Modular Forms: Example Sheet 1

Notation:
As in the lectures, $\mathcal{H}$ denotes the complex upper half plane $\{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. We adopt the standard notational conventions that $q = e^{2\pi i \tau}$ and that an element $\gamma \in \Gamma(1)$ is given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1. Show that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$

$$\mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}.$$

   Show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

   defines a group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$.

2. This exercise gives an alternative derivation of the q-expansion for the Eisenstein series

$$G_k(\tau) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{1}{(c\tau + d)^k}.$$

   (a) Show that $\pi \cot(\pi\tau) = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m$ (where $q = e^{2\pi i \tau}$)

   (b) Using the identity[1]

$$\pi \cot(\pi\tau) = \frac{1}{\tau} + \sum_{d=1}^{\infty}\left(\frac{1}{\tau - d} + \frac{1}{\tau + d}\right),$$

   show that for $k > 2$ an even integer

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

   where $\sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1}$.
   Hint: differentiate the two expressions for $\pi \cot(\pi\tau)$ given in parts a) and b).

3. Define the Bernoulli numbers B_k for $k \geq 0$ by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

   (a) Show that

$$\frac{t}{e^t - 1} + \frac{t}{2}$$

   is an even function and conclude that $B_1 = -1/2$ and $B_k = 0$ for $k > 1$ odd.

   (b) Show that $B_0 = 1$ and $\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$ for all $n > 1$. Deduce that $B_2 = 1/6$, $B_4 = -1/30$ and $B_6 = 1/42$.

[1] If you like that sort of thing, prove this identity using Poisson summation

(c) Using the first identity from Q2 b), show that

$$\pi\tau\cot(\pi\tau) = 1 - 2\sum_{k=1}^{\infty}\zeta(2k)\tau^{2k}$$

(d) Show that

$$\pi\tau\cot(\pi\tau) = \pi i\tau + \sum_{k=0}^{\infty} B_k \frac{(2\pi i\tau)^k}{k!}$$

and conclude that for even integers $k > 0$

$$2\zeta(k) = -B_k\frac{(2\pi i)^k}{k!}.$$

4. Let D be a negative integer, congruent to 0 or 1 mod 4. Consider binary quadratic forms $Q(x,y) = Ax^2 + Bxy + Cy^2$ with $A, B, C \in \mathbb{Z}$ and $B^2 - 4AC = D$. Suppose that $A > 0$ (i.e. the form is positive definite), and that $\gcd(A, B, C) = 1$. Denote by $\mathscr{S}$ the set of all such forms. The group $\Gamma(1)$ acts on $\mathscr{S}$ by $Q \circ \gamma(x,y) = Q(ax + by, cx + dy)$ (note that this is a right action). Denote by $h(D)$ the cardinality of the set of orbits $\mathscr{S}/\Gamma(1)$.

   (a) For $Q(x,y) \in S$ denote by τ_Q the unique element of $\mathcal{H}$ satisfying $Q(\tau_Q, 1) = 0$. Show that τ_Q determines Q uniquely and that $\tau_{Q\circ\gamma} = \gamma^{-1}\tau_Q$.

   (b) Denote by $\widetilde{\mathscr{F}}$ the subset of $\mathcal{H}$ given by

   $$\{|\tau| > 1, |\mathrm{Re}(\tau)| < 1/2\} \cup \{|\tau| \geq 1, \mathrm{Re}(\tau) = -1/2\} \cup \{|\tau| = 1, -1/2 \leq \mathrm{Re}(\tau) \leq 0\}.$$

   We showed in lectures that $\widetilde{\mathscr{F}}$ contains a unique representative of every $\Gamma(1)$-orbit in $\mathcal{H}$. Deduce that every orbit in $\mathscr{S}/\Gamma(1)$ contains a unique *reduced form* $Q^{red}(x,y) = Ax^2 + Bxy + Cy^2$ satisfying $-A < B \leq A < C$ or $0 \leq B \leq A = C$.

5. Let $\tau \in \widetilde{\mathscr{F}}$ (as defined in the previous question). Show that unless $\tau = i$ or $\tau = \omega = -1/2 + \frac{\sqrt{3}}{2}i$, the stabiliser of τ in $\Gamma(1)$ is the order two group $\{\pm I\}$. What are the stabilisers of i and ω?

   Describe the stabiliser of a general element $\tau \in \mathcal{H}$.

6. Suppose we have a set of $f_1, ..., f_n$ of weakly modular functions of level $\Gamma(1)$ with distinct weights $k_1, ..., k_n$. Show that $\{f_1, ..., f_n\}$ is a linearly independent subset of the vector space of meromorphic functions on $\mathcal{H}$.

7. We proved in lectures that the set $\{E_4^a E_6^b : a, b \geq 0, 4a + 6b = k\}$ is a basis for $M_k(\Gamma(1))$. Check that the cardinality of this basis is $\lfloor \frac{k}{12} \rfloor$ if $k = 2$ mod 12 and $\lfloor \frac{k}{12} \rfloor + 1$ otherwise.

8. Let k be an even integer. Show that $\dim_{\mathbb{C}} S_k(\Gamma(1)) = \max\{0, \dim_{\mathbb{C}} M_k(\Gamma(1)) - 1\}$, and moreover that multiplication by $\Delta = \frac{E_4^3 - E_6^2}{1728}$ gives an isomorphism from $M_k(\Gamma(1))$ to $S_{k+12}(\Gamma(1))$.

9. For even $k > 2$ denote by $E_k(\tau)$ the normalised Eisenstein series $\frac{G_k(\tau)}{2\zeta(k)}$. Show (using the dimension formula) that $E_4^2 = E_8$ and deduce that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{i=1}^{n-1}\sigma_3(i)\sigma_3(n-i)$$

for $n \geq 1$.

10. Define $G_2(\tau)$ by the series

$$G_2(\tau) = 2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

It's not hard to check that this series is absolutely and uniformly convergent on compact subsets of $\mathcal{H}$. We denote by $G_2^*(\tau)$ the continuous (but non-holomorphic) function $G_2(\tau) - \frac{\pi}{\mathrm{Im}(\tau)}$. We will make use of the fact that $G_2^*(\gamma\tau) = (c\tau + d)^2 G_2^*(\tau)$ in this exercise (the proof of this will appear in the lecture notes).

(a) Denote by $D(\tau)$ the function defined by the infinite product

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

It converges to a non-vanishing holomorphic function on $\mathcal{H}$. For $f(\tau)$ a non-zero holomorphic function on $\mathcal{H}$, we write $L(f(\tau))$ for $f'(\tau)/f(\tau)$, the logarithmic derivative of f — it is equal to $\frac{d}{d\tau} \log f(\tau)$ and defined wherever f is non-vanishing. If f, g are two non-zero holomorphic functions on $\mathcal{H}$ then $L(f(\tau)) = L(g(\tau))$ if and only if $f = cg$ for some constant c.

Prove that

$$\frac{L(D(\tau))}{2\pi i} = \frac{G_2(\tau)}{2\zeta(2)}.$$

(b) Show that for $\gamma \in \Gamma(1)$

$$L(D(\gamma\tau)) = L((c\tau + d)^{12} D(\tau))$$

and deduce that for each $\gamma \in \Gamma(1)$ there is a non-zero constant $c(\gamma)$ with $D(\gamma\tau) = c(\gamma)(c\tau + d)^{12} D(\tau)$.

(c) Show that the map $\gamma \mapsto c(\gamma)$ of the previous part is a group homomorphism from $\Gamma(1)$ to $\mathbb{C}^\times$. By considering $c(\gamma)$ for generators γ of $\Gamma(1)$, show that $c(\gamma) = 1$ for all γ, and conclude that $D \in S_{12}(\Gamma(1))$.

(d) Deduce from the dimension formula that $\Delta(\tau) = D(\tau)$.

11. Define a sub $\mathbb{Z}$-module $M_k(\Gamma(1), \mathbb{Z})$ of $M_k(\Gamma(1))$ to consist of those modular forms whose q-expansions have integer coefficients. We write $S_k(\Gamma(1), \mathbb{Z})$ for the submodule of cusp forms. We have $\Delta \in S_{12}(\Gamma(1), \mathbb{Z})$, $E_4 \in M_4(\Gamma(1), \mathbb{Z})$ and $E_6 \in M_4(\Gamma(1), \mathbb{Z})$.

(a) Show that multiplication by Δ gives an isomorphism between $M_k(\Gamma(1), \mathbb{Z})$ and $S_{k+12}(\Gamma(1), \mathbb{Z})$.

(b) Show that $M_k(\Gamma(1), \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank equal to the dimension of $M_k(\Gamma(1))$ and moreover a basis is given by:

- $\{E_4^a \Delta^b : a, b \geq 0, 4a + 12b = k\}$ if $k = 0 \bmod 4$
- $\{E_6 E_4^a \Delta^b : a, b \geq 0, 6 + 4a + 12b = k\}$ if $k = 2 \bmod 4$

12. Let Γ be the group generated by the matrices

$$\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$.

(a) Suppose that $c \neq 0$. Show that there is an $n \in \mathbb{Z}$ with

$$\gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

satisfying $|d_1| < |c_1|/2$.

(b) Suppose that $d \neq 0$. Show that there is an $n \in \mathbb{Z}$ with

$$\gamma \begin{pmatrix} 1 & 0 \\ 4n & 1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

satisfying $|c_2| < 2|d_2|$.

(c) Show that $\Gamma = \Gamma_0(4)$. [Hint: Repeatedly apply parts a) and b) to elements of $\Gamma_0(4)$.]

13. This exercise shows that the reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.

(a) Let $a, b, c, d \in \mathbb{Z}$ with a non-zero and $ad - bc = 1 \bmod N$ and consider the matrix

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

(the bars denote reduction mod N). Show that there is an integer b', congruent to b mod N, and coprime to a. [Hint: write $b' = b + tN$ and use the Chinese remainder theorem to find t satisfying some equations modulo the prime divisors of a]

(b) Show that there are integers k, l such that

$$\begin{pmatrix} a & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$$

with $c' = c + kN$ and $d' = d + lN$ and conclude that $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.

14. (a) Let p be prime and n a positive integer. Show that the finite group $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ has cardinality

$$p^{3e}(1 - 1/p^2).$$

(b) Let N be a positive integer. Show that $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ has cardinality

$$N^3 \prod_{p|N} (1 - 1/p^2).$$

(c) Show that the map $\Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b$$

is surjective with kernel $\Gamma(N)$.

(d) Show that the map $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$$

is surjective with kernel $\Gamma_1(N)$.

(e) Deduce that the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ is equal to

$$N \prod_{p|N} (1 + 1/p).$$

Comments and corrections can be sent by email to `jjmn2@cam.ac.uk`. *A number of the exercises are borrowed from Diamond and Shurman's textbook.*