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**Modular Forms: Example Sheet 2**

1. Let
$$R = \bigoplus_{n \geq 0} R_n$$
be a positively graded integral domain. Denote the fraction field $\mathrm{Frac}(R)$ by $K$ and denote the integral closure of $R$ in $K$ by $\widetilde{R}$.

   For $n \in \mathbb{Z}$, denote by $K_n$ the additive subgroup of $K$ consisting of elements of the form $\frac{f}{g}$ with $f \in R_{m+n}$ and $g \in R_m$, with $g$ non-zero and $m \in \mathbb{Z}_{\geq 0}$.

   (a) Show that $K_0$ is a field and that
   $$S := \bigoplus_{n \in \mathbb{Z}} K_n \subset K$$
   is a graded $K_0$-algebra.

   (b) Show that the set of $n \in \mathbb{Z}$ such that $K_n \neq 0$ is an additive subgroup of $\mathbb{Z}$ (hence equal to $q\mathbb{Z}$ for some $q \geq 0$).

   (c) Denote by $t$ a non-zero element of $K_q$. Show that the map $X \mapsto t$ induces an isomorphism
   $$K_0[X, X^{-1}] \cong S.$$
   In particular, $S$ is integrally closed in its field of fractions.

   (d) Show that $\widetilde{R}$ is a graded subring of $S$, i.e. that
   $$\widetilde{R} = \bigoplus_{n \geq 0} \widetilde{R} \cap K_n.$$
   You may use the fact that if $R \subset S$ is a graded inclusion of graded rings then the integral closure of $R$ in $S$ is a graded subring of $S$.

   (e) Finally, show that the graded ring of modular forms
   $$M(\Gamma) = \bigoplus_{k \geq 0} M_k(\Gamma)$$
   is integrally closed in its field of fractions.

2. Suppose a group $G$ acts (by holomorphic automorphisms) on a Riemann surface $X$.[1] Show that $G$ acts properly if and only if the map
$$\begin{aligned} \pi : G \times X &\to X \times X \\ (g, x) &\mapsto (x, gx) \end{aligned}$$
is *proper*. We give $G$ the discrete topology and the products $G \times X$ and $X \times X$ the product topology, and say that a continuous map between topological spaces is proper if the pre-image of every compact set is compact.

3. Let $(x : y)$ be a point of $\mathbb{P}^1(\mathbb{Q})$. We always assume that $x$ and $y$ are coprime integers.

   (a) Show that $\Gamma(N)(x : y) = \Gamma(N)(x' : y')$ if and only if $(x, y) = \pm(x', y') \bmod N$.

   (b) Suppose $x, y \in \mathbb{Z}$ and denote their reductions mod $N$ by $\overline{x}, \overline{y} \in \mathbb{Z}/N\mathbb{Z}$. Show that the following are equivalent:

[1] Alternatively, just assume that $G$ acts continuously on a locally compact and Hausdorff topological space.

- $\overline{x}, \overline{y}$ lift to coprime integers $x', y'$
- $\gcd(x, y, N) = 1$
- $(\overline{x}, \overline{y})$ has order $N$ in the additive group $(\mathbb{Z}/N\mathbb{Z})^2$.

(c) Deduce that the number of cusps of level $\Gamma(N)$ is

$$\frac{1}{2}N^2 \prod_{p|N}(1 - \frac{1}{p^2})$$

for $N > 2$ and 3 for $N = 2$.

(d) Show that each of these cusps has width $N$.

4. Show that for $N > 2$ the modular curve $Y(\Gamma(N))$ has no elliptic points (i.e. for every point $\tau$ of $\mathcal{H}$ the stabiliser of $\tau$ in $\Gamma(N)$ is trivial or $\{\pm I\}$). Deduce that the Riemann surfaces $X(\Gamma(N))$ have genus

$$1 + \frac{d_N(N-6)}{12N}$$

and that

$$\dim M_{2k}(\Gamma(N)) = \frac{(2k-1)d_N}{12} + \frac{d_N}{2N},$$

where

$$d_N = \frac{1}{2}N^3 \prod_{p|N}(1 - \frac{1}{p^2})$$

for $N > 2$ and $d_2 = 6$.

5. Let $N \in \mathbb{Z}_{\geq 1}$ and suppose $(\alpha, \beta) \in (\mathbb{Z}/N\mathbb{Z})^2$ has order $N$.
For integers $k \geq 3$ define

$$G_k^{(\alpha,\beta)}(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2-\{(0,0)\} \\ (m,n)=(\alpha,\beta) \bmod N}} \frac{1}{(m\tau + n)^k}.$$

(a) Show that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$G_k^{(\alpha,\beta)}|_{\gamma,k} = G_k^{(\alpha,\beta)\gamma}$$

where $(\alpha, \beta)\gamma$ is given by the action of $\mathrm{SL}_2(\mathbb{Z})$ on row vectors.
Deduce that $G_k^{(\alpha,\beta)}(\tau)$ is a modular form of weight $k$ and level $\Gamma(N)$, and moreover if $\alpha = 0$, it is of level $\Gamma_1(N)$.

(b) Show that for $k \geq 3$ and $N > 2$ the set $G_k^{(\alpha,\beta)}(\tau)$ gives a basis of the quotient vector space $M_k(\Gamma(N))/S_k(\Gamma(N))$. *You may assume that this space has dimension equal to the number of cusps of $X(\Gamma(N))$ for all $k \geq 3$. We proved this for even $k$ in lectures.*

6. Show that the genus of $X_0(p)$ is equal to $\lfloor \frac{p+1}{12} \rfloor - 1$ if $p = 1 \bmod 12$ and $\lfloor \frac{p+1}{12} \rfloor$ otherwise.

7. Let $f \in M_k(\Gamma(1)) - \{0\}$. By considering the meromorphic differential $\omega(f)$ on $X(\Gamma(1))$, give a proof (different from the one given in lectures) that

$$\mathrm{ord}_\infty(f) + \sum_{\Gamma(1)\tau\in\Gamma(1)\backslash\mathcal{H}} \frac{1}{n_\tau}\mathrm{ord}_\tau(f) = \frac{k}{12}.$$

8. Suppose $f : X \to Y$ is a holomorphic map of compact connected Riemann surfaces. For $y \in Y$, consider the divisor $[y] \in \mathrm{Div}(Y)$. Denote by $f^*[y]$ the divisor $\sum_{x\in f^{-1}(y)} e_x[x]$, where $e_x$ is the ramification degree of the map $f$ at $x$. We also denote by $f^*$ the unique extension of $f^*$ to a map of Abelian groups $\mathrm{Div}(Y) \to \mathrm{Div}(X)$. Let $\omega_0$ be a non-zero meromorphic differential of degree one on $Y$ and show that

$$\mathrm{div}(f^*(\omega_0)) = f^*\mathrm{div}(\omega_0) + \sum_{x\in X}(e_x - 1)[x].$$

Deduce the Riemann-Hurwitz formula for the genus of $X$.

9. Consider the function
$$\phi(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2.$$
Note that $\phi(\tau)^{12} = \Delta(\tau)\Delta(11\tau)$.

Show that for $f \in S_2(\Gamma_0(11))$, $f^{12} = c\phi^{12}$ for $c$ a constant, and deduce that $\phi$ is in $S_2(\Gamma_0(11))$ (moreover, it is a basis for this one-dimensional space).

10. In this question $\theta(\tau)$ denotes the Jacobi $\theta$ function $\sum_{n\in\mathbb{Z}} q^{n^2}$.

   (a) Show that $M_2(\Gamma_0(4))$ has a basis consisting of $\theta(\tau)^4$ and a form $F$ whose $q$-expansion is
   $$\sum_{n>0 \text{ odd}} \sigma_1(n)q^n.$$
   (b) Show that $f = \theta(\tau)^2 \in M_1(\Gamma_1(4))$, and moreover if $\gamma \in \Gamma_0(4)$ then $f|_{\gamma,1} = \chi(\gamma)f$, where $\chi$ is the unique non-trivial character of the group $\Gamma_0(4)/\Gamma_1(4) \cong (\mathbb{Z}/4\mathbb{Z})^\times$.

   (c) Show that $M_1(\Gamma_1(4))$ is one-dimensional, hence spanned by $\theta(\tau)^2$.

11. This long exercise gives a dimension formular for modular forms of odd weight ($\geq 3$).

Suppose $-I \notin \Gamma$. Show that for $\tau \in \mathcal{H}$ the stabiliser of $\tau$ in $\Gamma$ has order 1 or 3.

Let $x \in \mathbb{P}^1(\mathbb{Q})$ with $x = \alpha\infty$, $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, and let $s = \Gamma x$ be the associated cusp. Let $h$ be the width of $s$.

Show that the stabiliser of $x$ in $\Gamma$ is a cyclic group, generated by $\alpha\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\alpha^{-1}$ or $\alpha\begin{pmatrix}-1 & -h\\ 0 & -1\end{pmatrix}\alpha^{-1}$.

In the former case we say that the cusps $s$ is *regular*, in the latter that it is *irregular*. When the cusp is regular, a meromorphic form $f$ of odd weight and level $\Gamma$ has a Fourier expansion in the varaible $q_h = e^{2\pi i\tau/h}$, whilst if the cusp is regular, it only has a Fourier expansion in the variable $q_{2h} = e^{\pi i \tau/h}$. When $s$ is regular, we define $\mathrm{ord}_s(f)$ to be the order of vanishing (in the variable $q_h$) of $f|_{\alpha,k}$ at $\infty$, whilst if $s$ is irregular, we define $\mathrm{ord}_s(f)$ to be half the order of vanishing in the variable $q_{2h}$.

Suppose $-I \notin \Gamma$ and $k$ is odd. For $f$ a non-zero meromorphic form of weight $k$ and level $\Gamma$, define a divisor $\mathrm{div}(f)$ on $X(\Gamma)$ by
$$\mathrm{div}(f) = \sum_{x\in X(\Gamma)} a_x[x]$$
with $a_x = \mathrm{ord}_x(f)/n_x$ for $x \in Y(\Gamma)$ and $a_s = \mathrm{ord}_s(f)$ for $s$ a cusp.

Show that
$$\lfloor \mathrm{div}(f)\rfloor := \sum_{x\in X(\Gamma)} \lfloor a_x\rfloor[x] = \frac{1}{2}\mathrm{div}(\omega(f^2)) + \sum_{n_x=3}\lfloor\frac{k}{3}\rfloor[x] + \sum_{s \text{ regular}}\frac{k}{2}[s] + \sum_{s \text{ irregular}}\frac{k-1}{2}[s].$$

Suppose there exists a non-zero meromorphic form of weight $k$ and level $\Gamma$.[2] Deduce that for $k \geq 3$
$$\dim(M_k(\Gamma)) = (k-1)(g-1) + \lfloor\frac{k}{3}\rfloor r_3 + \frac{k}{2}r_\infty^{\mathrm{reg}} + \frac{k-1}{2}r_\infty^{\mathrm{irr}}$$
where $g$ is the genus of $X(\Gamma)$, $r_3$ is the number of elliptic points of order 3, $r_\infty^{\mathrm{reg}}$ is the number of regular cusps and $r_\infty^{\mathrm{irr}}$ is the number of irregular cusps.

*Comments, queries and corrections can be sent by email to* `jjmn2@cam.ac.uk`.

[2] A proof that such a form exists is described at the end of section 3.6 in Diamond and Shurman.