Partial Frobenius structures, the Tate conjecture, and BSD over function fields

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The Tate conjecture + Drinfeld's lemma

This is work in progress which focuses on marrying two phenomena:

- Tate's conjecture on algebraic cycles, and
- Drinfeld's lemma on varieties in characteristic p.

The Tate conjecture

Let k be a finitely generated field, and let X be a smooth projective variety over k.

Conjecture

The cycle class map

$$A^i(X) \otimes \mathbf{Q}_\ell \to H^{2i}(X_{k^s}, \mathbf{Q}_\ell)(i)^{\mathrm{Gal}(k^s/k)}$$

is surjective.

Here $A^i(X)$ is the Chow group of algebraic cycles of X of codimension i, modulo rational equivalence.
The conjecture is hard because algebraic cycles are difficult to construct.

The Tate conjecture and BSD over function fields

Theorem (Tate, Milne)

Assume the Tate conjecture. Let E be an elliptic curve over a function field K. Then the BSD conjecture holds for E:

$$\operatorname{ord}_{s=1} L(E/K, s) = \operatorname{rk} E(K).$$

The elliptic curve E/K corresponds to an *elliptic fibration* $\mathcal{E} \to X$, where $X/\mathbf{F}_q$ is the curve whose function field is K. The Tate conjecture gets applied to the surface $\mathcal{E}/\mathbf{F}_q$.

Factors of $(1 - q^{1-s})$ in $L(E/K, s)$ correspond to certain $G_{\mathbf{F}_q}$-invariant lines in $H^2(\mathcal{E}_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)(1)$; invoking the Tate conjecture produces cycles in $A^1(\mathcal{E}) = \operatorname{Pic} \mathcal{E}$, which maps onto $E(K)$.

Modularity for elliptic curves over function fields

Once again, let $X/\mathbf{F}_q$ be a curve with function field K, and let E/K be an elliptic curve. Assume E has split multiplicative reduction at a place $\infty \in |X|$, with conductor $N\infty$.

There is a curve $X_0^\infty(N)$ over K, the *Drinfeld modular curve*, parametrizing Drinfeld A-modules with $\Gamma_0(N)$ structure. $(A = H^0(X \backslash \{\infty\}, \mathcal{O}_X).)$

There is a *modular parametrization* $X_0^\infty(N) \to E$ over K, analogous to the case of elliptic curves over $\mathbf{Q}$.

There's even an analytic description of $X_0^\infty(N)_{K_\infty}$ as $\mathcal{H}/\Gamma_0(N)$, where $\mathcal{H} = \mathbf{P}^{1,\mathrm{an}}_{K_\infty} \backslash \mathbf{P}^1(K_\infty)$ is Drinfeld's upper half-plane.

Drinfeld-Heegner points

Recall our uniformization $X_0^\infty(N) \to E$. Let K'/K be a quadratic extension satisfying the Heegner condition with respect to E, so that $\mathrm{ord}_{s=1} L(E/K', s)$ is odd. There are "Drinfeld-Heegner points" $\xi_{K'} \in \mathrm{Div}\, X_0^\infty(N)$ for a quadratic extension K'/K, which can be pushed into E to obtain points $y_{K'} \in E(K')$.

Theorem (Brown, Ulmer, Yun-Zhang)

$L'(E/K', 1) = \mathrm{ht}(y_{K'})$ *up to an explicit nonzero constant. Therefore if E/K' has analytic rank 1, it has Mordell-Weil rank 1. (Tate had already observed that* $\mathrm{rk}_{\mathrm{an}}(E) \geq \mathrm{rk}_{\mathrm{MW}}(E)$*, so there is no need for a Kolyvagin-type theorem.)*

If $L'(E/K', 1) = 0$, then we expect $\mathrm{rk}_{\mathrm{MW}}(E/K') \geq 3$, but the unifomization seems to be of no help constructing points of $E(K')$.

Shtukas and their moduli spaces

In the function field setting, there exists a notion of shtukas with multiple legs, which currently does not exist over number fields. Recall our curve $X/\mathbf{F}_q$.

Definition

Let $S/\mathbf{F}_q$ be a scheme, and let $P, Q\colon S \to X$. An Drinfeld X-shtuka *over S is a pair $(\mathcal{F}, \phi)$, where:*

- *$\mathcal{F}$ is a vector bundle over $X \times_{\mathbf{F}_q} S$*
- *$\phi\colon (\mathrm{id} \times \mathrm{Fr}_S)^*\mathcal{F} \dashrightarrow \mathcal{F}$ is a rational map, which is an isomorphism away from the graphs $\Gamma_P, \Gamma_Q \subset X \times_{\mathbf{F}_q} S$.*

We require that ϕ have a "simple pole" at P and a "simple zero" at Q. These are the *legs* of the shtuka.
In this talk, our vector bundles will have rank 2.

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Let Sht^2 be the moduli stack of Drinfeld X-shtukas ($2 =$ number of legs). This is a Deligne-Mumford stack. The projection $\mathrm{Sht}^2 \to X \times X$ (sending a shtuka to its pair of legs) is relative dimension 2.

It is also possible to add level structures, *e.g.* $\mathrm{Sht}_0^2(N)$ for an effective divisor $N \subset X$ (this means $\Gamma_0(N)$-level structure).

Uniformization by spaces of shtukas?

Now let E/K be a non-isotrivial elliptic curve of conductor N. Let $\mathcal{E} \to X$ be the corresponding elliptic surface. Then $\mathcal{E} \times_{\mathbf{F}_q} \mathcal{E} \to X \times_{\mathbf{F}_q} X$ is a (relative) surface.

We also have the surface $\mathrm{Sht}_0^2(N) \to X \times_{\mathbf{F}_q} X$. Let η be the generic point of $X \times_{\mathbf{F}_q} X$.

Expectation

There exists a cycle in $A^2(\mathrm{Sht}_0^2(N) \times_{X \times X} \mathcal{E} \times_{\mathbf{F}_q} \mathcal{E})$, inducing a nontrivial $\mathrm{Gal}(\overline{\eta}/\eta)$-equivariant map $H^2(\mathrm{Sht}_0^2(N)_{\overline{\eta}}) \to H^2((\mathcal{E} \times_{\mathbf{F}_q} \mathcal{E})_{\overline{\eta}})$.

By Drinfeld, the cohomology of $H^1(\mathcal{E}) \otimes H^1(\mathcal{E}) \subset H^2(\mathcal{E} \times \mathcal{E})$ appears in $H^2(\mathrm{Sht}_0^2(N))$; by the Tate conjecture there should exist an algebraic correspondence inducing this.
Pedantic note: the Tate conjecture might not literally apply to our $\mathrm{Sht}_0^2(N)$, which is not even of finite type; to address this we might instead use a space of D-shtukas, where D/K is a nonsplit quaternion algebra.

Uniformization by spaces of shtukas

Expectation

There exists a nontrivial algebraic correspondence $\mathrm{Sht}_0^2(N) \dashrightarrow \mathcal{E} \times_{\mathbf{F}_q} \mathcal{E}$.

We might call such an E "2-modular".

The big questions are then:

1. Can we find examples of E/K which are 2-modular?
2. If E is 2-modular, can we use the uniformization by $\mathrm{Sht}_0^2(N)$ to solve BSD for E?

An example: $X = \mathbf{P}^1_{\mathbf{F}_2}$, $N = (0) + 2(1) + (\infty)$.

Let's look at the case $X = \mathbf{P}^1_{\mathbf{F}_2}$, $N = (0) + 2(1) + (\infty)$. There is a unique cuspidal automorphic form for GL_2 at this level, and it corresponds to an elliptic curve

$$E_t : \; y^2 + (t+1)xy = x^3.$$

Meanwhile, $\mathrm{Sht}^2_0(N)$ is birational to a K3 elliptic surface of rank 18, defined over $\eta = \operatorname{Spec} \mathbf{F}_2(P, Q)$, with equation

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2,$$

where

$$\begin{aligned} a_1(t) &= (P+1)(Q+1)t \\ a_2(t) &= (P+1)(Q+1)t(t+P)(t+Q) \\ a_3(t) &= (P+1)(Q+1)t(t+P)(t+Q)(t+1)(t+PQ) \end{aligned}$$

An example: $X = \mathbf{P}^1_{\mathbf{F}_2}$, $N = (0) + 2(1) + (\infty)$.

Elkies observed that $(0,0)$ is a 6-torsion section of $\mathrm{Sht}_0^2(N) \to \mathbf{P}^1_t$, and that in fact $\mathrm{Sht}_0^2(N)$ is the universal K3 elliptic surface with 6-torsion section.

Theorem (Elkies)

Working over $\eta = \operatorname{Spec} \mathbf{F}_2(P,Q)$, there exists a finite-to-one map from $\mathrm{Sht}_0^2(N)_\eta$ onto the Kummer surface $\mathrm{Km}(E_P \times E_Q)_\eta$.

Recall that $\mathrm{Km}(E_P \times E_Q)$ is the desingularization of $(E_P \times E_Q)/[-1]$. It is a K3 elliptic surface of rank 18. The cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & E_P \times E_Q \\ \downarrow & & \downarrow \\ \mathrm{Sht}_0^2(N) & \longrightarrow & \mathrm{Km}(E_P \times E_Q) \end{array}$$

now shows that E_t is 2-modular!

Heegner-Drinfeld cycles

Let E/K be an elliptic curve over a function field, and assume that E is 2-modular, so that we have a correspondence $c\colon \mathrm{Sht}_0^2(N) \dashrightarrow \mathcal{E} \times \mathcal{E}$ over $X \times X$.

Let K'/K be a quadratic extension satisfying the Heegner hypothesis. This time $L(E/K', s)$ is even. Then there exists a *Heegner-Drinfeld cycle* $\xi_{K'} \in Z^2(\mathrm{Sht}_0^2(N)_{X' \times X'})$; this is essentially the locus of shtukas with "CM by K'". Let $x_{K'} = c(\xi_{K'})$, so that $x_{K'} \in Z^2(\mathcal{E}' \times \mathcal{E}')$, where $\mathcal{E}' = \mathcal{E} \times_X X'$.

Theorem (Yun-Zhang)

We have $L^{(2)}(E/K', 1) = h(x_{K'})$ up to an explicit nonzero constant.

This is true regardless of whether $L(E/K', 1)$ is 0 or not!! There is a similar formula for higher derivatives.

Theorem (Yun-Zhang)

For the Heegner-Drinfeld cycle $x_{K'} \in A^2(\mathcal{E}' \times \mathcal{E}')$, we have $L^{(2)}(E/K', 1) = h(x_{K'})$ up to an explicit nonzero constant.

Let's suppose $\mathrm{rk}_{\mathrm{an}}(E/K') = 2$. The theorem says that $x_{K'} \in A^2(\mathcal{E}' \times \mathcal{E}')$ is nonzero. On the other hand, BSD would have us believe that there exist classes $R_1, R_2 \in A^2(\mathcal{E}')$ whose images span $E(K')$, and that $L^{(2)}(E/K', 1)$ should relate to the regulator $\det \langle R_i, R_j \rangle$.

This suggests that, up to a constant:

$$x_{K'} = R_1 \otimes R_2 - R_2 \otimes R_1 \in A^1(\mathcal{E}') \otimes A^1(\mathcal{E}') \subset A^2(\mathcal{E}' \times \mathcal{E}').$$

If we knew that $x_{K'}$ belonged to $A^1(\mathcal{E}') \otimes A^1(\mathcal{E}')$, this would be a way of constructing points of Mordell-Weil.
Yun-Zhang already imply that $x_{K'}$ is "alternating in the two legs", but this is not enough to imply that $x_{K'}$ belongs to $A^1(\mathcal{E}') \otimes A^1(\mathcal{E}')$.

Partial Frobenius

There is a feature of this story we have not yet leveraged. The surface $\mathrm{Sht}_0^2(N) \to X \times X$ has a *partial Frobenius structure*. (Away from the diagonal, anyway.) This means an endomorphism Φ_1 of $\mathrm{Sht}_0^2(N)$ making the diagram commute:

$$\begin{array}{ccc} \mathrm{Sht}_0^2(N) & \xrightarrow{\Phi_1} & \mathrm{Sht}_0^2(N) \\ \downarrow & & \downarrow \\ X \times X & \xrightarrow[\mathrm{Fr}_X \times \mathrm{id}]{} & X \times X. \end{array}$$

Similarly for Φ_2, and $\Phi_1\Phi_2 = \Phi_2\Phi_1$ equals absolute Frobenius.

The product $\mathcal{E} \times \mathcal{E}$ has an obvious partial Frobenius structure, namely $\Phi_1 = \mathrm{Fr}_{\mathcal{E}} \times \mathrm{id}$ and $\Phi_2 = \mathrm{id} \times \mathrm{Fr}_{\mathcal{E}}$.

Partial Frobenius structures

Let X_1 and X_2 be nice schemes over $\mathbf{F}_p$. For a scheme $Y \to X_1 \times_{\mathbf{F}_p} X_2$ we may talk of a partial Frobenius (PF) structure on Y: this means a $\mathrm{Fr}_{X_1} \times 1$-linear endomorphism $\Phi_1 \colon Y \to Y$, and similarly a Φ_2, such that $\Phi_1\Phi_2 = \Phi_2\Phi_1 = \mathrm{Fr}_Y$.

The obvious example is the *split structure* $Y = X_1 \times_{\mathbf{F}_p} X_2$, with $\Phi_1 = \mathrm{Fr}_{X_1} \times 1$, etc.

Another example: $X_i = \operatorname{Spec} \mathbf{F}_p[t_i^{\pm 1}]$, $i = 1, 2$, so that $X_1 \times_{\mathbf{F}_p} X_2 = \operatorname{Spec} \mathbf{F}_p[t_1^{\pm 1}, t_2^{\pm 2}]$. Let $Y = \operatorname{Spec} \mathbf{F}_p[t_1^{\pm 1}, t_2^{\pm 2}, y]/(y^{p-1} = t_1 t_2)$. There is a PF structure with $\Phi_1(y) = t_1 y$, $\Phi_2(y) = t_2 y$.

This PF structure is nonsplit. However it is a quotient of a split structure: $Y = \tilde{Y}/G$, where $\tilde{Y} = X_1' \times_{\mathbf{F}_p} X_2'$ carries the split PF structure, each $X_i' \to X_i$ is an étale μ_{p-1}-torsor, and $G = \mu_{p-1}$ acting diagonally.

Drinfeld's lemma

Theorem (Drinfeld)

Let $Y \to X_1 \times X_2$ be a finite étale morphism with PF structure. Then $Y \cong (\tilde{X}_1 \times \tilde{X}_2)/H$, where $\tilde{X}_i \to X$ are finite Galois, and $H \subset \mathrm{Gal}(\tilde{X}_1/X_1) \times \mathrm{Gal}(\tilde{X}_2/X_2)$.

In other words, finite étale PF structures are all quotients of split ones.

Other PF structures $Y \to X_1 \times X_2$, like our shtuka moduli spaces $\mathrm{Sht}_0^2(N)$, are much more complicated. But Drinfeld's lemma shows that the cohomology $H^i(Y_{\overline{\eta}}, \mathbf{Q}_\ell)$, a priori admitting only an action of $\pi_1(X_1 \times X_2)$, actually admits an action of $\pi_1(X_1) \times \pi_1(X_2)$.

PF structures on abelian varieties

Let X_1 and X_2 be (not necessariy projective) smooth curves over $\mathbf{F}_p$.

Here's a way to construct an abelian scheme over $X_1 \times X_2$ with PF structure: choose an abelian scheme $A_1 \to X_1$, and an étale G-torsor $X_2' \to X_2$. Get G to act on A_1, and define $Y = (A_1 \times X_2')/G$, with G acting diagonally. Then Y may not be split, but its pullback to $X_1 \times X_2'$ is.

In fact, any abelian scheme over $X_1 \times X_2$ with PF structure becomes split over $X_1' \times X_2'$, where $X_i' \to X_i$ is some finite étale cover! (Think about each irreducible summand of the $\pi_1(X_1) \times \pi_1(X_2)$-module $H^1(A_{\overline{\eta}}, \mathbf{Q}_\ell)$. It must be of the form $\rho_1 \boxtimes \rho_2$, where the weights of ρ_1 and ρ_2 sum up to 1.)

Set-up for the PF Tate conjecture

Let $Y \to X_1 \times X_2$ be a projective and smooth, and equipped with a PF structure. Then $H^i(Y_{\overline{\eta}}, \mathbf{Q}_\ell)$ is a representation of $\pi_1(X_1) \times \pi_1(X_2)$ by Drinfeld's lemma.

Define the Tate twist $\mathbf{Q}_\ell(r_1, r_2)$ as the exterior tensor product $\mathbf{Q}_\ell(r_1) \boxtimes \mathbf{Q}_\ell(r_2)$, a representation of $\pi_1(X_1) \times \pi_1(X_2)$.

On the algebraic cycle side, F_1 and F_2 act on $A^r(Y)$, with F_1F_2 acting as p^r. Let $A^{r_1,r_2}(Y)$ denote the subgroup where F_i acts as p^{r_i}.

Thus if $Y = Y_1 \times Y_2$ is split, then $A^{r_1,r_2}(Y)$ contains the image of $A^{r_1}(Y_1) \otimes A^{r_2}(Y_2)$.
For each pair (r_1, r_2) with $r_1 + r_2 = r$ we have the cycle class map

$$A^{r_1,r_2}(Y) \otimes \mathbf{Q}_\ell \to H^{2r}(Y_{\overline{\eta}}, \mathbf{Q}_\ell)(r_1, r_2)^{\pi_1(X_1)\times\pi_1(X_2)}.$$

The PF Tate conjecture / splitting of cycles

Let $Y \to X_1 \times X_2$ be a smooth projective PF structure. Evidence is admittedly scant, but I can't resist suggesting these two conjectures:

Conjecture (PF Tate)

The cycle class map

$$A^{r_1,r_2}(Y) \otimes \mathbf{Q}_\ell \to H^{2r}(Y_{\overline{\eta}}, \mathbf{Q}_\ell)(r_1, r_2)^{\pi_1(X_1)\times\pi_1(X_2)}$$

is surjective.

Conjecture (Splitting of cycles)

If $Y = Y_1 \times Y_2$ *is a split PF structure, then*

$$A^{r_1}(Y_1) \otimes A^{r_2}(Y_2) \to A^{r_1,r_2}(Y_1 \times Y_2)$$

is surjective.

PF Tate conjecture

Conjecture (PF Tate)

The cycle class map

$$A^{r_1,r_2}(Y) \otimes \mathbf{Q}_\ell \to H^{2r}(Y_{\overline{\eta}}, \mathbf{Q}_\ell)(r_1, r_2)^{\pi_1(X_1)\times\pi_1(X_2)}$$

is surjective.

At the very least, when $Y = Y_1 \times Y_2$ is a split structure, the Künneth formula shows that Tate implies PF Tate, and indeed that the RHS is spanned by $A^{r_1}(Y_1) \otimes A^{r_2}(Y_2)$.

For an abelian scheme Y with PF structure, we have seen that Y becomes split after passage to a finite étale cover $X_1' \times X_2'$. The Tate conjecture for divisors ($r = 1$) is known for abelian varieties (Faltings/Zarhin), so the PF Tate conjecture is true unconditionally in this case.

Consequences of the PF Tate conjecture

Let K be the function field of a curve $X/\mathbf{F}_q$, and let E/K be an elliptic curve of conductor N.

The PF Tate conjecture predicts an algebraic correspondence $c\colon \mathrm{Sht}_0^2(N) \dashrightarrow \mathcal{E} \times \mathcal{E}$ over $X \times X$, which is in a sense equivariant for the PF structures on either side. Let's call this state of affairs "E is 2-modular + EPF". Then c induces maps $A^{r_1,r_2}(\mathrm{Sht}_0^2(N)) \to A^{r_1,r_2}(\mathcal{E} \times \mathcal{E})$.

Now suppose K'/K is quadratic, satisfying the Heegner condition with respect to N, such that E/K' has analytic rank 2.

The Drinfeld-Heegner cycle $\xi_{K'}$ is PF-stable on the nose! So its class lies in $A^{1,1}(\mathrm{Sht}_0^2(N))$. Its image $x_{K'} = c(\xi_{K'})$ lies in $A^{1,1}(\mathcal{E} \times \mathcal{E})$.

Consequences of the PF Tate conjecture

Suppose that E/K' has analytic rank 2.

The Drinfeld-Heegner cycle $\xi_{K'}$ lies in $A^{1,1}(\mathrm{Sht}_0^2(N))$. Its image $x_{K'} = c(\xi_{K'})$ lies in $A^{1,1}(\mathcal{E} \times \mathcal{E})$.

Under the splitting cycles conjecture, $x_{K'}$ lies in the image of $A^1(\mathcal{E}) \otimes A^1(\mathcal{E})$. By Yun-Zhang, $x_{K'}$ actually lies in the antisymmetric part of $A^2(\mathcal{E} \times \mathcal{E}) \otimes \mathbf{Q}$, which means it comes from an element of $\wedge^2 A^1(\mathcal{E}) \otimes \mathbf{Q}$. (Reasoning: when the 2 legs collide, $x_{K'}$ becomes a Drinfeld-Heegner point coming from a space of shtukas with one leg. But since $L'(E/K', 1) = 0$, this latter point is torsion.)

Also by Yun-Zhang, the height of $x_{K'}$ is nonzero. This is enough to imply that the Mordell-Weil rank of E/K' is 2.

Conclusion

This kind of strategy should work for any rank r.

Theorem

Assume that E/K is r-modular + EPF, and that the splitting cycles conjecture holds for $\mathcal{E}^r$. If $\mathrm{rk}_{\mathrm{an}}(E/K') = r$, then $\mathrm{rk}_{\mathrm{MW}}(E/K') = r$.

Unfortunately, I do not know whether the example $E/\mathbf{F}_2(t)$ is 2-modular + EPF (someone please help me verify!).

Moral: assume a strong (PFE) version of the Tate conjecture (and in particular assume BSD). Let E/K' have rank r. Then the Heegner-Drinfeld cycle coming from $\mathrm{Sht}_0^r(N)$ (r-legged shtukas) spans the one-dimensional space $\wedge^r E(K')$.

Thank you for listening!