorphism of (2.15) extends (uniquely) to an isomorphism

$$\overline{Y}_k \xrightarrow{\cong} \overline{G}(\mathbb{Q})\backslash\Big(\left(G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}\right) \times \overline{\mathrm{Dr}}_k\Big).$$

The actions of Γ, G and $\mathbb{T}_U$ on Y_k all extend uniquely to $\overline{Y}_k$, and so coincide with the actions on the right hand side of this isomorphism induced by the actions on $\overline{\mathrm{Dr}}_k$.

We now define étale $\kappa^\times$-torsors over the scheme Dr_k and its compactification $\overline{\mathrm{Dr}}_k$. This will allow us to define $\kappa^\times$-torsors over Y_k and $\overline{Y}_k$. We first define a k-scheme

$$\mathrm{Dr}_k(1) = \coprod_{d\in D^\times/1+\mathfrak{p}_D} \mathrm{Dr}^d_k$$

where Dr^d_k is equal to the k-scheme Dr^0_k. We give $\mathrm{Dr}_k(1)$ the right $\Gamma \times G$-action induced by the right action of this group on Dr^0_k. We also consider a right $\kappa^\times$-action on $\mathrm{Dr}_k(1)$: $u \in \kappa^\times$ acts via the map

$$\iota^+(u) : \mathrm{Dr}^d_k \to \mathrm{Dr}^{du}_k$$

given by the right Γ-action on Dr^0_k. Finally, we have a left action of $D^\times$ on $\mathrm{Dr}_k(1)$ given by left multiplication on $D^\times/1 + \mathfrak{p}_D$. Now we define a map $\alpha : \mathrm{Dr}_k(1) \to \mathrm{Dr}_k$: on Dr^d_k, α is given by composing the identity map $\mathrm{Dr}^d_k \to \mathrm{Dr}^0_k$ with the inclusion $\mathrm{Dr}^0_k \hookrightarrow \mathrm{Dr}_k$ and then applying the left action of d on Dr_k as defined in 2.3. It is clear that α is equivariant with respect to the $D^\times$ action on its source and target. We can extend everything to smooth compactifications and obtain a map $\alpha : \overline{\mathrm{Dr}}_k(1) \to \overline{\mathrm{Dr}}_k$.

Lemma 2.2. *The map $\alpha : \overline{\mathrm{Dr}}_k(1) \to \overline{\mathrm{Dr}}_k$ defines an étale $\kappa^\times$-torsor.*

Proof. The map α is flat and of finite type, so by [Gro71, Exp. I, Proposition 5.7] and fpqc descent [Gro71, Exp. VIII Corollaire 5.4], it suffices to check that the fibre of α over each geometric point of $\overline{\mathrm{Dr}}_k$ is an étale $\kappa^\times$-torsor. First we show that α restricts to an étale $\kappa^\times$-torsor over the affine curve Dr_k. Observe that α restricts to a map $\coprod_{d\in\mathcal{O}_D^\times/1+\mathfrak{p}_D}\mathrm{Dr}_k^d \to \mathrm{Dr}_k^0$. Since α is $D^\times$ equivariant (and in particular equivariant with respect to the action of $\Pi^{\mathbb{Z}}$), it suffices to show that this restriction is a $\kappa^\times$ torsor. We can identify $\mathcal{O}_D^\times/1+\mathfrak{p}_D$ with $\kappa^\times$. Recall the well-known fact that the fppf quotient of Dr_k^0 by its right Γ-action is representable by a scheme P. In fact

$$P \cong \mathbb{P}^1_k\backslash\mathbb{P}^1(k_0).$$

In particular $\mathrm{Dr}_k^0 \to P$ is an étale Γ-torsor. If we let $u\in\kappa^\times$ act on Dr_k^0 by $\iota^+(u)$ we obtain an étale $\kappa^\times$-torsor. Consider the diagram

$$\begin{array}{ccc} \coprod_{d\in\mathcal{O}_D^\times/1+\mathfrak{p}_D}\mathrm{Dr}_k^d & \xrightarrow{pr} & \mathrm{Dr}_k^0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow \\ \mathrm{Dr}_k^0 & \longrightarrow & P \end{array}$$

where the top row is just the identity map $\mathrm{Dr}_k^d \to \mathrm{Dr}_k^0$ on each component (note that this map is $\kappa^\times$-equivariant). The diagram is Cartesian, by the definition of a torsor applied to $\mathrm{Dr}_k^0 \to P$, so the left hand vertical map is also an étale $\kappa^\times$-torsor. It remains to show that α gives a torsor over geometric points of the boundary $\overline{\mathrm{Dr}}_k\backslash\mathrm{Dr}_k$. This is easy to see from the explicit description of the boundary Δ of Dr_k^0 and its Γ-action given at the beginning of the section. ☐

Definition 2.3. Denote by $Y_k(1)$ the smooth affine scheme

$$\overline{G}(\mathbb{Q})\backslash\Big(\big(G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}\big)\times\mathrm{Dr}_k(1)\Big).$$

Similarly, denote by $\overline{Y}_k(1)$ the smooth proper scheme

$$\overline{G}(\mathbb{Q})\backslash\Big(\big(G^{\infty,\mathfrak{p}}/U^{\mathfrak{p}}\big)\times\overline{\mathrm{Dr}}_k(1)\Big).$$

Both these schemes have commuting right actions of $\kappa^\times$, G, Γ and $\mathbb{T}_U$. Denote by $\alpha:\overline{Y}_k(1)\to\overline{Y}_k$ the map induced by $\alpha:\overline{\mathrm{Dr}}_k(1)\to\overline{\mathrm{Dr}}_k$. It follows from lemma 2.2 that $\alpha:\overline{Y}_k(1)\to\overline{Y}_k$ is an étale $\kappa^\times$-torsor.

We have an isomorphism

$$\overline{Y}_k(1)\cong\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}^\infty)/U^{\mathfrak{p}}(1+\mathfrak{p}_D)\times\overline{\mathrm{Dr}}_k^0 \tag{2.19}$$

given by mapping $(g^{\mathfrak{p}},x)$ with $x\in\overline{\mathrm{Dr}}_k^d$ to $(g^{\mathfrak{p}}d,x)$. This isomorphism respects the G, Γ and $\mathbb{T}_U$ actions on each side – the $G\times\Gamma$ action on the right hand side comes from the action on $\overline{\mathrm{Dr}}_k^0$, the $\mathbb{T}_U$ action comes from its action on

$$\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}^\infty)/U^{\mathfrak{p}}(1+\mathfrak{p}_D).$$

The action of $u\in\kappa^\times$ on $\overline{Y}_k(1)$ translates to the action $(g,x)\mapsto(gu,x\iota^+(u))$ on the right hand side, where when we write gu we are thinking of u as an element of $\mathcal{O}_D^\times/1+\mathfrak{p}_D$.

In section 4.2, we will compute the de Rham cohomology of $\overline{Y}_k$ by relating it to the de Rham cohomology of $\overline{Y}_k(1)$ (which, thanks to (2.19), has a simple description in terms of the cohomology of $\overline{\mathrm{Dr}}_k^0$).

3. Representation theoretic preliminaries

We write $\mathbf{G}$ for the algebraic group $\mathrm{GL}_2/\mathbb{F}_p$, $\mathbf{B}$ for the upper triangular Borel and $\overline{\mathbf{B}}$ for the lower triangular Borel. Similarly, we write $\mathbf{N}, \overline{\mathbf{N}}$ for the upper and lower triangular matrices with diagonal entries 1. Write $\mathbf{T}$ for the diagonal matrices and $\mathbf{Z}$ for the centre of $\mathbf{G}$. We denote the k_0 points of each of these algebraic groups by the appropriate Roman letter. We abusively confuse a rational representation of $\mathbf{G}$ with its '$\overline{\mathbb{F}}_p$-points', i.e. the attached $\overline{\mathbb{F}}_p$-vector space with an action of $\mathbf{G}(\overline{\mathbb{F}}_p)$.

Definition 3.1. A *weight* is a character $\lambda : \mathbf{T} \to \mathbb{G}_m$. If $(a, b) \in \mathbb{Z}^2$, we write $\lambda_{a,b}$ for the weight

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mapsto \alpha^a \beta^b.$$

A weight $\lambda_{a,b}$ is *dominant* (with respect to $\overline{\mathbf{B}}$) if $b - a \geq 0$. Write w for the non-trivial element of the Weyl group of $\mathbf{G}$. The Weyl group acts on the set of weights, and in particular we have $w \cdot \lambda_{a,b} = \lambda_{b,a}$. We denote by V_0 the rational representation of $\mathbf{G}$ given by the *dual* of the standard representation. Explicitly, V_0 has a basis x, y (dual to the standard basis of the standard representation) with the action of $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ given by

$$g \cdot x = (\det g^{-1})(\delta x - \beta y)$$
$$g \cdot y = (\det g^{-1})(-\gamma x + \alpha y).$$

If $\lambda = \lambda_{a,b}$ is a dominant weight, we write $\boldsymbol{\nabla}(\lambda)$ for the rational representation of $\mathbf{G}$ given by $\det^b \otimes \mathrm{Symm}^{b-a} V_0$. We denote by $\nabla(\lambda)$ the $k_0[G]$-module obtained by restriction to G. These representations have as a basis $1 \otimes x^i y^j$ with $i + j = b - a$. These are the 'dual Weyl modules' or 'induced modules' for $\mathbf{G}$ (see Proposition 3.5). We denote by $\boldsymbol{\Delta}(\lambda)$ the rational representation $\boldsymbol{\nabla}(-w \cdot \lambda)^*$ and denote its restriction to G by $\Delta(\lambda)$. These are the 'Weyl modules' for $\mathbf{G}$.

If V is a rational representation of $\mathbf{G}$ then we say that a weight ν is a weight of V if $\mathrm{Hom}_{\mathbf{T}}(\nu, V) \neq 0$. The multiplicity of a weight in V is defined to be $\dim_{\overline{\mathbb{F}}_p}(\mathrm{Hom}_{\mathbf{T}}(\nu, V))$. If λ is a dominant weight, then the weights of $\boldsymbol{\nabla}(\lambda)$ are $(a, b), (a + 1, b - 1), ..., (b, a)$, each having multiplicity one.

If $0 \leq b - a \leq q - 1$ we say that λ is *restricted*. If $0 \leq b - a \leq p - 1$ we say that λ is *p-restricted*. Any restricted weight λ can be written as a sum $\sum_{i=0}^{f-1} p^i \lambda_i$, where the weights λ_i are p-restricted.

The p-power map σ induces a Frobenius endomorphism σ of $\mathbf{G}$. If $i \in \mathbb{Z}_{\geq 0}$ and V is a rational $\mathbf{G}$-representation we write $V^{(i)}$ for the rational $\mathbf{G}$-representation with the same underlying vector space as V but with the g action given by the action of $\sigma^i(g)$ on V.

The following are some well-known results in the representation theory of G and $\mathbf{G}$ in natural characteristic.

Proposition 3.2. *Let λ be a dominant weight. Then the representation $\boldsymbol{\nabla}(\lambda)$ contains a unique irreducible rational subrepresentation. Equivalently, $\boldsymbol{\Delta}(\lambda)$ has a unique irreducible quotient. Both of these representations are isomorphic to the irreducible representation of $\mathbf{G}$ with highest weight λ. We denote it by $\mathbf{L}(\lambda)$ and denote its restriction to G by $L(\lambda)$. If λ is p-restricted then $\mathbf{L}(\lambda) = \boldsymbol{\nabla}(\lambda)$. If λ is restricted then $L(\lambda)$ is an* *(absolutely) simple $k_0[G]$-module. This construction gives all the simple $k_0[G]$-modules. If $\lambda = \sum_{i=0}^{f-1} p^i \lambda_i$ with λ_i p-restricted, then*

$$\mathbf{L}(\lambda) \cong \bigotimes_i \mathbf{L}(\lambda_i)^{(i)}.$$

Proof. See, for example, chapter 2 of [Hum]. □

Lemma 3.3. *Let λ be a restricted weight. Then $\mathrm{Ext}^1_{k_0[G]}(L(\lambda), L(\lambda)) = 0$.*

Proof. This is a special case of [BP12, Corollary 5.6 (i)]. □

Definition 3.4. Let λ be a dominant weight. The algebraic induced representation $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(w \cdot \lambda)$ is a rational representation of $\mathbf{G}$ given by the subspace of the regular functions on $\overline{\mathbf{N}}\backslash\mathbf{G}$ consisting of functions f satisfying $f(tg) = w \cdot \lambda(t) f(g)$ for every $t \in \mathbf{T}$, $g \in \mathbf{G}$. The action of $\mathbf{G}$ is given by right translation of the argument of a function.

Proposition 3.5. *Let λ be a dominant weight. Then*

$$\mathbf{\nabla}(\lambda) \cong \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(w \cdot \lambda).$$

Proof. We can think of $\mathbf{\nabla}(\lambda_{a,b})$ as homogeneous polynomial functions f on the standard representation V_0^* of degree $b - a$. Denote by e_1 and e_2 the standard basis vectors of V_0^* (the $\mathbf{G}$-action is twisted by $\det^b$ from the standard one). Consider the map

$$\Phi : \mathbf{\nabla}(\lambda) \to \overline{\mathbb{F}}_p[\overline{\mathbf{N}}\backslash\mathbf{G}]$$

given by

$$\Phi(f) : g \mapsto \det(g)^b f(g^{-1}e_2).$$

This is $\mathbf{G}$-equivariant for the left $\mathbf{G}$-action on these spaces. Moreover, for $t \in \mathbf{T}$ we have $\Phi(f)(tg) = w \cdot \lambda(t)\Phi(f)(g)$. So Φ gives a map $\mathbf{\nabla}(\lambda) \to \mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(w \cdot \lambda)$. We can deduce that this is an isomorphism if we know that summing over all dominant weights gives an isomorphism

$$\bigoplus_{\lambda \text{ dominant}} \mathbf{\nabla}(\lambda) \cong \overline{\mathbb{F}}_p[\overline{\mathbf{N}}\backslash\mathbf{G}].$$

This is standard. □

Corollary 3.6. *Restricting functions from $\mathbf{G}(\overline{\mathbb{F}}_p)$ to G gives a G-equivariant map*

$$\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(w \cdot \lambda) \to \mathrm{Ind}_{B}^{G}(w \cdot \lambda) \otimes_{k_0} \overline{\mathbb{F}}_p,$$

which descends to a map $\nabla(\lambda) \to \mathrm{Ind}_B^G(w \cdot \lambda)$. If λ is restricted this map is injective.

Proof. The only thing requiring proof is the injectivity. We use the notation of the preceding proof. Suppose we have a polynomial function f on V_0^*, with coefficients f_i in k_0 and degree $b-a$, such that $\Phi(f)$ restricts to zero on G. We have $f(ue_1 + ve_2) = \sum_{i=0}^{b-a} f_i u^i v^{b-a-i}$. In particular, $f(ue_1 + ve_2)$ is a polynomial in u with at least q distinct roots (one for each $u \in k_0$). Hence $b - a \geq q$. □

Lemma 3.7. *We have an isomorphism*

$$\mathrm{Ind}_B^G(w \cdot \lambda)^* \cong \mathrm{Ind}_B^G(-w \cdot \lambda).$$

Proof. By [CL76, Proposition 3.5] $\mathrm{Ind}_B^G(w\cdot\lambda)$ contains a vector φ on which $\overline{B}$ acts via $w\cdot\lambda$ and such that φ generates the G-representation $\mathrm{Ind}_B^G(w\cdot\lambda)$. This gives us a $\overline{B}$-equivariant map $w\cdot\lambda \to \mathrm{Ind}_B^G(w\cdot\lambda)$ such that the image is contained in no proper $k_0[G]$-submodule. Dually we get a $\overline{B}$-equivariant map $\mathrm{Ind}_B^G(w\cdot\lambda)^* \to -w\cdot\lambda$ such that there is no proper $k_0[G]$-submodule contained in the kernel. Frobenius reciprocity gives us the desired map

$$\mathrm{Ind}_B^G(w\cdot\lambda)^* \to \mathrm{Ind}_B^G(-w\cdot\lambda),$$

and it is injective since its kernel is a G-stable submodule of the kernel of $\mathrm{Ind}_B^G(w\cdot\lambda)^* \to -w\cdot\lambda$. Since the dimensions are equal, the map is an isomorphism. □

Corollary 3.8. *Let $\lambda = \lambda_{a,b}$ be a restricted weight, and let $\lambda' = \lambda_{b-q+1,a}$ (note that λ' is again a restricted weight). Then the map of Corollary 3.6 together with its dual induce a short exact sequence*

$$0 \to \nabla(\lambda) \to \mathrm{Ind}_B^G(w\cdot\lambda) \to \Delta(\lambda') \to 0.$$

Proof. The above corollary and lemma give an injective map

$$\nabla(\lambda) \to \mathrm{Ind}_B^G(w\cdot\lambda)$$

and, dually, a surjective map

$$\mathrm{Ind}_B^G(w\cdot\lambda) \to \Delta(\lambda').$$

Since

$$\dim\Delta(\lambda') + \dim\nabla(\lambda) = q+1 = \dim \mathrm{Ind}_B^G(w\cdot\lambda)$$

we need only to show that the composite of the two maps is zero. In fact, we will show that $\mathrm{Hom}_G(\nabla(\lambda),\Delta(\lambda')) = 0$. First suppose that $b-a=0$. Then $\nabla(\lambda) = L(\lambda) = \det^b$ and $\Delta(\lambda') = L(\lambda') = \det^b \otimes \mathrm{Symm}^{q-1}(V_0)$ and it is clear that $\mathrm{Hom}_G(\nabla(\lambda),\Delta(\lambda')) = 0$. Now suppose that $b-a>0$. Let $\theta \in \mathrm{Hom}_G(\nabla(\lambda),\Delta(\lambda'))$. The weights of $\nabla(\lambda)$ are $\lambda_{a,b}, \lambda_{a+1,b-1}, \ldots, \lambda_{b,a}$ whilst the weights of $\Delta(\lambda')$ are $\lambda_{b-q+1,a}, \lambda_{b-q+2,a}, \ldots, \lambda_{a,b-q+1}$. Therefore $\ker(\theta)$ contains the weight spaces for $\lambda_{a+1,b-1}, \ldots, \lambda_{b-1,a+1}$ (since these weights of T do not appear in $\Delta(\lambda')$). In particular, $\theta(1\otimes x^{b-a-1}y) = \theta(1\otimes xy^{b-a-1}) = 0$. But we have

$$\begin{pmatrix}1&0\\1&1\end{pmatrix} 1\otimes x^{b-a-1}y = -1\otimes x^{b-a} + 1\otimes x^{b-a-1}y$$

$$\begin{pmatrix}1&1\\0&1\end{pmatrix} 1\otimes xy^{b-a-1} = 1\otimes xy^{b-a-1} - 1\otimes y^{b-a}.$$

Since $\ker(\theta)$ is G-stable, we have shown that $\ker(\theta)$ is all of $\nabla(\lambda)$, as required. □

Proposition 3.9. *Suppose $\lambda = \lambda_{a,b}$ is restricted. Suppose $0 < b-a < q-1$. Then*

$$\mathrm{soc}_G(\mathrm{Ind}_B^G(w\cdot\lambda)) = L(\lambda).$$

Proof. This follows from [CL76, Theorem 7.4]. □

Remark. If $b-a=0$ or $q-1$ then $\mathrm{Ind}_B^G(w\cdot\lambda) = L(\lambda)\oplus L(\lambda')$. Note that Proposition 3.9 implies that the short exact sequence of Corollary 3.8 is non-split when $0 < b-a < q-1$.

We will need to know the socle of a Weyl module in one particular case:

Lemma 3.10. *Let $\lambda_{a,b}$ be a restricted weight. Suppose $b-a=q-2$. Then*

$$\mathrm{soc}_G(\Delta(\lambda)) = L(\lambda_{a+p^{f-1}-1,a+p^{f-1}(p-1)-1})$$

Proof. This is a consequence of [CC76], [Der81]. In fact, these references completely describe the submodule structure of $\Delta(\lambda)$. We refer to [Dot85, 2.3] for an elegant description of the submodule structure of certain dual Weyl modules for SL_n, and explain the necessary computation in the case $b-a=q-2$. For $0 \le r \le q-2$ and $0 \le i \le f-2$, set $\alpha_i(r)$ to be the carry into the coefficient of p^{i+1} when adding (the p-adic expansions of) r and $q-2-r$. This defines an element $\alpha(r)$ of $\{0,1\}^{f-1}$. Denote by E the subset of $\{0,1\}^{f-1}$ comprising $\alpha(r)$ for each $0 \le r \le q-2$. A partial ordering on E is given by defining $\alpha \preceq \alpha'$ if $\alpha_i \le \alpha_i$ for all $0 \le i \le f-2$. We say that a subset S of E is closed if S contains the predecessors of all of its elements. For closed S we get a submodule F_S of $\nabla(\lambda_{a,b})$ by taking the submodule generated by the vectors of weights $\lambda_{a+r,b-r}$ for $0 \le r \le q-2$ and $\alpha(r) \in S$. It is proved in the references cited above that every submodule of $\nabla(\lambda_{a,b})$ has this form. To compute $\mathrm{cosoc}(\nabla(\lambda_{a,b}))$, or equivalently $\mathrm{soc}(\Delta(\lambda_{a,b}))$, we must determine the maximal elements in the poset E. We observe that $\alpha(p^{f-1}-1) = (1,1,...,1)$ so this is clearly the greatest element in E, which implies that $\mathrm{cosoc}(\nabla(\lambda_{a,b}))$ is irreducible. Moreover, it is straightforward to check that $p^{f-1}-1$ is the smallest integer r satisfying $\alpha(r) = (1,1,...,1)$, so

$$\mathrm{cosoc}(\nabla(\lambda_{a,b})) = L(\lambda_{a+p^{f-1}-1,b-p^{f-1}+1}) = L(\lambda_{a+p^{f-1}-1,a+p^{f-1}(p-1)-1}).$$

Dualising gives the desired result for $\mathrm{soc}_G(\Delta(\lambda))$. □

4. Cohomology

We return to the setting of section 2. The aim of this section is to describe the first de Rham cohomology groups of $\overline{Y}_k$ and $\overline{Z}_k$ as $\Gamma \times G$-modules (and as $\mathbb{T}_U$-modules).

4.1. Igusa part of the cohomology.

The connected components of $\overline{Z}_k$ are parameterised by pairs (ξ, P), with $\xi \in k_0^\times$ labelling the connected components X_ξ of $\tilde{X}$ and P denoting a one dimensional k_0-vector subspace of $\mathfrak{p}^{-1}\Lambda/\Lambda$. Denote such an irreducible component by $\mathrm{Ig}_{\xi,P}$. Each k-scheme $\mathrm{Ig}_{\xi,P}$ is isomorphic to the relative moduli over X_k of Drinfeld level structures $\eta : \mathfrak{p}^{-1}\Lambda/\Lambda \to \mathcal{G}_{1,k}$ such that $P \subset \ker(\eta)$. The group G acts on $\overline{Z}_k$ from the right. The description of this action on the components is that the map $\eta \mapsto \eta \circ g$ induces an isomorphism $\mathrm{Ig}_{\xi,P} \to \mathrm{Ig}_{\xi \det(g), g^{-1}P}$, which gives the action of g on the component $\mathrm{Ig}_{\xi,P}$. The action of $\zeta \in \Gamma$ on $\overline{Z}_k$ is given by the identity map $\mathrm{Ig}_{\xi,P} \to \mathrm{Ig}_{\xi\zeta^{-(q+1)},P}$.

Denote by P_0 the unique P such that $g^{-1}P = P$ for all $g \in \overline{B}$ — recall that $\overline{B}$ is the lower triangular matrices in G. Write Ig_0 for $\coprod_\xi \mathrm{Ig}_{\xi,P_0}$.

Lemma 4.1. *We have an isomophism of $G \times \Gamma$-modules*

$$H^1_{dR}(\overline{Z}_k/k) \cong \mathrm{Ind}_{\overline{B}}^G H^1_{dR}(\mathrm{Ig}_0/k).$$

Proof. This is clear from the description of the action of G on the components, and the fact that $\overline{B}$ stabilises Ig_0. □

Lemma 4.2. *The action of $\overline{B}$ on Ig_0 factors through the diagonal matrices T.*

Proof. See [KM85, Theorem 13.10.3 (3)]. □

For $\lambda_{a,b}$ a restricted weight, write $H^1_{dR}(\mathrm{Ig}_0/k)_{a,b}$ for the direct summand of $H^1_{dR}(\mathrm{Ig}_0/k)$ on which T acts via the character $\lambda_{a,b}$ (recall that k is an extension of k_0).

Lemma 4.3. *The group* Γ *acts on* $H^1_{dR}(\mathrm{Ig}_0/k)_{a,b}$ *via the character* $\zeta \mapsto \zeta^{-b(q+1)}$.

Proof. This follows immediately from the fact that $(\zeta, \begin{pmatrix} 1 & 0 \\ 0 & \zeta^{q+1} \end{pmatrix}) \in \Gamma \times T$ acts trivially on Ig_0. □

Proposition 4.4. *We have a* $\Gamma \times G$ *and* $\mathbb{T}_U$ *equivariant isomorphism*

$$H^1_{dR}(\overline{Z}_k/k) \cong \bigoplus_{(a,b)} [\mathrm{Ind}_B^G H^1_{dR}(\mathrm{Ig}_0/k)_{a,b}].$$

The action of Γ *on* $H^1_{dR}(\mathrm{Ig}_0/k)_{a,b}$ *is given by the character* $\zeta \mapsto \zeta^{-b(q+1)}$. *As a* $k[G]$*-module, we have an isomorphism*

$$\mathrm{Ind}_B^G H^1_{dR}(\mathrm{Ig}_0/k)_{a,b} \cong (\mathrm{Ind}_B^G \lambda_{a,b})^{\oplus m} \otimes_{k_0} k$$

for some multiplicity m.

Proof. This follows immediately from the above lemmas. □

Remark. In fact, by considering the relative Frobenius morphism $\overline{Z}_k/k \to \overline{Z}_k^{(p)}/k$ we can see that the above isomorphism sends the action of φ on the left hand side to the action on the right hand side induced from the σ-linear maps $\varphi : H^1_{dR}(\mathrm{Ig}_0/k)_{a,b} \to H^1_{dR}(\mathrm{Ig}_0/k)_{ap,bp}$. As a consequence, Fil^1 of the Hodge filtration on the left hand side is given by

$$\bigoplus_{(a,b)} [\mathrm{Ind}_B^G \mathrm{Fil}^1(H^1_{dR}(\mathrm{Ig}_0/k)_{a,b})].$$

However, we will not use these facts in what follows.

Note that since $\lambda_{a,b}$ is restricted, so is $\lambda_{b,a+q-1}$, and $w \cdot \lambda_{b,a+q-1} = \lambda_{a,b}$ as weights of T. So we have

$$\mathrm{Ind}_B^G(\lambda_{a,b}) = \mathrm{Ind}_B^G(w \cdot \lambda_{b,a+q-1}).$$

Recall that the structure of these representations was discussed in Corollary 3.8 and Proposition 3.9.

4.2. **Drinfeld part of the cohomology.** In this subsection we use the construction of the $\kappa^\times$ torsor over $\overline{Y}_k$ in section 2.4 to relate the de Rham cohomology of $\overline{Y}_k$ to a space of mod p modular forms for $\overline{G}$. We write $H^1_{dR}(\overline{Y}_k/k)$ for the hypercohomology of the de Rham complex of coherent sheaves $\Omega^\bullet_{\overline{Y}_k/k}$ of the smooth proper curve $\overline{Y}_k/k$. This cohomology group is a k-vector space with the following additional structures:

- Commuting left actions of $\mathbb{T}_U$, Γ, G.
- A one step Hodge filtration Fil^1.
- A σ-linear map φ.

We can similarly consider a cohomology group $H^1_{dR}(\overline{Y}_k(1)/k)$, which is a filtered φ-module over k with commuting left actions of $\mathbb{T}_U$, Γ, G and $\kappa^\times$. The cohomology group $H^1_{dR}(\overline{\mathrm{Dr}}^0_k/k)$ is a filtered φ-module over k with commuting left actions of Γ, G and $\kappa^\times$ – the left action of $\kappa^\times$ is given by composing ι^+ and the left action of Γ. Recall that we have an étale $\kappa^\times$ torsor

$$\alpha : \overline{Y}_k(1) \to \overline{Y}_k$$

which restricts to a torsor over the affine curve Y_k. The map α induces k-linear maps

$$\alpha^* : H^1_{dR}(\overline{Y}_k/k) \to H^1_{dR}(\overline{Y}_k(1)/k)$$

which commute with all the additional structures.

Lemma 4.5. *The map α^* gives an isomorphism (respecting all the structures listed above)*

$$\alpha^* : H^1_{dR}(\overline{Y}_k/k) \cong H^1_{dR}(\overline{Y}_k(1)/k)^{\kappa^\times}$$

where the superscript $\kappa^\times$ denotes taking invariants under the left $\kappa^\times$-action.

Proof. Let $\Omega^\bullet(1)$ denote the de Rham complex for $\overline{Y}(1)/k$ and let $\Omega^\bullet$ denote the de Rham complex for $\overline{Y}_k$. Since α is étale, we have an isomorphism of complexes $\alpha^*\Omega^\bullet \cong \Omega^\bullet(1)$. Pulling back an acyclic resolution of $\Omega^\bullet$ by α gives an acyclic, $\kappa^\times$-equivariant resolution of $\Omega^\bullet(1)$. Since $\overline{Y}_k(1) \to \overline{Y}_k$ is a torsor, for any quasicoherent sheaf $\mathcal{F}$ on $\overline{Y}_k$ we have $\alpha_*(\alpha^*\mathcal{F})^{\kappa^\times} = \mathcal{F}$ (by étale descent for quasicoherent sheaves), so by considering the double complex obtained by applying $\alpha_*()^{\kappa^\times}$ to our acyclic $\kappa^\times$-equivariant resolution of $\Omega^\bullet(1)$ we obtain a spectral sequence

$$E_2^{i,j} : H^i(\kappa^\times, H^j_{dR}(\overline{Y}(1)/k)) \Rightarrow H^{i+j}_{dR}(\overline{Y}_k/k).$$

Recall that $\kappa^\times$ is a cyclic group of order q^2-1. Since multiplication by q^2-1 is an isomorphism on the k-vector space $H^{i+j}_{dR}(\overline{Y}_k/k)$, the $E_2^{i,j}$ term of the spectral sequence vanishes except when $i=0$, and we obtain the desired isomorphism from the exact sequence of low degree terms, since the map $E^1_\infty \to E_2^{0,1}$ coincides with the composition of α^* and the inclusion

$$H^1_{dR}(\overline{Y}_k(1)/k)^{\kappa^\times} \hookrightarrow H^1_{dR}(\overline{Y}_k(1)/k).$$

□

Remark. Note that we could abbreviate the above proof by computing the hypercohomology of the de Rham complex on the étale site and using the Hochschild-Serre spectral sequence in étale cohomology.

We can now use lemma 4.5 together with equation (2.19) to give a simple description of the cohomology groups of interest.

Definition 4.6. Suppose we have a finite set Σ and a k-vector space V. Then we denote by $\mathcal{L}(\Sigma, V)$ the k-vector space of functions from Σ to V. Note that if V is finite dimensional there is a canonical isomorphism $\mathcal{L}(\Sigma, k) \otimes_k V \to \mathcal{L}(\Sigma, V)$.

We write Σ_U for the finite set $\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}^\infty)/U^{\mathfrak{p}}(1+\mathfrak{p}_D)$. Recall that this finite set has a right action (by correspondences) of $\mathbb{T}_U$ and a right action of $\kappa^\times = \mathcal{O}_D^\times/1+\mathfrak{p}_D$ by right multiplication. We suppose V is a filtered φ-module over k, equipped with commuting left actions of $\kappa^\times$ and $\Gamma\times G$. Then we endow $\mathcal{L}(\Sigma_U, V)$ with

- the obvious commuting left actions of $\mathbb{T}_U$ and $\Gamma\times G$
- a left action of $\kappa^\times$ defined by

$$u : \mathcal{L}(\Sigma_U, V) \to \mathcal{L}(\Sigma_U, V)$$
$$f \mapsto (s \mapsto uf(su)),$$

We also make $\mathcal{L}(\Sigma_U, V)$ into a filtered φ-module by giving it

- a filtration $\mathrm{Fil}^1(\mathcal{L}(\Sigma_U, V)) = \mathcal{L}(\Sigma_U, \mathrm{Fil}^1(V))$
- a σ-linear map φ given by $(\varphi f)(\sigma) = \varphi(f(\sigma))$

Note that $H^1_{dR}(\overline{\mathrm{Dr}}^0_k/k)$ has the structures required of V in the above definition.

Lemma 4.7. *We have an isomorphism*

$$H^1_{dR}(\overline{Y}(1)/k) \cong \mathcal{L}(\Sigma_U, H^1_{dR}(\overline{\mathrm{Dr}}^0_k/k))$$

which respects all the structures in the above itemized list.

Proof. This follows from equation (2.19). □

Putting together lemma 4.5 and lemma 4.7, we obtain the following Proposition:

Proposition 4.8. *We have isomorphisms of filtered φ-modules*

$$H^1_{dR}(\overline{Y}_k/k) \cong \mathcal{L}(\Sigma_U, H^1_{dR}(\overline{\mathrm{Dr}}^0_k/k))^{\kappa^\times}$$

compatible with the $\mathbb{T}_U$ and $\Gamma\times G$ actions.

We think of the right hand sides of the isomorphisms in this Proposition as spaces of mod p modular forms for $\overline{G}$, with coefficients in the filtered φ-module (with $\kappa^\times$ action) $H^1_{dR}(\overline{\mathrm{Dr}}^0_k/k)$.

4.3. **De Rham cohomology of the Drinfeld curve.** Proposition 4.8 reduces the computation of the de Rham cohomology of $\overline{Y}_k$ to the computation of the cohomology of $\overline{\mathrm{Dr}}^0_k$, which is essentially done in [HJ90]. Throughout this subsection k will denote any finite extension of k_0 containing a quadratic extension of k_0. To simplify notation we will denote the proper curve $\overline{\mathrm{Dr}}^0_k$ by $\overline{C}$ and denote the affine curve Dr^0_k by C.

Proposition 4.9. *We have a decomposition*

$$H^1_{dR}(\overline{C}/k) = \bigoplus_{\substack{i\in\mathbb{Z}/(q^2-1)\mathbb{Z} \\ i\neq 0 \bmod q+1}} H^1_{dR}(\overline{C}/k)(i),$$

where $H^1_{dR}(\overline{C}/k)(i)$ is the direct summand on which Γ acts via the character $\zeta\mapsto\zeta^{-i}$. For every i there is a short exact sequence

$$0 \longrightarrow H^0(\overline{C}, \Omega^1_{\overline{C}/k})(i) \longrightarrow H^1_{dR}(\overline{C}/k)(i) \longrightarrow H^1(\overline{C}, \mathcal{O}_{\overline{C}})(i) \longrightarrow 0. \tag{4.1}$$

If we write $i = i_0 + i_1(q+1)$, with $1 \leq i_0 \leq q$ and $0 \leq i_1 \leq q-2$ then we have isomorphisms of $k[G]$-modules

$$H^0(\overline{C}, \Omega^1_{\overline{C}/k})(i) \cong \nabla(\lambda_{i_1+1, i_0+i_1-1}) \otimes_{k_0} k$$
$$H^1(\overline{C}, \mathcal{O}_{\overline{C}})(i) \cong \Delta(\lambda_{i_0+i_1, i_1+q-1}) \otimes_{k_0} k,$$

where if λ is not a dominant weight we set $\nabla(\lambda) = \Delta(\lambda) = 0$. Note that the weights appearing in the above are restricted. The extension in (4.1) is non-split (unless $i_0 = 1$ or q, in which case one of the terms in (4.1) is zero).

Proof. This is a mild generalisation of [HJ90, Proposition 2.8, Proposition 4.8]. It can be deduced directly from loc. cit. by keeping track of connected components. Note also that the central characters of the G-representations appearing in $H^1_{dR}(\overline{C}/k)(i)$ are determined by the fact that the action of the centre of G is inverse to the action of $k_0^\times \subset \mu_{q-1}(\Gamma) \subset \Gamma$. Therefore the weights $\lambda_{a,b}$ appearing in $H^1_{dR}(\overline{C}/k)(i)$ satisfy $a + b = i = i_0 + 2i_1 \mod (q-1)$. □

Corollary 4.10. *For each i, the weights of $H^1_{dR}(\overline{C}/k)(i)$ all appear with multiplicity one. They are*

$$\lambda_{a,b} : 0 \leq a, b \leq q-2, a+b = i \textit{ mod } q-1.$$

In fact, [HJ90, Proposition 4.7] allows us to determine the G-socle of $H^1_{dR}(\overline{C}/k)(i)$. To state things precisely, we need the following

Definition 4.11. Let $\lambda = \lambda_{a,b}$ be a restricted weight. The weights of $\Delta(\lambda)$ are $\lambda_{a+c,b-c}$ where c runs over integers satisfying $0 \leq c \leq b-a$. We define $\Delta(\lambda)_1$ to be 0 if $b - a \neq -2$ mod p, and otherwise to be the direct summand of $\Delta(\lambda)$ (as a $k_0[T]$-module) given by taking the direct sum of the weight spaces for $\lambda_{a+c,b-c}$ where $0 \leq c \leq b-a$ and $c = -1$ mod p.

Remark. Note that the condition $b - a = -2$ mod p implies that the set of weights appearing in $\Delta(\lambda)_1$ is closed under the action of the Weyl group (since $b - a - c = -1$ mod p).

Lemma 4.12. *$\Delta(\lambda)_1$ is a G-stable submodule of $\Delta(\lambda)$.*

Proof. This can be shown by an elementary calculation. It also follows from [HJ90, Proposition 4.8]. □

Lemma 4.13. *Suppose $\mu = \lambda_{a,b}$ is a restricted weight such that $L(\mu)$ is an irreducible constituent of $\Delta(\lambda)_1$ for a restricted weight λ. Then $b - a = 0$ mod p.*

Proof. Suppose $L(\mu)$ is an irreducible constituent of $\Delta(\lambda)_1$. It follows from the definition of $\Delta(\lambda)_1$ that the weights of $L(\mu)$ are a subset of

$$\{\lambda_{a+kp,b-kp} : 0 \leq k \leq \frac{q-1}{p}\}.$$

In particular, $(b,a) = (a+kp, b-kp)$ (an equality in $(\mathbb{Z}/(q-1)\mathbb{Z})^2$) for some k, but since $0 \leq kp < q-1$ we have $b - a = kp$ in $\mathbb{Z}$. So $b - a = 0$ mod p, as required. □

Definition 4.14. We use the notation of Proposition 4.9. Let $i \in \mathbb{Z}/(q^2-1)\mathbb{Z}$, $i \neq 0 \bmod q+1$ and let $a = i_0 + i_1$, $b = i_1 + q - 1$. We set $H^1_{dR}(\overline{C}/k)(i)_1$ to be 0 if $i_0 \neq 1 \bmod p$, and otherwise to be the direct summand of $H^1_{dR}(\overline{C}/k)(i)$ (as a $k[T]$-module) given by taking the direct sum of the weight spaces for $\lambda_{a+c,b-c}$ where $0 \leq c \leq b-a$ and $c = -1 \bmod p$.

It is immediate from the above definition that the image of $H^1_{dR}(\overline{C}/k)(i)_1$ in $H^1(\overline{C}, \mathcal{O}_{\overline{C}})(i)$ is G-stable, and isomorphic to $\Delta(\lambda_{a,b})_1 \otimes_{k_0} k$.

Proposition 4.15. *The $k[T]$-module $H^1_{dR}(\overline{C}/k)(i)_1$ is a G-stable subspace of $H^1_{dR}(\overline{C}/k)(i)$. If $i_0 \geq 2$, it is the maximal G-stable subspace of $H^1_{dR}(\overline{C}/k)(i)$ which injects into $H^1(\overline{C}, \mathcal{O}_{\overline{C}})(i)$.*

Proof. This follows from [HJ90, Proposition 4.7], in particular the displayed equations 4), 5), 6) of that Proposition, which explicitly describe the action of $\mathrm{SL}_2(k_0) \times \mu_{q+1}(\Gamma)$ on the de Rham cohomology of a connected component of $\overline{C}$. □

Remark. Note that if $q = p$, we have $H^1_{dR}(\overline{C}/k)(i)_1 = 0$ for all i.

Putting everything together, we obtain the key result of this section:

Corollary 4.16. *If $i_0 \geq 2$ then we have*

$$\mathrm{soc}_G H^1_{dR}(\overline{C}/k)(i) = [L(\lambda_{i_1+1,i_0+i_1-1}) \oplus \mathrm{soc}_G \Delta(\lambda_{i_0+i_1,i_1+q-1})_1] \otimes_{k_0} k.$$

In particular, if $i_0 \geq 2$ and $\Delta(\lambda_{i_0+i_1,i_1+q-1})_1 = 0$, then

$$\mathrm{soc}_G H^1_{dR}(\overline{C}/k)(i) = L(\lambda_{i_1+1,i_0+i_1-1}) \otimes_{k_0} k.$$

If $i_0 = 1$ then we have

$$\mathrm{soc}_G H^1_{dR}(\overline{C}/k)(i) = \mathrm{soc}_G \Delta(\lambda_{i_0+i_1,i_1+q-1}) \otimes_{k_0} k.$$

5. Breuil–Kisin modules

We assume from now on that $p > 2$. For brevity, we will refer to Breuil–Kisin modules as Kisin modules for the rest of this section.

5.1. **Kisin modules with coefficients and tame descent data.** In this section we let K be any finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$, and fix a uniformiser ϖ. We write k for the residue field. Denote the ramification index of K by e. Denote $[k : \mathbb{F}_p]$ by d. Let E be a finite field and denote $[E : \mathbb{F}_p]$ by d_E. We write $\mathfrak{S}_1$ for the ring $k[[u]]$. Since $\mathfrak{S}_1$ is an $\mathbb{F}_p$-algebra we have a p-power ring homomorphism $\sigma : \mathfrak{S}_1 \to \mathfrak{S}_1$. For an $\mathfrak{S}_1$-module M we write $\sigma^* M$ for the $\mathfrak{S}_1$-module $\mathfrak{S}_1 \otimes_\sigma M$.

Definition 5.1. A Kisin module over K is a free $\mathfrak{S}_1$-module $\mathfrak{M}$, equipped with a $\mathfrak{S}_1$-linear map $\varphi : \sigma^*\mathfrak{M} \to \mathfrak{M}$ which satisfies $u^e\mathfrak{M} \subset \mathrm{Im}\,(\varphi)$. Define morphisms between Kisin modules over K to be $\mathfrak{S}_1$-module morphisms respecting φ. We denote the resulting category by $\mathfrak{Mod}_K$.

Write Gr_K for the category of finite flat group schemes over $\mathcal{O}_K$ which are killed by p. It follows from results of Breuil and Kisin (see [Kis09, Theorem 1.1.3, Proposition 1.1.11]) that there is an anti-equivalence of categories $Gr : \mathfrak{Mod}_K \to \mathrm{Gr}_K$ given by first forming the Breuil module $S_1 \otimes_\sigma \mathfrak{M}$ and then taking the corresponding finite flat group scheme. For $\mathcal{G} \in \mathrm{Gr}_K$ we let $\mathfrak{Mod}(\mathcal{G})$ denote a quasi-inverse of $\mathcal{G}$ with respect to Gr.

Our discussion of coefficients and descent data will be entirely parallel to the exposition of [Sav08] in the setting of Breuil modules.

Fix an algebraic closure $\overline{K}$ of K, together with a choice of compatible p-power roots $\varpi^{1/p^n} \in \overline{K}$. Let $K_\infty = \cup_{n\geq 1} K(\varpi^{1/p^n})$ and write $G_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty)$. Denote by $\mathrm{Rep}_E(G_{K_\infty})$ the category of continuous representations of G_{K_∞} on finite-dimensional E-vector spaces. Denote by $\varphi\mathrm{Mod}_K$ the category of finite-dimensional $k((u))$-vector spaces V equipped with a bijective σ-linear map φ. As discussed in [Kis09, 1.1.12] there is an equivalence of categories

$$T : \varphi\mathrm{Mod}_K \to \mathrm{Rep}_{\mathbb{F}_p}(G_{K_\infty}).$$

More explicitly, the functor T is given by sending $M \in \varphi\mathrm{Mod}_K$ to $[M \otimes_{k((u))} k_{\widehat{\mathscr{E}}^{ur}}]^{\varphi=1}$, where $k_{\widehat{\mathscr{E}}^{ur}}$ is a separable closure of $k((u))$ with an action of G_{K_∞} inducing the action of G_{K_∞} on $T(M)$. In particular the action satisfies the following: suppose $v \in k_{\widehat{\mathscr{E}}^{ur}}$ satisfies $v^r = u^s$ for coprime integers r, s with $s > 0$ and $p \nmid s$. Then for $g \in G_{K_\infty}$ we have $gv = \frac{g\varpi_v}{\varpi_v} v$ where $\varpi_v \in \mathcal{O}_{\overline{K}}$ is some element satisfying $\varpi_v^r = \varpi^s$. If $\mu_r(\overline{k}) \subset k$ then this determines gv uniquely. Note that the field $k_{\widehat{\mathscr{E}}^{ur}}$ comes equipped with a G_{K_∞}-equivariant embedding $\overline{k} \hookrightarrow k_{\widehat{\mathscr{E}}^{ur}}$, where G_{K_∞} acts on $\overline{k}$ via $\mathrm{Gal}(\overline{k}/k)$.

If $\mathfrak{M} \in \mathfrak{Mod}_K$ then $\mathfrak{M}[\frac{1}{u}] \in \varphi\mathrm{Mod}_K$, and it follows from [Kis09, Proposition 1.1.13] that there is a canonical isomorphism of G_{K_∞}-representations

$$T(\mathfrak{M}[\frac{1}{u}]) \xrightarrow{\sim} Gr(\mathfrak{M})^\vee(\mathcal{O}_{\overline{K}})(-1)|_{G_{K_\infty}} = Gr(\mathfrak{M})(\mathcal{O}_{\overline{K}})^*|_{G_{K_\infty}},$$

where $^\vee$ denotes the Cartier dual, * denotes the dual representation and (-1) denotes a Tate twist.

Definition 5.2. Denote by $\mathfrak{Mod}_K(E)$ the category whose objects are Kisin modules $\mathfrak{M}$ over K equipped with an $\mathbb{F}_p$-algebra map $E \to \mathrm{End}_{\mathfrak{Mod}_K}(\mathfrak{M})$. The morphisms are given by morphisms in $\mathfrak{Mod}_K$ compatible with the E action.

The functor Gr induces an anti-equivalence of categories between $\mathfrak{Mod}_K(E)$ and the category $\mathrm{Gr}_K(E)$ of finite flat E-module schemes over $\mathcal{O}_K$. We refer to the objects of $\mathfrak{Mod}_K(E)$ as Kisin modules over K with coefficients in E. The E action on $\mathfrak{M}$ gives it the structure of a finite free module over $\mathfrak{S}_E := E \otimes_{\mathbb{F}_p} \mathfrak{S}_1$. We set $\mathrm{rk}(\mathfrak{M}) = \mathrm{rk}_{\mathfrak{S}_E}(\mathfrak{M})$.

We assume from now on that k embeds in E. We fix such an embedding

$$\sigma_0 : k \hookrightarrow E.$$

For $i \in \mathbb{Z}/d\mathbb{Z}$ we recursively define $\sigma_{i+1} = \sigma_i \circ \sigma$. The embeddings σ_i run over the d different embeddings $k \hookrightarrow E$. We have a decomposition $E \otimes_{\mathbb{F}_p} k = \prod_i E \otimes_{\sigma_i, k} k$ so for $\mathfrak{M} \in \mathfrak{Mod}_K(E)$ we get a similar decomposition $\mathfrak{M} = \oplus_{i \in \mathbb{Z}/d\mathbb{Z}} \mathfrak{M}_i$ where $\mathfrak{M}_i$ is a finite free $E[[u]] \cong E \otimes_{\sigma_i, k} \mathfrak{S}_1$-module and φ restricts to maps $\varphi_i : \mathfrak{M}_i \to \mathfrak{M}_{i-1}$ for each i. $\mathfrak{M}_i$ is the piece of $\mathfrak{M}$ where the action of $1 \otimes x \in E \otimes_{\mathbb{F}_p} k$ is the same as the action of $\sigma_i(x) \otimes 1$. We again denote by σ_i the E-linear extension of σ_i to a map $E \otimes_{\mathbb{F}_p} k \to E$.

Now we discuss tame descent data. Let K/L be a tamely ramified Galois extension, with ramification degree $e(K/L)$, residue field k_L and Galois group $\mathrm{Gal}(K/L)$. We assume that the fixed uniformiser ϖ of K satisfies $\varpi^{e(K/L)} \in L$, and denote by L_∞ the field $\cup_{n\geq 0} L(\varpi^{e(K/L)/p^n})$. Note that K_∞/L_∞ is a finite Galois extension with Galois group canonically isomorphic to $\mathrm{Gal}(K/L)$. The group $\mathrm{Gal}(K/L)$ acts on $E \otimes_{\mathbb{F}_p} k$ via the trivial action on the first factor and via $\mathrm{Gal}(k/k_L)$ on the second factor. Denote by η the character $\mathrm{Gal}(K/L) \to \mathcal{O}_K^\times$ given by $\gamma \mapsto \gamma(\varpi)/\varpi$ and let $\overline{\eta} : \mathrm{Gal}(K/L) \to k^\times$ be the reduction of η mod ϖ.

The choice of uniformiser ϖ such that $\varpi^{e(K/L)} \in L$ induces an identification of $\mathrm{Gal}(K/L)$ with $\mathrm{Gal}\left(k((u))/k_L((u^{e(K/L)}))\right)$. The choice of compatible p-power roots $\varpi^{e(K/L)/p^n} \in \overline{K}$ allows us to extend the natural action of G_{K_∞} on $k_{\widehat{\mathscr{E}}^{ur}}$ to a natural action of G_{L_∞}.

We set $D = [k_L : \mathbb{F}_p]$. For $i \in \mathbb{Z}/d\mathbb{Z}$ denote by $[i] \in \mathbb{Z}/D\mathbb{Z}$ the residue class of i mod D. We have $[i] = [j]$ if and only if $\sigma_i|_{k_L} = \sigma_j|_{k_L}$.

Definition 5.3. Denote by $\mathfrak{Mod}_K(E, L)$ the category whose objects are Kisin modules $\mathfrak{M}$ over K with coefficients in E equipped with additive bijections $[\gamma] : \mathfrak{M} \to \mathfrak{M}$ for each $\gamma \in \mathrm{Gal}(K/L)$ satisfying

- $[\gamma]$ commutes with φ for each $\gamma \in \mathrm{Gal}(K/L)$
- $[1]$ is the identity and $[\gamma\gamma'] = [\gamma] \circ [\gamma']$
- $[\gamma](au^i m) = \gamma(a) \cdot (1 \otimes \overline{\eta}(\gamma)^i) \cdot u^i \cdot [\gamma](m)$ for $a \in E \otimes_{\mathbb{F}_p} k$ and $m \in \mathfrak{M}$.

Note that the latter condition implies that $\gamma : \mathfrak{M}_i \to \mathfrak{M}_{\gamma i}$, where $\sigma_{\gamma i} = \sigma_i \circ \gamma^{-1}$. The embedding $\sigma_{\gamma i}$ only depends on the image of γ in $\mathrm{Gal}(k/k_L)$.

If $\mathfrak{M} \in \mathfrak{Mod}_K(E, L)$ we obtain $\mathfrak{M}[\frac{1}{u}] \in \varphi\mathrm{Mod}_K$, equipped with an action of $\mathrm{Gal}(K/L)$ which is semi-linear with respect to the action of $\mathrm{Gal}(K/L)$ on $k((u))$ via its identification with

$$\mathrm{Gal}\left(k((u))/k_L((u^{e(K/L)}))\right).$$

Galois descent gives us a φ-module M_L over $k_L((u^{e_{K/L}}))$, so we obtain a G_{L_∞} action on

$$T(\mathfrak{M}[\frac{1}{u}]) = T(M_L) = [M_L \otimes_{k_L((u^{e_{K/L}}))} k_{\widehat{\mathscr{E}}^{ur}}]^{\varphi=1}.$$

Proposition 5.4. *The functor Gr induces an anti-equivalence of categories*

$$\mathfrak{Mod}_K(E, L) \to \mathrm{Gr}_K(E, L)$$

where $\mathrm{Gr}_K(E, L)$ *denotes the category of finite flat* E*-module schemes* $\mathcal{G}$ *over* $\mathcal{O}_K$ *with descent data relative to* L *(in the sense of* [BCDT01, 4.1]*). There is a canonical isomorphism*

$$T(\mathfrak{M}[\frac{1}{u}]) \xrightarrow{\sim} Gr(\mathfrak{M})^\vee(\overline{L})(-1)|_{G_{L_\infty}} = Gr(\mathfrak{M})(\overline{L})^*|_{G_{L_\infty}}.$$

Proof. This can be deduced from [Sav08, Proposition 3.2] using the comparison between Breuil and Kisin modules. Alternatively, observe that giving a tame descent datum on the generic fibre of the finite flat group scheme is equivalent to giving a Galois descent datum to $k_L((u^{e_{K/L}}))$ on $\mathfrak{M}[\frac{1}{u}] \in \varphi\mathrm{Mod}_K$. This is equivalent to a semilinear action of $\mathrm{Gal}(K/L)$ on $\mathfrak{M}[\frac{1}{u}]$, as we have described — the relevant cocycle is given by $\gamma \mapsto [\gamma] - \mathrm{id}$. The property that the descent datum extends to isomorphisms of group schemes $[\gamma] : \mathcal{G} \cong {}^{\gamma}\mathcal{G}$ is equivalent to the property that the $[\gamma]$ give bijections on $\mathfrak{M}$. □

The following is obtained by translating [Sav08, Theorem 3.5] from Breuil modules to Kisin modules:

Theorem 5.5. *Let $\mathfrak{M} \in \mathfrak{Mod}_K(E, L)$. Then there exist integers $0 \leq r_i \leq e$, $0 \leq k_i < e(K/L)$ for each $i \in \mathbb{Z}/D\mathbb{Z}$, satisfying $pk_i = r_i + k_{i-1} \bmod e(K/L)$, together with $c \in (E \otimes_{\mathbb{F}_p} k_L)^\times$ such that $\mathfrak{M}$ has the following form:*

- $\mathfrak{M}_i = \mathfrak{S}_1 \cdot m_i$ *for* $i \in \mathbb{Z}/d\mathbb{Z}$
- $\varphi(\sigma^* m_i) = \sigma_{i-1}(c) u^{r_{[i]}} m_{i-1}$
- $[\gamma] m_i = \overline{\eta}(\gamma)^{k_{[i]}} m_{\gamma i}$.

Suppose we have integers $0 \leq r_i \leq e$, $0 \leq k_i < e(K/L)$ for each $i \in \mathbb{Z}/D\mathbb{Z}$, satisfying $s_i := r_i - pk_i + k_{i-1} = 0 \bmod e(K/L)$, together with $c \in (E \otimes_{\mathbb{F}_p} k_L)^\times$ and let $\mathfrak{M}$ be the Kisin module described in the above Proposition. We denote by m the generator $(m_0, ..., m_{d-1})$ of $\mathfrak{M}$ as an $\mathfrak{S}_E$-module. Denote the corresponding finite flat E-module scheme over $\mathcal{O}_K$ by $\mathcal{G}$. The descent datum gives us a finite flat E-module scheme $\mathcal{G}_L$ over L. Denote the unramified extension of K corresponding to $\sigma_0 : k \hookrightarrow E$ by K_1, so the field K_1 has residue field E. We will compute the representation ψ of I_L given by $\mathcal{G}_L(\overline{L})$. Since it is a character $\psi : I_L \to E^\times$, it is necessarily finite and tamely ramified, so it is determined by its restriction to I_{L_∞}. So we just need to work with the Kisin module $\mathfrak{M} \otimes_{\sigma_0, k} E \in \mathfrak{Mod}_{K_1}(E, L)$. For this reason, from now on we may assume that $k = E$.

Lemma 5.6. *Choose $\alpha \in (E \otimes_{\mathbb{F}_p} \overline{k})^\times$ such that $[(1 \otimes \sigma)(\alpha)]c = \alpha$. We claim that such an α exists and is unique up to scaling by $\epsilon \otimes 1$ for $\epsilon \in E^\times$. Choose $u_D \in k_{\widehat{\mathscr{E}}^{ur}}$ such that $u_D^{p^D-1} = u$. This is unique up to scaling by an element of $\mu_{p^D-1}(\overline{k}) = E_L^\times$. Let x_0 be the integer*

$$x_0 = -\sum_{i=0}^{D-1} s_{i+1} p^i = (p^D - 1)k_0 - \sum_{i=0}^{D-1} r_{i+1} p^i$$

and recursively define $x_{j-1} = (p^D - 1)s_j + px_j$ for $j \in \mathbb{Z}/D\mathbb{Z}$. The definition of x_0 ensures that the definition makes sense even for $j = 1 - D$. Denote by β the element $(u_D^{x_{[i]}} u^{-k_{[i]}})_{i \in \mathbb{Z}/d\mathbb{Z}}$ of $(E \otimes_{\mathbb{F}_p} k_{\widehat{\mathscr{E}}^{ur}})^\times$. Then the one-dimensional E-vector space $(\mathfrak{M} \otimes_{\mathfrak{S}_1} k_{\widehat{\mathscr{E}}^{ur}})^{\varphi=1}$ is spanned by $\alpha\beta m$.

Proof. It follows from the definitions (by an elementary computation) that $\alpha\beta m$ lies in $(\mathfrak{M} \otimes_{\mathfrak{S}_1} k_{\widehat{\mathscr{E}}^{ur}})^{\varphi=1}$. The only thing we need to justify is the existence of α. The map $\alpha \mapsto \frac{\alpha}{(1\otimes\sigma)\alpha}$ comes from a morphism of connected smooth reductive group schemes

$$\mathrm{Res}^E_{\mathbb{F}_p} \mathbb{G}_m \to \mathrm{Res}^E_{\mathbb{F}_p} \mathbb{G}_m$$
$$x \mapsto \frac{x}{Fx}$$

where F denotes absolute Frobenius. It suffices to show that this map is an epimorphism, but this follows from the fact that it is an isomorphism on the d-dimensional formal group. The kernel is the constant group scheme $E^\times$, which gives the claim about uniqueness. □

Corollary 5.7. *Choose $\varpi_D \in \overline{K}$ such that $\varpi_D^{p^D-1} = \varpi$. Then the character $\psi : I_L \to E^\times$ is given by*

$$\psi(g) = \frac{\varpi_D^{x_0}}{g\varpi_D^{x_0}}.$$

Note that the right hand side of this expression is independent of the choice of ϖ_D, since G_L acts trivially on $\mu_{p^D-1}(\overline{k}) = k_L^\times$.

Proof. Let $g \in I_{L_\infty}$. Proposition 5.4 tells us that $g(\alpha\beta m) = (\psi^{-1}(g) \otimes 1)(\alpha\beta m)$. We have

$$g(\alpha\beta m) = \alpha g(\beta m)$$

since g is in the inertia subgroup. Since $\beta m = \sum_i u_D^{x[i]} u^{-k_{[i]}} m_i$, we have

$$g(\beta m) = \sum_i \frac{g\varpi_D^{x[i]}}{\varpi_D^{x[i]}} u_D^{x[i]} u^{-k_{[i]}} m_{gi}$$

so, since $[g0] = [0]$, the m_0 component of $g(\alpha\beta m)$ is

$$\frac{g\varpi_D^{x[0]}}{\varpi_D^{x[0]}} u_D^{x[0]} u^{-k_{[0]}} \alpha_0 m_0.$$

On the other hand, the m_0 component of $(\psi^{-1}(g) \otimes 1)(\alpha\beta m)$ is

$$\psi^{-1}(g) u_D^{x[0]} u^{-k_{[0]}} \sigma_0(\alpha) m_0.$$

Equating these, we get the desired statement. □

The above results can also be deduced from [Sav08].

5.2. **Kisin modules and crystalline cohomology.**

Definition 5.8. Suppose $\mathfrak{M} \in \mathfrak{Mod}_K$. Then we denote by $\mathfrak{M}_{dR}$ the k-vector space $\sigma^*\mathfrak{M}/u(\sigma^*\mathfrak{M})$. If $\mathfrak{M} \in \mathfrak{Mod}_K(E)$, then $\mathfrak{M}_{dR}$ is a free $k \otimes_{\mathbb{F}_p} E$-module. If $\mathfrak{M} \in \mathfrak{Mod}_K(E, L)$ then $\mathfrak{M}_{dR}$ is a free $k \otimes_{\mathbb{F}_p} E$-module equipped with a semi-linear action of $\mathrm{Gal}(K/L)$ (the semi-linearity comes from the action of $\mathrm{Gal}(k/k_L)$ on k).

Definition 5.9. Let $\mathcal{G}_k$ be a finite flat group scheme over $\mathrm{Spec}(k)$. Denote by $\mathbf{D}(\mathcal{G}_k)$ the classical contravariant Dieudonné module of $\mathcal{G}_k$. We define the crystalline Dieudonné module $\mathbb{D}^*(\mathcal{G}_k)$ of $\mathcal{G}_k$ to be the Frobenius twist $k \otimes_{\sigma,k} \mathbf{D}(\mathcal{G}_k)$. Our terminology is justified by [BBM82, 4.2.14].

Remark. If $\mathcal{G}_k = \mathcal{A}_k[p]$ for an Abelian variety $\mathcal{A}/\mathrm{Spec}(k)$ then there is a canonical isomorphism $\mathbb{D}^*(\mathcal{G}_k) = H^1_{cris}(A_k/k)$ sending F to φ, by [BBM82, 2.5].

Proposition 5.10. *Suppose $\mathfrak{M} \in \mathfrak{Mod}_K$. Recall that we have contravariantly associated to $\mathfrak{M}$ a finite flat group scheme $Gr(\mathfrak{M})$ over $\mathcal{O}_K$. Then there is a canonical isomorphism*

$$\mathfrak{M}_{dR} = \mathbb{D}^*(Gr(\mathfrak{M})_k)$$

which sends $1 \otimes \varphi$ to F.

Proof. This is [BCDT01, Theorem 5.1.3, Part 3] together with the relationship between the Breuil module and the Kisin module attached to a finite flat group scheme killed by p ([Kis09, Proposition 1.1.11]). □

5.3. **Kisin modules with G-action.**

Definition 5.11. For a finite group H, we denote by $\mathfrak{Mod}_K^H(E,L)$ the category whose objects are objects of $\mathfrak{M} \in \mathfrak{Mod}_K(E,L)$ equipped with a homomorphism $H \to \mathrm{Aut}(\mathfrak{M})$. The morphisms are given by morphisms in $\mathfrak{Mod}_K(E,L)$ which are compatible with the H-action.

We will find the following elementary lemma useful:

Lemma 5.12. *Let M be a free $E[[u]]$-module of finite rank. Suppose M has a $E[[u]]$-linear action of the torus $T \subset G$. Then*

$$M = \bigoplus_{\lambda} M[\lambda]$$

where λ runs over weights of T and $M[\lambda]$ denotes the subspace of M on which T acts via the character $\sigma_0 \circ \lambda$.

Remark. Note that the direct summands $M[\lambda]$ are again free $E[[u]]$ modules, since $E[[u]]$ is a PID.

Proof. It is enough to show that the action of a generator ξ of the finite group $k_0^\times$ on M can be diagonalised. We have a $E[[u]]$-linear map $[\xi] : M \to M$, which satisfies $[\xi]^{q-1} = \mathrm{id}$. For each $i = 0, ..., q-2$ we define a linear map $e_i : M \to M$ by

$$e_i := \prod_{\substack{j=0 \\ j\neq i}}^{j=q-2} ([\xi] - \xi^j).$$

Then $[\xi]$ restricted to $e_i M$ is multiplication by the scalar ξ^i. We have

$$\bigoplus_{i=0}^{q-2} e_i M = M$$

since if $m \in M$ we observe that the polynomial identity

$$\sum_{i=0}^{q-2} \prod_{\substack{j=0 \\ j\neq i}}^{q-2} (X - \xi^j) = (q-1)x^{q-1}$$

entails

$$\frac{1}{q-1} \sum_{i=0}^{q-2} e_i m = m.$$

□

Now we suppose k is an extension of k_0, that k embeds into E via a fixed embedding σ_0, and that k_L contains k_0. Suppose $\mathfrak{M} \in \mathfrak{Mod}_K^T(E,L)$. Then lemma 5.12 gives weight space decompositions $\mathfrak{M}_i = \bigoplus_\lambda \mathfrak{M}_i[\lambda]$, where the sum is over weights λ of T. Restricting to these weight spaces the descent datum gives maps $[\gamma] : \mathfrak{M}_i[\lambda] \to \mathfrak{M}_{\gamma i}[\lambda]$ (since $k_0 \subset k_L$ the weights are preserved by the action of $\mathrm{Gal}(K/L)$). The map φ restricts to maps $\varphi_i : \mathfrak{M}_i[\lambda] \to \mathfrak{M}_{i-1}[\lambda]$.

Definition 5.13. Let $\lambda : T \to k_0^\times$ be a weight. Then for $\mathfrak{M} \in \mathfrak{Mod}_K^T(E,L)$ we define $\mathfrak{M}[\lambda]$ to be

$$\bigoplus_i \mathfrak{M}_i[\lambda].$$

The weight space $\mathfrak{M}[\lambda]$ is the piece of $\mathfrak{M}$ on which T acts via $\sigma_0 \circ \lambda$.

Lemma 5.14. *With the structures induced from $\mathfrak{M}$, $\mathfrak{M}[\lambda] \in \mathfrak{Mod}_K^T(E, L)$ and we have a direct sum decomposition in $\mathfrak{Mod}_K^T(E, L)$:*

$$\mathfrak{M} = \bigoplus_{\lambda} \mathfrak{M}[\lambda].$$

Proof. This follows from lemma 5.12. ☐

Proposition 5.15. *Let $\mathfrak{M} \in \mathfrak{Mod}_K^G(E, L)$ and suppose we have a $G_{L_\infty} \times G$-equivariant isomorphism $T(\mathfrak{M}[\frac{1}{u}]) \cong \rho \otimes_E \pi$, where ρ and π are representations of G_{L_∞} and G on E-vector spaces and ρ is one-dimensional. Moreover, assume $\pi = L(\mu) \otimes_{k_0, \sigma_0} E$ for some restricted weight $\mu = \lambda_{a,b}$. Then the non-zero weight spaces $\mathfrak{M}[\lambda]$ are isomorphic to each other in $\mathfrak{Mod}_K(E, L)$ and for $i \in \mathbb{Z}/d\mathbb{Z}$ we have $(\mathfrak{M}_{dR})_i \cong L(\mu) \otimes_{k_0, \sigma_0} E$ as an $E[G]$-module.*

Proof. First we show that $(\mathfrak{M}_{dR})_i \cong L(\mu) \otimes_{k_0, \sigma_0} E$ as an $E[G]$-module. Let $\mathbf{k}$ be a finite extension of $k((u))$ inside $k_{\widehat{\mathscr{E}}^{ur}}$ such that

$$T(\mathfrak{M}[\frac{1}{u}]) = (\mathfrak{M}[\frac{1}{u}] \otimes_{k((u))} \mathbf{k})^{\varphi=1}.$$

Observing that we are free to prove the Proposition after extending the coefficient field E, we may assume that the residue field of $\mathbf{k}$, which we denote $k_{\mathbf{k}}$, embeds in E. Note that $\mathbf{k}$ is a $k_{\mathbf{k}}$-algebra, since it is an equicharacteristic local field. We have a canonical isomorphism

$$T(\mathfrak{M}[\frac{1}{u}]) \otimes_{\mathbb{F}_p} \mathbf{k} \cong \mathfrak{M}[\frac{1}{u}] \otimes_{k((u))} \mathbf{k}$$

of free $E \otimes_{\mathbb{F}_p} \mathbf{k}$-modules with G-action. Since

$$E \otimes_{\mathbb{F}_p} \mathbf{k} = \prod_{\tau : k_{\mathbf{k}} \hookrightarrow E} \mathbf{k} \otimes_{k_{\mathbf{k}}, \tau} E$$

(note this is a direct product of local fields) we obtain isomorphisms

$$\mathbf{k} \otimes_{k_{\mathbf{k}}, \tau} T(\mathfrak{M}[\frac{1}{u}]) \cong (\mathfrak{M}[\frac{1}{u}] \otimes_{k((u))} \mathbf{k})_\tau$$

where on the right hand side we take the direct summand corresponding to the factor $\mathbf{k}_{\tau,E} := \mathbf{k} \otimes_{k_{\mathbf{k}}, \tau} E$ of $E \otimes_{\mathbb{F}_p} \mathbf{k}$. Denote the ring of integers of $\mathbf{k}$ by $\mathbf{o}$. The ring of integers in $\mathbf{k}_{\tau,E}$ is then given by $\mathbf{o}_{\tau,E} := \mathbf{o} \otimes_{k_{\mathbf{k}}, \tau} E$. The residue field of $\mathbf{k}_{\tau,E}$ can be naturally identified with E. We define two $\mathbf{o}_{\tau,E}$-lattices $\mathbf{L}_1, \mathbf{L}_2 \subset (\mathfrak{M}[\frac{1}{u}] \otimes_{k((u))} \mathbf{k})_\tau$ by taking $\mathbf{L}_1 = (\mathfrak{M} \otimes_{\mathfrak{S}_1} \mathbf{o})_\tau$, $\mathbf{L}_2 = \mathbf{o} \otimes_{k_{\mathbf{k}}, \tau} T(\mathfrak{M}[\frac{1}{u}])$. Our desired statement follows if we can prove that the reductions of each of these lattices are isomorphic as $E[G]$-modules. But this is immediate from the Brauer–Nesbitt Theorem, since π is irreducible.

Now we can deduce that the non-zero weight spaces $\mathfrak{M}[\lambda]$ are isomorphic to each other in $\mathfrak{Mod}_K(E, L)$. We construct isomorphisms in $\mathfrak{Mod}_K(E, L)$ between $\mathfrak{M}[\mu]$ and each $\mathfrak{M}[\lambda]$ for the other weights λ (recall μ is the highest weight of $L(\mu)$). Observe that $\mathfrak{M}[\lambda] \in \mathfrak{Mod}_K(E, L)$ has rank one. Fix a generator m_μ of this free rank one $\mathfrak{S}_E$-module. For some $x_\mu \in (E \otimes_{\mathbb{F}_p} k_{\widehat{\mathscr{E}}^{ur}})^\times$ we have

$$x_\mu m_\mu \in T(\mathfrak{M}[\mu][\frac{1}{u}]) = T(\mathfrak{M}[\frac{1}{u}])[\mu].$$

Write e_λ for an element of the group algebra $E[G]$ which sends a basis for the highest weight space of $L(\mu)$ to a basis for the λ-weight space (we can assume that the coefficients of e_λ actually lie in $k_0 \subset E$).

Therefore, possibly after rescaling x_λ by an element of $E^\times$, we have $e_\lambda(x_\mu m_\mu) = x_\lambda m_\lambda$. We deduce that $e_\lambda(m_\mu) = (x_\lambda/x_\mu)m_\lambda \in \mathfrak{M}$, so $x_\lambda/x_\mu \in \mathfrak{S}_E$ (and it is a unit in $E \otimes_{\mathbb{F}_p} k((u))$).

Moreover, $e_\lambda \in k_0[G]$ takes the basis vector $(1 \otimes m)_i$ of $(\mathfrak{M}_{dR}[\mu])_i$ to a basis vector of $(\mathfrak{M}_{dR}[\lambda])_i$, so we see that x_λ/x_μ has u-adic valuation 0 for each λ. Therefore we have $x_\lambda/x_\mu \in (\mathfrak{S}_E)^\times$. So e_λ gives an isomorphism $\mathfrak{M}[\mu] \cong \mathfrak{M}[\lambda]$ in $\mathfrak{Mod}_K(E, L)$, since it is an isomorphism of $\mathfrak{S}_E$-modules, commuting with all the other structures. □

6. Applications to regular Serre weights

We again return to the setting of Section 2. We fix uniformisers $\varpi_0 \in K_0$ and $\varpi \in K$ satisfying $\varpi^{q^2-1} = -\varpi_0$. We work with the theory of Kisin modules with respect to the uniformiser ϖ of K — note that this means we are implicitly working with the uniformiser $-\varpi_0$ of K_0 when we discuss Kisin modules with descent data to K_0.

Definition 6.1. Suppose L is a subextension of K/K_0 with residue field containing a quadratic extension of k_0. Define the fundamental character of niveau $2f$ to be

$$\begin{aligned} \omega_{2f} : \mathrm{Gal}(K/L) &\to \mu_{q^2-1}(k) \subset k_L^\times \\ \gamma &\mapsto \gamma(\varpi)/\varpi \bmod \varpi \end{aligned}$$

Similarly, we define a fundamental character of niveau f

$$\begin{aligned} \omega_f : \mathrm{Gal}(K/K_0) &\to \mu_{q-1}(k) \subset k_0^\times \\ \gamma &\mapsto \gamma(\varpi^{1+q})/\varpi^{1+q} \bmod \varpi \end{aligned}$$

and the mod p cyclotomic character

$$\begin{aligned} \omega_1 : \mathrm{Gal}(K/\mathbb{Q}_p) &\to \mu_{p-1}(k) = \mathbb{F}_p^\times \\ \gamma &\mapsto \gamma(1-\zeta_p)/(1-\zeta_p) \bmod \varpi. \end{aligned}$$

Note that if $f = 1$ and $\varpi_0 = p$ our definitions of ω_f and ω_1 coincide, and we have $\omega_f = \omega_{2f}^{1+q}$.

6.1. **Finite flat group schemes and Jacobians of semistable curves.** Let L/K_0 be a finite extension, with $L \subset K_0'$ and recall the semistable model $\widetilde{X}/\mathcal{O}_K$ for $X(\mathfrak{p})/K_0$. Denote the semi-Abelian scheme $\mathrm{Pic}^0(\widetilde{X}/\mathcal{O}_K)$ by $\mathcal{A}^0$ (it is the connected component of the identity in the Néron model of $A := \mathrm{Pic}^0(X(\mathfrak{p})_K)$ over $\mathcal{O}_K$) and let $\mathcal{G}$ be the quasi-finite separated flat group scheme over $\mathcal{O}_K$ given by the p-torsion $\mathcal{A}^0[p]$. We denote by $\mathcal{G}_t \subset \mathcal{G}_f \subset \mathcal{G}$ the toric and finite parts of $\mathcal{G}$. They are finite flat group schemes over $\mathcal{O}_K$, with descent data relative to K_0, since $\mathcal{A}_K^0$ descends to K_0 (the Néron model, together with the toric and finite parts, are all canonical, so are preserved by the descent datum on the generic fibre).

The Grothendieck orthogonality Theorem [Gro72, Exp. IX, Proposition 5.6] tells us that the Weil pairing (composed with the principal polarisation in one of the factors)

$$A[p] \times A[p] \to \mu_p$$

makes the generic fibres $\mathcal{G}_{t,K}$ and $\mathcal{G}_{f,K}$ exact annihilators. We have a canonical isomorphism of G_{K_0}-modules (using the descent datum to give the action of G_{K_0} on the left hand side)

$$\mathcal{G}_K(\overline{K})^* = H^1_{et}(X(\mathfrak{p})_{\overline{K}_0}, \mathbb{Z}/p\mathbb{Z}).$$

Let E be a finite extension of $\mathbb{F}_p$ and suppose we have a $E[G_L]$-submodule V of $H^1_{et}(X(\mathfrak{p})_{\overline{K}_0}, E)$. Since we have $\mathcal{G}_K(\overline{K}) = H^1_{et}(X(\mathfrak{p})_{\overline{K}_0}, \mathbb{Z}/p\mathbb{Z})(1)$ we have $V(1) \subset (\mathcal{G}_K \otimes E)(\overline{K})$. Write H_V for the finite flat E-submodule scheme, with descent datum to L, of $\mathcal{G}_K \otimes E$ with $H_V(\overline{K}) = V(1)$ which we obtain from this embedding. If $H_V \subset \mathcal{G}_{f,K} \otimes E$ we say that H_V is finite.

If H_V is finite, we denote the Zariski closure of H_V in $\mathcal{G}_f \otimes E$ by $\mathcal{H}_V$. It is a finite flat E-module scheme over $\mathcal{O}_K$, with descent datum to L.

Denote by $\mathcal{A}^{ab}_k$ the maximal Abelian quotient of $\mathcal{A}^0_k$. We have a short exact sequence in the Abelian category of fppf sheaves of Abelian groups over $\mathrm{Spec}(k)$.

$$0 \to T \to \mathcal{A}^0_k \to \mathcal{A}^{ab}_k \to 0.$$

Applying the snake lemma to the diagram made up of two copies of this short exact sequence with multiplication by p maps between them we get a short exact sequence

$$0 \to \mathcal{G}_{t,k} \to \mathcal{G}_k \to \mathcal{A}^{ab}_k[p] \to 0.$$

Lemma 6.2. *Suppose H_V is finite. We have*

$$\mathcal{H}_{V,k} \times_{\mathcal{G}_k \otimes E} \mathcal{G}_{t,k} \otimes E = 0$$

if and only if

$$\mathcal{H}_{V,K} \times_{\mathcal{G}_{f,K} \otimes E} \mathcal{G}_{t,K} \otimes E = 0.$$

If these equivalent conditions hold, then there is a natural embedding of crystalline Dieudonné modules with coefficients in E

$$\mathbb{D}^*(\mathcal{H}^\vee_{V,k}) \hookrightarrow \mathbb{D}^*(\mathcal{A}^{ab}_k[p]) \otimes_{\mathbb{F}_p} E.$$

Proof. The equivalence of the two conditions follows from the fact that passing to the closed fibre gives an equivalence of categories between multiplicative finite flat E-module schemes over $\mathcal{O}_K$ and multiplicative finite flat E-module schemes over k. Moreover, the finite flat E-module scheme $\mathcal{H}_V \times_{\mathcal{G}_f \otimes E} \mathcal{G}_t \otimes E$ is zero if and only if its generic fibre is zero.

Now the condition

$$\mathcal{H}_{V,k} \times_{\mathcal{G}_k \otimes E} \mathcal{G}_{t,k} \otimes E = 0$$

implies that $\mathcal{H}_{V,k} \subset \mathcal{A}^{ab}_k[p] \otimes E$, so dually we have a surjection $\mathcal{A}^{ab}_k[p] \otimes E \to \mathcal{H}^\vee_{V,k}$ (we use auto-duality of the Jacobian $\mathcal{A}^{ab}_k$). This gives the desired result. □

We finish this section by recalling the description of T and $\mathcal{A}^{ab}_k$ in terms of the curve $\widetilde{X}_k$. Denote by $\daleth$ the dual graph of $\widetilde{X}_k$ and recall that the normalisation of $\widetilde{X}_k$ is $\overline{Y}_k \coprod \overline{Z}_k$. We have $\mathcal{A}^{ab}_k = \mathrm{Pic}^0(\overline{Y}_k) \times \mathrm{Pic}^0(\overline{Z}_k)$ and $T = H^1(\daleth, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{G}_m$.

Lemma 6.3. *There is a natural identification $\mathbb{D}^*(\mathcal{G}_{t,k}) = k \otimes_{\sigma,k} H^1(\daleth, k)$.*

Proof. Since $\mathcal{G}_{t,k} = H^1(\beth, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mu_p$ we have $\mathbf{D}(\mathcal{G}_{t,k}) = H^1(\beth, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k = H^1(\beth, k)$ (with F acting as $1 \otimes \sigma$). The lemma follows from the definition of $\mathbb{D}^*$. □

Definition 6.4. We say that a $k[G]$-module is *Steinberg* if every irreducible constituent is isomorphic to $L(\lambda_{a,a+q-1}) \otimes_{k_0} k$ for some $a \in \mathbb{Z}$.

Remark. Being Steinberg is invariant under twisting by σ.

The following lemma is well-known in the l-adic case. In our setting it follows from the same explicit computation of the cohomology of the dual graph $\beth$, which we describe below.

Lemma 6.5. *The $k[G]$-module $\mathbb{D}^*(\mathcal{G}_{t,k})$ is Steinberg.*

Proof. This follows from describing the action of G on $H^1(\beth, k)$, since we can apply lemma 6.3. Recall that the connected components of $\overline{Z}_k$ are indexed by pairs (ξ, P) with $\xi \in k_0^\times$ and $P \in \mathbb{P}^1(k_0)$. The connected components of $\overline{Y}_k$ are indexed by pairs (ξ, x) with $\xi \in k_0^\times$ and $x \in X_{ss}(k)$. For each ξ the irreducible components of $\widetilde{X}_k$ corresponding to (ξ, P) and (ξ, x) intersect (for all P and x), and these are the only intersections. We have $\beth = \coprod_\xi \beth_\xi$, where $\beth_\xi$ denotes the dual graph of the connected component of $\widetilde{X}_k$ corresponding to ξ. Fix an orientation of these dual graph by making the components labelled by (ξ, P) the target and those labelled by (ξ, x) the source. Write M_ξ for the vector space of k-valued functions on the set of vertices of $\beth_\xi$. Write N_ξ for the vector space of k-valued functions on the set of edges of $\beth_\xi$, and write d_ξ for the map $M_\xi \to N_\xi$ satisfying $d_\xi f(e) = f(s(e)) - f(t(e))$, where $s(e)$ and $t(e)$ denote the source and target of an edge e. Then $H^1(\beth_\xi, k) = \bigoplus_{\xi \in k_0^\times} \mathrm{Coker}\,(d_\xi)$. It is sufficient to show that every irreducible constituent of the $k[\mathrm{SL}_2(k_0)]$-module $\mathrm{Coker}\,(d_\xi)$ is isomorphic to $L(\lambda_{0,q-1})|_{\mathrm{SL}_2(k_0)} \otimes_{k_0} k$.

We have an $\mathrm{SL}_2(k_0)$-equivariant isomorphism

$$N_\xi = \oplus_{x \in X_{ss}(k)} k^{\mathbb{P}^1(k_0)}$$

where the action of $\mathrm{SL}_2(k_0)$ is given by its action on $k^{\mathbb{P}^1(k_0)}$. The map is given by sending a function f to $(P \mapsto f(e_{x,P}))_x$ where $e_{x,P}$ denotes the edge joining (ξ, x) and (ξ, P). We also have $M_\xi = k^{\mathbb{P}^1(k_0)} \oplus k^{X_{ss}(k)}$. The image of $k^{X_{ss}(k)}$ in N_ξ is given by $\oplus_{x \in X_{ss}(k)} k \cdot \mathbf{1} \subset \oplus_{x \in X_{ss}(k)} k^{\mathbb{P}^1(k_0)}$, where $\mathbf{1} \in k^{\mathbb{P}^1(k_0)}$ denotes the constant function on $\mathbb{P}^1(k_0)$ with value 1. On the other hand, the image of $f \in k^{\mathbb{P}^1(k_0)}$ in N_ξ is given by $(f)_x$. So we see that we have an $\mathrm{SL}_2(k_0)$-equivariant isomorphism

$$\mathrm{Coker}\,(d_\xi) \cong \mathrm{Coker}\left(L(\lambda_{0,q-1}) \otimes_{k_0} k \xrightarrow{\mathrm{diag}} \bigoplus_{x \in X_{ss}(k)} L(\lambda_{0,q-1}) \otimes_{k_0} k \right).$$

□

Corollary 6.6. *The $k_0[G]$ representation $\mathcal{G}_{t,K}(\overline{K}) \otimes_{\mathbb{F}_p} k_0$ is Steinberg. If $V \subset H^1_{et}(X(\mathfrak{p})_{\overline{K}_0}, E)$ contains no irreducible G-constituent isomorphic to $L(\lambda_{a,a+q-1}) \otimes_{k_0} k$ for any $a \in \mathbb{Z}$ then H_V is finite and there is a G, φ and $\mathrm{Gal}(K/K_0')$-equivariant embedding*

$$\mathfrak{Mod}(\mathcal{H}_V^\vee)_{dR} \hookrightarrow [H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)] \otimes_k E.$$

Proof. The first part of the Corollary follows from the proof of lemma 6.5 and fact that the étale finite flat group scheme $\mathcal{G}_t^\vee$ over $\mathcal{O}_K$ satisfies $\mathcal{G}_t^\vee(\overline{K}) = \mathcal{G}_t^\vee(\overline{k}) = H^1(\daleth, \mathbb{F}_p)^*$.

It follows from the Grothendieck orthogonality theorem that the $k_0[G]$ representation

$$(\mathcal{G}_K(\overline{K})/\mathcal{G}_{f,K}(\overline{K})) \otimes_{\mathbb{F}_p} k_0$$

is Steinberg. Therefore the map

$$H_V(\overline{K}) \to (\mathcal{G}_K(\overline{K})/\mathcal{G}_{f,K}(\overline{K})) \otimes_{\mathbb{F}_p} E$$

is the zero map, and $H_V \subset \mathcal{G}_{f,K} \otimes E$.

Finally, we apply lemma 6.2 to conclude. □

6.2. **Serre weights: supersingular case.** In this section L will be the unramified quadratic extension of K_0 inside K. The coefficient field E is, as usual, assumed to be an extension of k. We fix an embedding $\sigma_0 : k \hookrightarrow E$. We have $e(K/L) = q^2 - 1$. Note that $\mathrm{Gal}(K/K_0') = \Gamma$ is the inertia subgroup of $\mathrm{Gal}(K/K_0')$, and is therefore canonically isomorphic to the inertia subgroup of $\mathrm{Gal}(K/L)$. We therefore regard Γ as a subgroup of $\mathrm{Gal}(K/L)$. We denote $e(K_0/\mathbb{Q}_p)$ by e_0 and $e(K/\mathbb{Q}_p)$ by e. We have $e = (q^2 - 1)e_0$.

Definition 6.7. Let $\mu = \lambda_{a,b}$ be a restricted weight. Write $b - a = y = \sum_{i=0}^{f-1} y_i p^i$ with $0 \le y_i < p$. If $f > 1$, we say that μ is *weakly regular* if $y \neq q - 1$ and $y_i \neq 0$ for every $0 \le i \le f - 1$. If $f = 1$, we say that μ is weakly regular if $0 \le y \le p - 3$.

With notation as in the above definition, write $y[i]$ for the integer $\sum_{j=0}^{f-1} y_{j-i} p^j$ where we interpret y_j as y_{f+j} when $j < 0$. We have an isomorphism of G-representations

$$L(\lambda_{a,b})^{(i)} \cong L(\lambda_{ap^i, ap^i + y[i]}).$$

If $\mu = \lambda_{a,b}$ we write $\mu[i]$ for the restricted weight $\lambda_{ap^i, ap^i+y[i]}$. Note that $\mu[i]$ is weakly regular if and only if μ is weakly regular.

Lemma 6.8. *Suppose $\mu = \lambda_{a,b}$ is weakly regular, with $y = b - a$.*

Then any vector in the image of a map $L(\mu) \otimes_{k_0} k \to H^1_{dR}(\overline{Y}_k/k)$ lies in $\mathrm{Fil}^1(H^1_{dR}(\overline{Y}_k/k))$ and has Γ acting via the character $\zeta \mapsto \zeta^{-(y+2+(q+1)(a-1))}$.

Similarly, any vector in the image of a map $L(\mu) \otimes_{k_0} k \to H^1_{dR}(\overline{Z}_k/k)$ has Γ acting via the character $\zeta \mapsto \zeta^{-(q+1)a}$.

Proof. By Proposition 4.8, it suffices to prove the first part of the lemma with $H^1_{dR}(\overline{Y}_k/k)$ replaced by $H^1_{dR}(\overline{C}/k)$. Since $y \neq 0 \bmod p$, $L(\mu)$ does not appear as a constituent of $\Delta(\lambda)_1$ for any restricted weight λ, by lemma 4.13. Also, by lemma 3.10 $L(\mu)$ is not isomorphic to $\mathrm{soc}_G(\Delta(\lambda))$ for $\lambda = \lambda_{a,b}$ with $b - a = q - 2$.

Now corollary 4.16 implies that the vector in question lies in $\mathrm{Fil}^1(H^1_{dR}(\overline{C}/k))(i)$, with

$$L(\mu) = L(\lambda_{i_1+1, i_0+i_1-1}).$$

So we have $y = i_0 - 2$ and $a = 1 + i_1 \mod (q-1)$, which gives the desired answer for the action of Γ.

The second part of the lemma follows from Propositions 4.4 and 3.9. □

We will also need the following stronger regularity condition:

Definition 6.9. We say that a restricted weight $\mu = \lambda_{a,b}$ is *strongly e_0-regular* if $y = b - a = \sum_{i=0}^{f-1} y_i p^i$ satisfies $e_0 \leq y_i \leq p - 1 - e_0$ for every i.

Note that if $e_0 > \frac{p-1}{2}$ then there are no strongly e_0-regular weights. If μ is strongly e_0-regular, it is weakly regular. When $e_0 = 1$ the strongly e_0-regular condition is a little less restrictive than the 'regular' condition appearing in [Gee11] (this corresponds to $1 \leq y_i \leq p - 3$ for all i). On the other hand, the strongly regular condition is more restrictive than the conditions appearing in [Sch08a] (which correspond to $y_i \leq p - 1 - e_0$ for all i).

Proposition 6.10. *Suppose $\mathfrak{M} \in \mathfrak{Mod}_K(E, L)$ and that we have a $G_{L_\infty} \times G$-equivariant isomorphism $T(\mathfrak{M}[\frac{1}{u}]) \cong \rho \otimes_E \pi$, where ρ and π are representations of G_{L_∞} and G on E-vector spaces, ρ is one-dimensional and $\pi = L(\mu) \otimes_{k_0,\sigma_0} E$ for some weakly regular restricted weight $\mu = \lambda_{a,b}$. We write $y = b - a = \sum_{i=0}^{f-1} y_i p^i$ with $0 \leq y_i < p$.*

Moreover, suppose we have a $E \otimes_{\mathbb{F}_p} k$-module embedding

$$\mathfrak{M}_{dR} \hookrightarrow E \otimes_{\mathbb{F}_p} [H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)]$$

which is G-equivariant and $\mathrm{Gal}(K/K'_0) = \Gamma$-equivariant — the action of Γ on $\mathfrak{M}_{dR}$ comes from restricting the descent datum action of $\mathrm{Gal}(K/L)$.

Then the integers $0 \leq k_i < e(K/L) = q^2 - 1$ for each $i \in \mathbb{Z}/2f\mathbb{Z}$ attached to the rank one module $\mathfrak{M}[\mu]$ by Theorem 5.5 satisfy

- *if $k_{[i]} = 0$ mod $q + 1$ then $k_{[i]} = -(q + 1)ap^{f-i}$ mod $q^2 - 1$*
- *if $k_{[i]} \neq 0$ mod $q + 1$ then $k_{[i]} = p^{-1}(y[1 - i] + 2 + (q + 1)(q - 2 - ap^{f+1-i} - y[1 - i]))$ mod $q^2 - 1$.*

Definition 6.11. If $k_{[i]} = 0$ mod $q + 1$ we say that $i - 1$ is an Igusa index. If $k_{[i]} \neq 0$ mod $q + 1$ we say that $i - 1$ is a Drinfeld index.

Proof. For $i \in \mathbb{Z}/d\mathbb{Z}$ denote by v_i the image of $1 \otimes m_i$ in $\mathfrak{M}[\mu]_{dR}$. Since $m_i \in \mathfrak{M}_i$ we have $v_i \in \mathfrak{M}[\mu]_{dR,i-1}$. It follows from Proposition 5.15 that v_i has image a highest weight vector in an $E[G]$-submodule of

$$[H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)] \otimes_{k,\sigma_{i-1}} E$$

which is isomorphic to

$$L(\mu) \otimes_{k_0,\sigma_0} E = L(\mu)^{(f+1-i)} \otimes_{k_0,\sigma_{i-1}} E.$$

If $k_{[i]} = 0$ mod $q + 1$ then v_i has image in

$$[H^1_{dR}(\overline{Z}_k/k)] \otimes_{k,\sigma_{i-1}} E$$

and $i - 1$ is an Igusa index. If $k_{[i]} \neq 0$ mod $q + 1$ then v_i has image in

$$[H^1_{dR}(\overline{Y}_k/k)] \otimes_{k,\sigma_{i-1}} E$$

and $i-1$ is a Drinfeld index. Suppose $i-1$ is an Igusa index. By lemma 6.8 the Γ action on v_i is given by the character $\overline{\eta}^{-(q+1)ap^{f+1-i}}$. So we conclude that $k_{[i]} = -(q+1)ap^{f-i} \bmod q^2-1$ (we divide by p because the Γ-action on $1 \otimes m_i$ is the σ-twist of the Γ action on m_i).

Similarly, suppose $i-1$ is a Drinfeld index. Then by lemma 6.8 the Γ action on v_i is given by the character $\overline{\eta}^{-\left(y[1-i]+2+(q+1)(ap^{f+1-i}-1)\right)}$. So we conclude that

$$k_{[i]} = p^{-1}(q-1-y[1-i]) - (q+1)(ap^{f-i}) \bmod q^2-1.$$

□

We now consider $\mathfrak{M}$ as in the statement of Proposition 6.10, and denote the set of Drinfeld indices by J. For simplicity, we assume $a=0$. We have a congruence $r_{i+1} = pk_{i+1} - k_i \bmod q^2-1$ for every i, and we have defined $s_{i+1} = r_{i+1} - (pk_{i+1} - k_i)$, where we assume $0 \le k_i < q^2-1$ for every i. We want to compute $pk_{i+1} - k_i$ in the four different cases depending on the type of the indices $(i-1,i)$. If $i \in J$ then, mod q^2-1, we have

$$k_{i+1} = p^{-1}(q-1-y[-i]) = p^{-1}\sum_{j=0}^{f-1}(p-1-y_{j+i})p^j = \sum_{j=0}^{f-2}(p-1-y_{j+i+1})p^j + (p-1-y_i)p^{2f-1},$$

whilst if $i \notin J$ we have $k_{i+1} = 0$.

We therefore compute that

- If neither of $i-1, i$ are in J, then $r_{i+1} = s_{i+1} = (q^2-1)\delta_i$ for some δ_i satisfying $0 \le \delta_i \le e_0$.
- If $(i-1,i)$ are both in J then

  $$pk_{i+1} - k_i = (p-1-y_i)(q^2-1) - (p-1-y_{i-1})(p^{2f-1} - p^{f-1}).$$

  Therefore, if $y_{i-1} < p-1$ we get

  $$r_{i+1} = (q^2-1)\delta_i - (p-1-y_{i-1})(p^{2f-1} - p^{f-1})$$

  and

  $$s_{i+1} = -(p-1-y_i)(q^2-1) + (q^2-1)\delta_i$$

  for some δ_i satisfying $1 \le \delta_i \le e_0$.
- If $(i-1,i)$ are both in J and $y_{i-1} = p-1$ then $pk_{i+1} - k_i = (p-1-y_i)(q^2-1)$, $r_{i+1} = (q^2-1)\delta_i$ for some δ_i satisfying $0 \le \delta_i \le e_0$, and $s_{i+1} = -(p-1-y_i)(q^2-1) + (q^2-1)\delta_i$. In fact, $\delta_i \neq 0$, since $r_{i+1} = 0$ contradicts the fact that our vectors lie in Fil^1 (by lemma 6.8). So we obtain exactly the same result as in the previous item.
- If $i \in J$ and $i-1 \notin J$ then

  $$pk_{i+1} - k_i = \sum_{j=0}^{f-2}(p-1-y_{j+i+1})p^{j+1} + (p-1-y_i)q^2$$

  and we conclude that

  $$r_{i+1} = \sum_{j=0}^{f-1}(p-1-y_{j+i})p^j + (q^2-1)\delta_i$$

  for some δ_i satisfying $0 \le \delta_i \le e_0 - 1$, and $s_{i+1} = -(p-1-y_i)(q^2-1) + (q^2-1)\delta_i$.

- if $i \notin J$ and $i-1 \in J$ then

$$pk_{i+1} - k_i = -\sum_{j=0}^{f-2}(p-1-y_{j+i+1})p^j - (p-1-y_i)p^{2f-1}$$

and we conclude that

$$r_{i+1} = -\sum_{j=0}^{f-2}(p-1-y_{j+i+1})p^j - (p-1-y_i)p^{2f-1} + (q^2-1)\delta_i$$

for some δ_i satisfying $1 \leq \delta_i \leq e_0$ (we use the fact that not every y_i is equal to $p-1$), and $s_{i+1} = (q^2-1)\delta_i$.

Definition 6.12. For a subset $J \subset \mathbb{Z}/2f\mathbb{Z}$, and a set of integers $(\delta_i)_{i\in\mathbb{Z}/2f\mathbb{Z}}$, we say that (δ_i) is *allowable* for J if all the following conditions are verified

- $0 \leq \delta_i \leq e_0$ for every i
- if $i-1 \in J$, $i \notin J$ then $\delta_i \geq 1$
- if $i \in J$, $i-1 \notin J$ then $\delta_i \leq e_0 - 1$.

We say that $(J, (\delta_i))$ is a *pre-Serre datum* if (δ_i) is allowable for J, J bijects with $\mathbb{Z}/f\mathbb{Z}$ on reduction mod f and $\delta_i + \delta_{i+f} = e_0$ for all i. We say that a pre-Serre datum $(J, (\delta_i))$ is a *Serre datum* if $\delta_i \geq 1$ whenever $i-1 \in J$ (equivalently, if $\delta_i \leq e_0 - 1$ whenever $i-1 \notin J$).

Given a set J, (δ_i) allowable for J, and a restricted weight $\mu = \mu_{a,b}$ with $b - a = \sum_{i=0}^{f-1} y_i p^i$, we say that a character ψ of I_{K_0} is attached to $(J, (\delta_i), \mu)$ if

$$\psi = \omega_f^{-a} \prod_{i\in J}(\omega_{2f}^{p^i})^{p-1-y_i-\delta_i} \prod_{i\notin J}(\omega_{2f}^{p^i})^{-\delta_i}.$$

The following lemma follows from the computations preceding the above definition (keeping track of a also).

Lemma 6.13. *Let $\mathfrak{M}$ be as in the statement of Proposition 6.10, and denote the set of Drinfeld indices by J. Then there is a set of integers δ_i attached such that (δ_i) is allowable for J and $\rho|_{I_L}$ is attached to $(J, (\delta_i), \mu)$.*

Proof. Everything follows from the preceding computations and Corollary 5.7 (we just discuss the case $a = 0$ for simplicity). We only need to observe that

$$(q^2-1)^{-1}x_0 = -(q^2-1)^{-1}\sum_{i=0}^{2f-1} s_{i+1}p^i = \sum_{i\in J}(p-1-y_i-\delta_i)p^i + \sum_{i\notin J}(-\delta_i p^i).$$

□

The definitions of pre-Serre and Serre data are explained by the following lemma:

Lemma 6.14. *Suppose that $(J, (\delta_i))$ is a pre-Serre datum, $\mu = \lambda_{a,b}$ is a restricted weight and the character ψ of I_{K_0} is attached to $(J, (\delta_i), \mu)$.*

Then J bijects with $\mathbb{Z}/f\mathbb{Z}$ when we reduce its elements mod f, and there exist integers $-1 \leq \epsilon_i \leq e_0 - 1$ for $i \in \mathbb{Z}/2f\mathbb{Z}$, which depend only on i mod f, such that

$$\psi = \omega_f^{-a} \prod_{i \in J} (\omega_{2f}^{p^i})^{-(1+y_i+\epsilon_i)} \prod_{i \notin J} (\omega_{2f}^{p^i})^{-(e_0-1-\epsilon_i)}.$$

The integers ϵ_i are ≥ 0 unless i and $i-1$ are both in J or both in the complement of J.

Moreover, if $(J, (\delta_i))$ is a Serre datum, the ϵ_i are all ≥ 0.

Proof. We set $\epsilon_i = \delta_i - 1$ if both $i-1, i$ are in J and $\epsilon_i = \delta_i$ if $i \in J$, $i - 1 \notin J$. The requirement that $\epsilon_i = \epsilon_{i+f}$ then determines the ϵ_i with $i \not\in J$. The lemma then follows from the definitions. □

We will use the following easy lemma a few times in some calculations:

Lemma 6.15. *Let D be a positive integer and suppose that for $i = 0, ..., D-1$ we have integers α_i with $|\alpha_i| \leq p-1$ such that $\sum_{i=0}^{D-1} \alpha_i p^i = 0$ mod $p^D - 1$. Then one of the following statements holds:*

(1) $\alpha_i = 0$ *for all* i
(2) $\alpha_i = p-1$ *for all* i
(3) $\alpha_i = -(p-1)$ *for all* i.

Proof. Write Σ for the sum $\sum_{i=0}^{D-1} \alpha_i p^i$. We have $|\Sigma| \leq \sum_{i=0}^{D-1} |\alpha_i| p^i$ with equality if and only if the α_i all have the same sign (we regard 0 as positive and negative). We also have $\sum_{i=0}^{D-1} |\alpha_i| p^i \leq \sum_{i=0}^{D-1} (p-1)p^i = p^D - 1$. Since $|\Sigma|$ is a multiple of $p^D - 1$ we therefore have $\Sigma = 0$ or we are in cases (2) or (3).

So we assume $\Sigma = 0$. Suppose some α_i is non-zero. Let I be the largest i such that α_i is non-zero. Then $\alpha_I p^I = \alpha_I p^I - \Sigma = -\sum_{i=0}^{I-1} \alpha_i p^i$, and the modulus of the right hand side satisfies $|\sum_{i=0}^{I-1} \alpha_i p^i| \leq \sum_{i=0}^{I-1} (p-1)p^i = p^I - 1 < |\alpha_I p^I|$. This is a contradiction, so every α_i is zero. □

The next Proposition gives a combinatorial argument, similar to [GS11b, Proposition 4.13], to produce a pre-Serre datum from the allowable set (δ_i) provided by lemma 6.13.

Proposition 6.16. *Suppose $\mathfrak{M}$ is as in the statement of Proposition 6.10. Moreover suppose that the inertial character $\rho\rho^q|_{I_L}$ is given by $\omega_f^{-a-b}\omega_1^{-e_0}$, that $e_0 < p-1$, and that $\rho|_{I_L}$ is not a power of ω_f. Then there exists a pre-Serre datum $(\widetilde{J}, (\widetilde{\delta}_i))$ such that $\rho|_{I_L}$ is attached to $(\widetilde{J}, (\widetilde{\delta}_i), \mu)$.*

Proof. We first claim that it is enough to find an allowable $(\widetilde{J}, (\widetilde{\delta}_i))$ such that $\rho|_{I_L}$ is attached to $(\widetilde{J}, (\widetilde{\delta}_i), \mu)$ and $\widetilde{J}$ bijects with $\mathbb{Z}/f\mathbb{Z}$. We must show that in this situation the condition that $\widetilde{\delta}_i + \widetilde{\delta}_{i+f} = e_0$ is automatically satisfied. The condition on $\rho\rho^q$ implies that

$$\omega_f^{-2a} \prod_{i=0}^{f-1} (\omega_f^{p^i})^{p-1-y_i-\widetilde{\delta}_i-\widetilde{\delta}_{i+f}} = \omega_f^{-a-b}\omega_1^{-e_0}$$

where the left hand side is equal to $\omega_f^{-a-b} \prod_{i=0}^{f-1} (\omega_f^{p^i})^{-\widetilde{\delta}_i-\widetilde{\delta}_{i+f}}$. Therefore we have

$$\prod_{i=0}^{f-1} (\omega_f^{p^i})^{e_0-(\widetilde{\delta}_i+\widetilde{\delta}_{i+f})} = 1.$$

Now $e_0 - (\widetilde{\delta}_i + \widetilde{\delta}_{i+f}) \in [-e_0, e_0]$ and $e_0 < p-1$, so lemma 6.15 implies that $\widetilde{\delta}_i + \widetilde{\delta}_{i+f} = e_0$ for all i. This justifies the claim.

Now we turn to the rest of the Proposition. Lemma 6.13 provides us with an allowable $J, (\delta_i)$ with attached character $\rho|_{I_L}$. Suppose that J does not already biject with $\mathbb{Z}/f\mathbb{Z}$. It suffices to produce an allowable $\widetilde{J}, (\widetilde{\delta}_i)$, with the same attached inertial character, such that $\widetilde{J}$ has more pairs $(i, i+f)$ satisfying $i \in \widetilde{J}$ if and only if $i+f \notin \widetilde{J}$ than J does (repeating this step finishes the proof). Let S be the set of indices i such that $0 \leq i \leq f-1$, $i \in J$ and $i+f \in J$, and let T be the set of indices i such that $0 \leq i \leq f-1$, $i \notin J$ and $i+f \notin J$.

The condition on $\rho\rho^q$ is equivalent to

$$\prod_{i=0}^{f-1}(\omega_f^{p^i})^{\delta_i+\delta_{i+f}}\prod_{i\in S}(\omega_f^{p^i})^{y_i+1-p}\prod_{i\in T}(\omega_f^{p^i})^{p-1-y_i} = \prod_{i=0}^{f-1}(\omega_f^{p^i})^{e_0}$$

which we rewrite as

$$\prod_{i=0}^{f-1}(\omega_f^{p^i})^{\delta_i+\delta_{i+f}-e_0+c_i} = 1,$$

where $c_i = y_i + 1 - p$ if $i \in S$ and $c_i = p - 1 - y_i$ if $i \in T$ (it is zero otherwise).

Each exponent in this product lies in the interval $[-e_0 - p + 1, e_0 + p - 1] \subset [3-2p, 2p-3]$ since $e_0 < p-1$. If follows, as in the proof of [GS11b, Proposition 4.13], that there are integers $a_i \in \{-1, 0, 1\}$ for $i \in [0, f-1]$ such that

$$\delta_i + \delta_{i+f} - e_0 + c_i = -a_i + pa_{i+1}$$

for each i (we always interpret the subscript of an a_i modulo f).

First we suppose that $N := S\coprod T \neq [0, f-1]$. Then we have an interval $I = [j', j] \subset N$ with $j'-1, j+1 \notin N$, where again the index set is interpreted modulo f. We permit the case $j'-1 = j+1$. Then we define a set $\widetilde{J}$ by removing every element of $I \cap J$ from J and then adding every element of $I \cap J^c$ (J^c denotes the complement of J in $[0, 2f-1]$). In other words, we switch the type of all the indexes in I. We define new integers $\widetilde{\delta}_i$ as follows

- If $i \in (j', j]$ set $\widetilde{\delta}_i = \delta_i + c_i + a_i - pa_{i+1}$.
- Set $\widetilde{\delta}_{j'} = \delta_{j'} + c_{j'} - pa_{j'+1}$.
- Set $\widetilde{\delta}_{j+1} = \delta_{j+1} + a_{j+1}$.
- For all other $i \in \{0, ..., 2f-1\}$ set $\widetilde{\delta}_i = \delta_i$.

It is easy to check that $\widetilde{J}, (\widetilde{\delta}_i)$ has the same attached inertial character as $J, (\delta_i)$. It remains to check that it is allowable. For $i \in (j', j]$ we have $\widetilde{\delta}_i + \delta_{i+f} = e_0$, so the conditions imposed on δ_{i+f} by Definition 6.12 imply that $\widetilde{\delta}_i$ satisfies the same conditions (with respect to $\widetilde{J}$). It remains to check that the conditions are verified at j' and $j+1$. Note that we have $c_{j'-1} = c_{j+1} = 0$. So $\delta_i + \delta_{i+f} - e_0 = -a_i + pa_{i+1}$ for $i = j'-1, j+1$. The left hand side of this equality lies in the interval $[-e_0, e_0] \subset [2-p, p-2]$, which forces the right hand side to be equal to zero or ± 1 and so we have $a_{j'} = a_{j+2} = 0$. Now

$$\widetilde{\delta}_{j'} = \delta_{j'} + c_{j'} - pa_{j'+1} = e_0 - \delta_{j'+f}$$

and it is easy to check that $\widetilde{\delta}_{j'}$ will satisfy the desired conditions. Finally, we also have $\widetilde{\delta}_{j+1} = \delta_{j+1} + a_{j+1} = e_0 - \delta_{j+1+f}$, and again we satisfy the desired conditions.

Now we just need to deal with the case where $N = [0, f-1]$. Since $\rho|_{I_L}$ is not a power of ω_f, we must have $\delta_j \neq \delta_{j+f}$ for some j. It can be easily verified that one of the following two constructions gives the desired allowable $\widetilde{J}$, $(\widetilde{\delta}_i)$

- We define $\widetilde{J}$ by switching the type of the indexes $j, ..., j+f-1$, and set $\widetilde{\delta}_i = \delta_i + c_i + a_i - pa_{i+1}$ for i in $(j, j+f-1]$, $\widetilde{\delta}_j = \delta_j + c_j - pa_j$ and $\widetilde{\delta}_{j+f} = \delta_{j+f} + a_j$.
- We define $\widetilde{J}$ by switching the type of the indexes $j+f, ..., j+2f-1$, and set $\widetilde{\delta}_i = \delta_i + c_i + a_i - pa_{i+1}$ for i in $(j+f, j+2f-1]$, $\widetilde{\delta}_{j+f} = \delta_{j+f} + c_{j+f} - pa_j$ and $\widetilde{\delta}_{j+f} = \delta_{j+f} + a_j$.

The point is that if both of these constructions fail to be allowable, we can deduce that $\delta_j = \delta_{j+f}$. ☐

Corollary 6.17. *Let L be the unramified quadratic extension of K_0 inside K. Let $\mu = \lambda_{a,b}$ be a weakly regular restricted weight. Write $b - a = y = \sum_{i=0}^{f-1} y_i p^i$.*

Suppose r is an absolutely irreducible two dimensional representation of G_F with coefficients in a finite extension E of $\mathbb{F}_p$, with $k \subset E$, such that r appears as an $E[G_F]$-submodule of $\mathrm{Hom}_G(L(\mu) \otimes_{k_0} E, H^1_{et}(X(\mathfrak{p})_{\overline{F}}, E))$.

Suppose that $r|_{G_{K_0}}$ is absolutely irreducible. Then there exists a pre-Serre datum $(J, (\delta_i))$ such that $r|_{G_L} = \rho \oplus \rho^q$ where $\rho|_{I_{K_0}}$ is attached to $(J, (\delta_i), \mu)$.

Proof. We have $r|_{G_L} = \rho \oplus \rho^q$ for some E-valued character ρ of niveau $2f$. Our assumptions imply that we have an injective $E[G_{K_0} \times G]$-module map

$$f : r \otimes_{k_0} L(\mu) \hookrightarrow H^1_{et}(X(\mathfrak{p})_{\overline{K}_0}, \mathbb{Z}/p\mathbb{Z}) \otimes E.$$

Set $V = \mathrm{Im}\,(f|_{\rho \otimes_{k_0} L(\mu)})$. Then Corollary 6.6 gives us a finite flat E-module over $\mathcal{O}_K$, with G-action and descent datum to L, denoted $\mathcal{H}_V$, such that if we set $\mathfrak{M} = \mathfrak{Mod}(\mathcal{H}_V^\vee)$, then $\mathfrak{M}$ satisfies the hypotheses of Proposition 6.16 (using [BDJ10, Corollary 2.11 (1)] and [Sch08a, Lemma 3.1] for the condition on $\rho\rho^q$). In particular, since $\mathcal{H}_V(\overline{\mathcal{O}}_K)(-1) = V$, we have $T(\mathfrak{M}[\frac{1}{u}]) = V \cong \rho \otimes_E (L(\mu) \otimes_{k_0} E)$.

We may assume $e_0 < p-1$, since, as observed by Schein [Sch08a], the statement is vacuous for $e_0 \geq p-1$. We now conclude using Proposition 6.16. ☐

Remark. To recover Theorem 1.1 (i), we apply the above corollary together with lemma 6.14.

Under the strong e_0-regularity assumption we can avoid the use of Proposition 6.16 and strengthen the above corollary. The proof is completely elementary given the definition of strongly e_0-regular, our description of $H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)$ and the classification of rank one Kisin modules with coefficients and descent datum (Theorem 5.5).

Theorem 6.18. *We put ourselves in the situation of Proposition 6.10, and make the following additional assumptions:*

- *μ is strongly e_0-regular*
- *the inertial character $\chi_0(\rho\rho^q)|_{I_L}$ is given by $\omega_f^{-a-b}\omega_1^{-e_0}$*

- *$\rho|_{I_L}$ is not a power of ω_f.*

Let $J \subset \mathbb{Z}/2f\mathbb{Z}$ be the set of Drinfeld indices. Then there exist integers $(\delta_i)_{i\in\mathbb{Z}/2f\mathbb{Z}}$ such that $(J,(\delta_i))$ is a Serre datum and $\rho|_{I_L}$ is attached to $(J,(\delta_i),\mu)$.

Proof. Since there are no strongly e_0-regular weights when $e_0 > \frac{p-1}{2}$ we assume that $e_0 \le \frac{p-1}{2}$. We follow the proof of Proposition 6.16, but we will not have to change the allowable $J,(\delta_i)$ given to us by lemma 6.13. As a consequence, we can make use of the fact that if $i-1, i$ are both in J we have $1 \le \delta_i \le e_0$. As before, we obtain $a_i \in \{-1,0,1\}$ for $i \in [0,f-1]$ such that

$$\delta_i + \delta_{i+f} - e_0 + c_i = -a_i + pa_{i+1} \tag{6.1}$$

for each i. We again denote by S the elements of $[0,f-1]$ such that i and $i+f$ are in J, denote by T the elements such that i and $i+f$ are not in J, and denote by N the union $S\coprod T$.

The strongly e_0-regular hypothesis implies that $c_i \in [e_0, p-1-e_0]$ if $i \in T$, $c_i \in [e_0+1-p, -e_0]$ if $i \in S$. We have $c_i = 0$ otherwise. So the left hand side of the equality (6.1) is in the intervals $[0,p-1]$, $[1-p,0]$, $[-e_0,e_0]$ for (respectively) i in T, i in S and i in neither. Applying lemma 6.15 to $\alpha_i = -a_i + pa_{i+1}$ shows that the a_i are all equal. If they are all 1 or all -1, we have $T = [0,f-1]$ and $S = [0,f-1]$ respectively. If $S = [0,f-1]$ then $J = \mathbb{Z}/2f\mathbb{Z}$, so $\delta_i + \delta_{i+f} \ge 2$ for every i and we cannot have $\delta_i + \delta_{i+f} - e_0 + c_i = 1-p$, which is a contradiction. If $T = [0,f-1]$ then for every $i \in \mathbb{Z}/2f\mathbb{Z}$, i is an Igusa index, $y_i = e_0$ and $\delta_i = e_0$. This implies that $\rho|_{I_L}$ is a power of ω_f, contradicting our assumptions. So we conclude that $a_i = 0$ for every i. This forces $\delta_i = \delta_{i+f} = 0$ for $i \in T$ and $\delta_i = \delta_{i+f} = e_0$ for $i \in S$. Since $J,(\delta_i)$ is allowable, this implies that if T is non-empty then $T = [0,f-1]$, and similarly if S is non-empty then $S = [0,f-1]$. We deduce that both S and T are empty, otherwise $\rho|_{I_L}$ is a power of ω_f.

We have now shown the $i \notin J$ if and only if $i+f \in J$ (i.e. i is an Igusa index if and only if $i+f$ is a Drinfeld index), and hence that $\delta_i + \delta_{i+f} = e_0$ (by the argument at the beginning of the proof of Proposition 6.16). This suffices to deduce the Theorem. ☐

Corollary 6.19. *Let L be the unramified quadratic extension of K_0 inside K. Let $\mu = \lambda_{a,b}$ be a strongly e_0-regular weight. Write $b-a = y = \sum_{i=0}^{f-1} y_i p^i$.*

Suppose r is an absolutely irreducible two dimensional representation of G_F with coefficients in a finite extension E of $\mathbb{F}_p$, with $k \subset E$, such that r appears as an $E[G_F]$-submodule of $\mathrm{Hom}_G(L(\mu)\otimes_{k_0} E, H^1_{et}(X(\mathfrak{p})_{\overline{F}}, E))$.

Suppose that $r|_{G_{K_0}}$ is absolutely irreducible. Then there exists a subset $J \subset \mathbb{Z}/2f\mathbb{Z}$, which bijects with $\mathbb{Z}/f\mathbb{Z}$ when we reduce its elements mod f, together with integers $0 \le \epsilon_i \le e_0 - 1$ for $i \in \mathbb{Z}/2f\mathbb{Z}$, which depend only on i mod f, such that $r|_{G_L} = \rho \oplus \rho^q$ where ρ satisfies

$$\rho|_{I_{K_0}} = \omega_f^{-a} \prod_{i\in J} (\omega_{2f}^{p^i})^{-(1+y_i+\epsilon_i)} \prod_{i\notin J} (\omega_{2f}^{p^i})^{-(e_0-1-\epsilon_i)}.$$

Proof. The proof is as for Corollary 6.17, replacing Proposition 6.16 with Theorem 6.18. ☐

6.3. **Serre weights: ordinary case.** We can also deduce results about Serre weights in the ordinary case. In fact, the proofs are almost identical to the supersingular case — this is a result of us working with 'induced' and 'cuspidal' parts of cohomology simultaneously. The fact that the local Galois representation may be non-split introduces a couple of extra steps in the arguments.

Suppose $\mathfrak{M} \in \mathfrak{Mod}_K(E, K_0)$ and that we have a $G_{L_\infty} \times G$-equivariant isomorphism $T(\mathfrak{M}[\frac{1}{u}]) \cong \rho \otimes_E \pi$, where ρ and π are representations of $G_{K_{0,\infty}}$ and G on E-vector spaces, ρ is two-dimensional reducible and $\pi = L(\mu) \otimes_{k_0} E$ for some weakly regular restricted weight $\mu = \lambda_{a,b}$. We write $y = b - a = \sum_{i=0}^{f-1} y_i p^i$ with $0 \le y_i < p$. If $\rho' \subset \rho$ with quotient ρ'' then we get a corresponding sub $k((u))$-vector space of $\mathfrak{M}[\frac{1}{u}]$ and define $\mathfrak{M}'$ to be the intersection of $\mathfrak{M}$ with this subspace in $\mathfrak{M}[\frac{1}{u}]$. This gives a subobject in $\mathfrak{Mod}_K(E, K_0)$, and it is saturated as a $\mathfrak{S}_1$-submodule of $\mathfrak{M}$. We have $T(\mathfrak{M}'[\frac{1}{u}]) \cong \rho' \otimes_E \pi$. The quotient $\mathfrak{M}'' := \mathfrak{M}/\mathfrak{M}'$ is again an object of $\mathfrak{Mod}_K(E, K_0)$, and $T(\mathfrak{M}''[\frac{1}{u}]) \cong \rho'' \otimes_E \pi$. Note that these constructions have well-known analogues for the corresponding finite flat group schemes (taking the Zariski closure of a sub-group scheme of $\mathcal{G}_K$ inside $\mathcal{G}$).

Proposition 6.20. *Suppose we have a $k \otimes_{\mathbb{F}_p} E$-module embedding*

$$\mathfrak{M}_{dR} \hookrightarrow [H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)] \otimes_{\mathbb{F}_p} E$$

which is G-equivariant and Γ-equivariant — the action of Γ on $\mathfrak{M}_{dR}$ comes from restricting the descent datum action of $\mathrm{Gal}(K/K_0)$.

Then the integers $0 \le k_i < e(K/K_0) = q^2 - 1$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ attached to the rank one module $\mathfrak{M}'[\mu]$ by Theorem 5.5 satisfy

- *if $k_{[i]} = 0 \bmod q + 1$ then $k_{[i]} = -(q+1)ap^{f-i} \bmod q^2 - 1$*
- *if $k_{[i]} \neq 0 \bmod q + 1$ then $k_{[i]} = p^{-1}(q - 1 - y[1-i]) - (q+1)(ap^{f-i}) \bmod q^2 - 1$*

and the same for $\mathfrak{M}''[\mu]$.

Proof. For $\mathfrak{M}'$ this immediate from Proposition 6.10. Now we have a $k \otimes_{\mathbb{F}_p} E$-module embedding

$$\mathfrak{M}''_{dR} \hookrightarrow [H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)] \otimes_{\mathbb{F}_p} E/\mathfrak{M}'_{dR}$$

which is G-equivariant and Γ-equivariant. It follows from lemma 3.3 that the argument of lemma 6.8 still applies, since when we quotient out by $(\mathfrak{M}'_{dR})_i$ we do not see any new copies of $L(\mu) \otimes_{k_0, \sigma_0} E$ in the socle of

$$\left([H^1_{dR}(\overline{Y}_k/k) \oplus H^1_{dR}(\overline{Z}_k/k)] \otimes_{\mathbb{F}_p} E/\mathfrak{M}'_{dR}\right)_i .$$

We conclude that we have the same description of the possible $k_{[i]}$'s for $\mathfrak{M}''$. □

Definition 6.21. For a subset $J \subset \mathbb{Z}/f\mathbb{Z}$, and a set of integers $(\delta_i)_{i \in \mathbb{Z}/f\mathbb{Z}}$, we say that (δ_i) is *allowable* for J if all the following conditions are verified

- $0 \le \delta_i \le e_0$ for every i
- if $i - 1 \in J$, $i \notin J$ then $\delta_i \ge 1$
- if $i \in J$, $i - 1 \notin J$ then $\delta_i \le e_0 - 1$.

Given two sets $J', J'' \subset \mathbb{Z}/f\mathbb{Z}$, and sets of integers $(\delta'_i)_{i \in \mathbb{Z}/f\mathbb{Z}}, (\delta''_i)_{i \in \mathbb{Z}/f\mathbb{Z}}$ which are allowable by J' and J'' respectively, we say that $(J', J'', (\delta'_i), (\delta''_i))$ is an *ordinary pre-Serre datum* if J'' is the complement of J' and $\delta'_i + \delta''_i = e_0$ for all i. Note that an ordinary pre-Serre datum is determined by either of the allowable pairs $J', (\delta'_i)$ or $J'', (\delta''_i)$.

We say that an ordinary pre-Serre datum $(J', J'', (\delta'_i), (\delta''_i))$ is an *ordinary Serre datum* if $\delta'_i \geq 1$ for all i such that $i-1 \in J'$ and $\delta'_i \leq e_0 - 1$ for all i such that $i - 1 \notin J'$ (this is equivalent to the corresponding condition on the (δ''_i)).

Given a set J, (δ_i) allowable for J, and a restricted weight $\mu = \lambda_{a,b}$ with $b - a = \sum_{i=0}^{f-1} y_i p^i$, we say that a character ψ of I_{K_0} is attached to $(J, (\delta_i), \mu)$ if

$$\psi = \omega_f^{-a} \prod_{i \in J} (\omega_f^{p^i})^{p-1-y_i-\delta_i} \prod_{i \notin J} (\omega_f^{p^i})^{-\delta_i}.$$

As in the supersingular case, we obtain some allowable sets $J, (\delta_i)$ whenever we are in the setting of Proposition 6.20.

Lemma 6.22. *Let $\mathfrak{M}$ be as in the statement of Proposition 6.20, denote the set of Drinfeld indices for $\mathfrak{M}'$ by J' and the set of Drinfeld indices for $\mathfrak{M}''$ by J''. Then there are sets of integers (δ'_i), (δ''_i) such that $J', (\delta'_i)$ and $J'', (\delta''_i)$ are allowable, $\rho'|_{I_{K_0}}$ is attached to $(J', (\delta'_i), \mu)$ and $\rho''|_{I_{K_0}}$ is attached to $(J'', (\delta''_i), \mu)$.*

We obtain the following analogue of Proposition 6.16:

Proposition 6.23. *Suppose $\mathfrak{M}$ is as in the statement of Proposition 6.20. Moreover suppose that the inertial character $\rho'\rho''|_{I_{K_0}}$ is given by $\omega_f^{-a-b}\omega_1^{-e_0}$, and that $e_0 < p - 1$,*

Then there exists an ordinary pre-Serre datum $\widetilde{J}', \widetilde{J}'', \widetilde{\delta}'_i, \widetilde{\delta}''_i$ such that $\rho'|_{I_{K_0}}$ is attached to $(\widetilde{J}', (\delta'_i), \mu)$ and $\rho''|_{I_{K_0}}$ is attached to $(\widetilde{J}'', (\delta''_i), \mu)$.

Proof. The proof is a simple modification of the proof of Proposition 6.16. The proof that we can ignore the requirement that $\widetilde{\delta}'_i + \widetilde{\delta}''_i = e_0$ is essentially identical to the first part of the proof of Proposition 6.16. Lemma 6.22 provides us with two allowable sets J', δ'_i and J'', δ''_i with attached inertial characters $\rho'|_{I_{K_0}}$ and $\rho''|_{I_{K_0}}$. Let S be the set of indices $i \in \mathbb{Z}/f\mathbb{Z}$ such that $i \in J'$ and $i \in J''$, and let T be the set of indices i such that $i \notin J'$ and $i \notin J''$. When $N := S \coprod T \neq \mathbb{Z}/f\mathbb{Z}$ we proceed exactly as in the proof of Proposition 6.16 and construct the desired set $\widetilde{J}'$ and integers $\widetilde{\delta}'_i, \widetilde{\delta}''_i$.

When $N = \mathbb{Z}/f\mathbb{Z}$, the proof is simpler in the ordinary case. We just define $\widetilde{J}' = \mathbb{Z}/f\mathbb{Z} \backslash J'$, and set $\widetilde{\delta}'_i = \delta'_i + c_i + a_i - pa_{i+1}$ for i in $\mathbb{Z}/f\mathbb{Z}$ (here the integers c_i and a_i are defined exactly as in the proof of Proposition 6.16 — in particular, we could just have written $\widetilde{\delta}'_i = e_0 - \delta''_i$). □

Corollary 6.24. *Let $\mu = \lambda_{a,b}$ be a weakly regular restricted weight. Write $b - a = y = \sum_{i=0}^{f-1} y_i p^i$.*

Suppose r is an absolutely irreducible representation of G_F with coefficients in a finite extension E of $\mathbb{F}_p$, with $k \subset E$, such that r appears as an $E[G_F]$-submodule of

$$\mathrm{Hom}_G(L(\mu) \otimes_{k_0} E, H^1_{et}(X(\mathfrak{p})_{\overline{F}}, E)).$$

Suppose that $r|_{G_{K_0}}$ is reducible. Then there exists a subset $J' \subset \mathbb{Z}/f\mathbb{Z}$ together with integers δ'_i for $i \in \mathbb{Z}/f\mathbb{Z}$, such that we have an ordinary pre-Serre datum $(J', J'', \delta'_i, \delta''_i)$ (here J'' is necessarily the *complement of J' and $\delta''_i = e_0 - \delta'_i$) and $r|_{G_{K_0}} = \begin{pmatrix} \rho' & * \\ 0 & \rho'' \end{pmatrix}$ where $\rho'|_{I_{K_0}}$ is attached to $(J', (\delta'_i), \mu)$ and $\rho''|_{I_{K_0}}$ is attached to $(J'', (\delta''_i), \mu)$.*

Proof. The assumptions of the corollary imply that we have an injective $E[G_{K_0} \times G]$-module map

$$f : r \otimes_{k_0} L(\mu) \hookrightarrow H^1_{et}(X(\mathfrak{p})_{\overline{K}_0}, \mathbb{Z}/p\mathbb{Z}) \otimes E.$$

Set $V = \mathrm{Im}\,(f)$. Then Corollary 6.6 gives us a finite flat E-module over $\mathcal{O}_K$, with G-action and descent datum to K_0, denoted $\mathcal{H}_V$, such that if we set $\mathfrak{M} = \mathfrak{Mod}(\mathcal{H}_V^\vee)$, then $\mathfrak{M}$ satisfies the hypotheses of Theorem 6.25 (we are again using [BDJ10, Corollary 2.11 (1)] and its generalisation to the ramified case, which is proved in the same way as [Sch08a, Lemma 3.1]). In particular, since $\mathcal{H}_V(\overline{\mathcal{O}}_K)(-1) = V$, we have $T(\mathfrak{M}[\frac{1}{u}]) = V \cong r \otimes_{k_0} L(\mu)$.

We may assume $e_0 < p - 1$, since, as observed by Schein [Sch08a], the statement is vacuous for $e_0 \geq p - 1$. We now conclude using Proposition 6.23. □

Remark. To recover the statement of theorem 1.1 (ii) we just apply lemma 6.14.

Theorem 6.25. *Suppose we are in the situation of Proposition 6.20 and make the following additional assumptions:*

- *μ is strongly e_0-regular*
- *if $y = (e_0)\sum_{i=0}^{f-1} p^i$ then we do not have $\rho'|_{I_{K_0}} = \rho''|_{I_{K_0}} = \omega_f^{-b}$*
- *if $y = (p-1-e_0)\sum_{i=0}^{f-1} p^i$ then we do not have $\rho'|_{I_{K_0}} = \rho''|_{I_{K_0}} = \omega_f^{-a}$*
- *the inertial character $\rho'\rho''|_{I_{K_0}}$ is given by $\omega_f^{-a-b}\omega_1^{-e_o}$.*

Let $J' \subset \mathbb{Z}/f\mathbb{Z}$ be the Drinfeld indices for $\mathfrak{M}'$ and let $J'' \subset \mathbb{Z}/f\mathbb{Z}$ be the Drinfeld indices for $\mathfrak{M}''$. Then there exist sets of integers $(\delta'_i)_{i\in\mathbb{Z}/f\mathbb{Z}}, (\delta''_i)_{i\in\mathbb{Z}/f\mathbb{Z}}$ such that $(J', J'', (\delta'_i), (\delta''_i))$ is an ordinary Serre datum, $\rho'|_{I_{K_0}}$ is attached to $(J', (\delta'_i), \mu)$ and $\rho''|_{I_{K_0}}$ is attached to $(J'', (\delta''_i), \mu)$.

Proof. We proceed exactly as in theorem 6.18. Again we assume $e_0 \leq \frac{p-1}{2}$. Lemma 6.22 provides us with allowable sets $J', (\delta'_i), J'', (\delta''_i)$. As in the supersingular case, under the strong regularity hypothesis (together with the additional hypotheses of this theorem), we will not have to modify these sets to obtain the conclusion of Proposition 6.23. As in the proof of theorem 6.18, we obtain $a_i \in \{-1, 0, 1\}$ for $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$\delta'_i + \delta''_i - e_0 + c_i = -a_i + pa_{i+1} \tag{6.2}$$

for each i. Set $S = J' \cap J''$ and set $T = (\mathbb{Z}/f\mathbb{Z}\backslash J') \cap (\mathbb{Z}/f\mathbb{Z}\backslash J'')$. Denote by N the union $S \coprod T$.

The strongly e_0-regular hypothesis implies that $c_i \in [e_0, p-1-e_0]$ if $i \in T$, $c_i \in [e_0+1-p, -e_0]$ if $i \in S$. We have $c_i = 0$ otherwise. So the left hand side of the equality (6.2) is in the intervals $[0, p-1]$, $[1-p, 0]$, $[-e_0, e_0]$ for (respectively) i in T, i in S and i in neither. Therefore lemma 6.15 implies that the a_i are all equal. If they are all 1 or all -1, we have $T = \mathbb{Z}/f\mathbb{Z}$ and $S = \mathbb{Z}/f\mathbb{Z}$ respectively.

If $S = \mathbb{Z}/f\mathbb{Z}$ then $J' = J'' = \mathbb{Z}/f\mathbb{Z}$, so $\delta'_i + \delta''_i \geq 2$ for every i and we cannot have $\delta_i + \delta_{i+f} - e_0 + c_i = 1 - p$, which is a contradiction. If $T = \mathbb{Z}/f\mathbb{Z}$ then for every $i \in \mathbb{Z}/f\mathbb{Z}$, i is an Igusa index for $\mathfrak{M}'$ and $\mathfrak{M}''$,

$y_i = e_0$ and $\delta_i' = \delta_i'' = e_0$. This implies that $\rho'|_{I_{K_0}} = \rho''|_{I_{K_0}} = \omega_f^{-b}$, so we again have a contradiction. So we conclude that $a_i = 0$ for every i. This forces $\delta_i' = \delta_i'' = 0$ and $y_i = p - 1 - e_0$ for $i \in T$ and $\delta_i' = \delta_i'' = e_0$ and $y_i = p - 1 - e_0$ for $i \in S$. Since $J, (\delta_i)$ is allowable, this implies that if T is non-empty then $T = \mathbb{Z}/f\mathbb{Z}$ (hence $\rho'|_{I_{K_0}} = \rho''|_{I_{K_0}} = \omega_f^{-a}$), and similarly if S is non-empty then $S = \mathbb{Z}/f\mathbb{Z}$ (hence $\rho'|_{I_{K_0}} = \rho''|_{I_{K_0}} = \omega_f^{-a}$). We therefore contradict our assumptions unless N is empty.

We have now shown that $i \in J'$ if and only if $i \notin J''$, and hence that $\delta_i' + \delta_i'' = e_0$ (by the argument at the beginning of the proof of Proposition 6.16). This suffices to deduce the Theorem. □

Corollary 6.26. *Assume the following*

- $\mu = \lambda_{a,b}$ *is a strongly* e_0*-regular restricted weight*
- *if* $y = b - a = (e_0)\sum_{i=0}^{f-1} p^i$ *then we do not have* $\rho'|_{I_{K_0}} = \rho''|_{I_{K_0}} = \omega_f^{-b}$
- *if* $y = b - a = (p - 1 - e_0)\sum_{i=0}^{f-1} p^i$ *then we do not have* $\rho'|_{I_{K_0}} = \rho''|_{I_{K_0}} = \omega_f^{-a}$

Suppose r *is an absolutely irreducible representation of* G_F *with coefficients in a finite extension* E *of* $\mathbb{F}_p$*, with* $k \subset E$*, such that* r *appears as an* $E[G_F]$*-submodule of*

$$\mathrm{Hom}_G(L(\mu) \otimes_{k_0} E, H^1_{et}(X(\mathfrak{p})_{\overline{F}}, E)).$$

Suppose that $r|_{G_{K_0}}$ *is reducible. Then there exists a subset* $J \subset \mathbb{Z}/f\mathbb{Z}$ *together with integers* $0 \leq \epsilon_i \leq e_0 - 1$ *for* $i \in \mathbb{Z}/f\mathbb{Z}$ *, such that* $r|_{G_{K_0}} = \begin{pmatrix} \rho' & * \\ 0 & \rho'' \end{pmatrix}$ *where*

$$\rho'|_{I_{K_0}} = \omega_f^{-a} \prod_{i \in J} (\omega_f^{p^i})^{-(1+y_i+\epsilon_i)} \prod_{i \notin J} (\omega_f^{p^i})^{-(e_0-1-\epsilon_i)}$$

and

$$\rho''|_{I_{K_0}} = \omega_f^{-a} \prod_{i \notin J} (\omega_f^{p^i})^{-(1+y_i+\epsilon_i)} \prod_{i \in J} (\omega_f^{p^i})^{-(e_0-1-\epsilon_i)}.$$

Proof. The proof is as for Corollary 6.24, replacing Proposition 6.23 with Theorem 6.25. We set $\epsilon_i = \widetilde{\delta}_i' - 1$ if both $i-1, i$ are in J, $\epsilon_i = \widetilde{\delta}_i'' - 1$ if neither of $i-1, i$ are in J, $\epsilon_i = \widetilde{\delta}_i'$ if $i \in J, i-1 \notin J$ and $\epsilon_i = \widetilde{\delta}_i''$ if $i \notin J, i-1 \in J$. □

7. Serre weights and finite flat models

We present the results on finite flat models for modular Galois representations which we can deduce from the work of the previous section. For simplicity we assume that $e_0 = 1$. We will need the following definition:

Definition 7.1. Suppose $\mathfrak{M} \in \mathfrak{Mod}_K(E, L)$ and π is an $E[G]$-module. Then we define an object $\pi \otimes_E \mathfrak{M} \in \mathfrak{Mod}_K(E, L)$ with G action, by letting the underlying $\mathfrak{S}_E$-module be $\pi \otimes_E \mathfrak{M}$ and defining φ to be $\mathrm{id} \otimes \varphi$. The action of $g \in G$ is given by $g \otimes \mathrm{id}$ and the action of $\gamma \in \mathrm{Gal}(K/L)$ is given by $\mathrm{id} \otimes [\gamma]$.

7.1. **Finite flat models: supersingular case.** We put ourselves in the setting of corollary 6.17, and assume that $e_0 = 1$. Therefore, we let $\mu = \lambda_{a,b}$ be a strongly 1-regular restricted weight — in other words, $b - a = y = \sum_{i=0}^{f-1} y_i p^i$ with $1 \leq y_i \leq p-2$. Suppose r is an absolutely irreducible representation of G_F with coefficients in a finite extension E of $\mathbb{F}_p$, with $k \subset E$, such that r appears as an $E[G_F]$-submodule of

$$\mathrm{Hom}_G(L(\mu) \otimes_{k_0} E, H^1_{et}(X(\mathfrak{p})_{\overline{F}}, E)).$$

Suppose that $r|_{G_{K_0}}$ is absolutely irreducible. We have shown that there exists a subset $J \subset \mathbb{Z}/2f\mathbb{Z}$, such that $r|_{G_L} = \rho \oplus \rho^q$ where ρ satisfies

$$\rho|_{I_{K_0}} = \omega_f^{-a} \prod_{i \in J}^{2f-1} (\omega_{2f}^{p^i})^{-(1+y_i)}.$$

The assumptions of corollary 6.17 imply that we have an injective $E[G_{K_0} \times G]$-module map

$$f : r \otimes_{k_0} L(\mu) \hookrightarrow H^1_{et}(X(\mathfrak{p})_{\overline{K}_0}, \mathbb{Z}/p\mathbb{Z}) \otimes E.$$

Set $W = \mathrm{Im}\,(f)$. Then Corollary 6.6 gives us a finite flat E-module over $\mathcal{O}_K$, with G-action and descent datum to K_0, denoted $\mathcal{H}_W$. We have a sub-module scheme (with descent datum to L) $\mathcal{H}_V \subset \mathcal{H}_W$ corresponding to $\rho \otimes_{k_0} L(\mu) \subset r \otimes_{k_0} L(\mu)$. Similarly, we obtain $\mathcal{H}_{V'} \subset \mathcal{H}_W$ corresponding to $\rho^q \otimes_{k_0} L(\mu)$.

Set $\mathfrak{N} = \mathfrak{Mod}(\mathcal{H}_W^\vee)$, $\mathfrak{M} = \mathfrak{Mod}(\mathcal{H}_V^\vee)$ and $\mathfrak{M}' = \mathfrak{Mod}(\mathcal{H}_{V'}^\vee)$. The proof of Theorem 6.18 determines the invariants attached to the rank one Kisin modules $\mathfrak{M}_0 := \mathfrak{M}[\mu], \mathfrak{M}'_0 := \mathfrak{M}[\mu]$ by Theorem 5.5. In particular, i is an Igusa index for $\mathfrak{M}_0$ if and only if it is a Drinfeld index for $\mathfrak{M}'_0$.

Theorem 7.2. *There is a G-equivariant isomorphism in $\mathfrak{Mod}_K(E, L)$*

$$\mathfrak{N} \cong (L(\mu) \otimes_{k_0} E) \otimes_E \mathfrak{M}_0 \oplus (L(\mu) \otimes_{k_0} E) \otimes_E \mathfrak{M}'_0.$$

Proof. It follows from Proposition 5.15 that $\mathfrak{M} \cong (L(\mu) \otimes_{k_0} E) \otimes_E \mathfrak{M}_0$ and $\mathfrak{M}' \cong (L(\mu) \otimes_{k_0} E) \otimes_E \mathfrak{M}'_0$ (each weight space is isomorphic as a Kisin module and the action of G is determined by its action on $\mathfrak{M}[\frac{1}{u}]$ and $\mathfrak{M}'[\frac{1}{u}]$). It remains to prove that we have an isomorphism $\mathfrak{N} \cong \mathfrak{M} \oplus \mathfrak{M}'$. Since $\mathcal{H}_V$ and $\mathcal{H}_{V'}$ are both closed subschemes of $\mathcal{H}_W$, we have injective maps $\mathfrak{M} \hookrightarrow \mathfrak{N}$ and $\mathfrak{M}' \hookrightarrow \mathfrak{M}$. So we obtain a map $\mathfrak{M} \oplus \mathfrak{M}' \to \mathfrak{N}$ which is an isomorphism when we invert u, since $\mathcal{H}_{V,K} \oplus \mathcal{H}_{V',K} \cong \mathcal{H}_{W,K}$. So it suffices to check that this map is injective when we reduce mod u, or equivalently that it induces an injective map $\mathfrak{M}_{dR} \oplus \mathfrak{M}'_{dR} \to \mathfrak{N}_{dR}$. This follows from the fact that each map $\mathfrak{M}_{dR} \hookrightarrow \mathfrak{N}_{dR}, \mathfrak{M}'_{dR} \hookrightarrow \mathfrak{N}_{dR}$ is injective (since we have a closed immersion of finite flat group schemes over $\mathcal{O}_K$) and the fact that i is an Igusa index for $\mathfrak{M}$ if and only if it is a Drinfeld index for $\mathfrak{M}'$ (and vice versa), so the images of $\mathfrak{M}_{dR}$ and $\mathfrak{M}'_{dR}$ in $\mathfrak{N}_{dR}$ have zero intersection. □

7.2. **Finite flat models: ordinary case.** We put ourselves in the setting of Corollary 6.24, and again assume that $e_0 = 1$. Therefore, we let $\mu = \lambda_{a,b}$ be a strongly 1-regular restricted weight. Write $b - a = y = \sum_{i=0}^{f-1} y_i p^i$ with $1 \leq y_i \leq p-2$ and moreover suppose that $y \neq \sum_{i=0}^{f-1} p^i$ and $y \neq (p-2)\sum_{i=0}^{f-1} p^i$.

Suppose r is an absolutely irreducible representation of G_F with coefficients in a finite extension E of $\mathbb{F}_p$, with $k \subset E$, such that r appears as an $E[G_F]$-submodule of

$$\mathrm{Hom}_G(L(\mu) \otimes_{k_0} E, H^1_{et}(X(\mathfrak{p})_{\overline{F}}, E)).$$

Suppose that $r|_{G_{K_0}}$ is reducible. We have shown that there exists a subset $J \subset \mathbb{Z}/f\mathbb{Z}$ such that $r|_{G_{K_0}} = \begin{pmatrix} \rho' & * \\ 0 & \rho'' \end{pmatrix}$ where

$$\rho'|_{I_{K_0}} = \omega_f^{-a} \prod_{i \in J} (\omega_f^{p^i})^{-(1+y_i)}$$

and

$$\rho''|_{I_{K_0}} = \omega_f^{-a} \prod_{i \notin J} (\omega_f^{p^i})^{-(1+y_i)}.$$

The assumption of the corollary implies that we have an injective $E[G_{K_0} \times G]$-module map

$$f : r \otimes_{k_0} L(\mu) \hookrightarrow H^1_{et}(X(\mathfrak{p})_{\overline{K}_0}, \mathbb{Z}/p\mathbb{Z}) \otimes E.$$

Set $V = \mathrm{Im}\,(f)$. Then Corollary 6.6 gives us a finite flat E-module over $\mathcal{O}_K$, with G-action and descent datum to K_0, denoted $\mathcal{H}_V$. We have a sub-module scheme $\mathcal{H}_{V'} \subset \mathcal{H}'_V$ corresponding to $\rho' \otimes_{k_0} L(\mu) \subset r \otimes_{k_0} L(\mu)$. We denote the quotient by $\mathcal{H}_{V''}$ — we have $\mathcal{H}_{V''}(\overline{K}) = \rho'' \otimes_{k_0} L(\mu)$.

Set $\mathfrak{M} = \mathfrak{Mod}(\mathcal{H}_V^\vee)$, $\mathfrak{M}' = \mathfrak{Mod}(\mathcal{H}_{V'}^\vee)$ and $\mathfrak{M}'' = \mathfrak{Mod}(\mathcal{H}_{V''}^\vee)$. We have a short exact sequence

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$$

where the outer two terms satisfy the assumptions of Proposition 5.15. Set $\mathfrak{M}'_0 = \mathfrak{M}'[\mu]$, $\mathfrak{M}''_0 = \mathfrak{M}''[\mu]$. In the proof of Theorem 6.25 we completely determine the invariants associated to these rank one modules by Theorem 5.5.

Proposition 7.3. *There is a G-equivariant short exact sequence in $\mathfrak{Mod}_K(E, K_0)$*

$$0 \to (L(\mu) \otimes_{k_0} E) \otimes_E \mathfrak{M}'_0 \to \mathfrak{M} \to (L(\mu) \otimes_{k_0} E) \otimes_E \mathfrak{M}''_0 \to 0.$$

Proof. This follows from Proposition 5.15. □

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