A geometric Jacquet–Langlands for mod p Hilbert modular forms, following Diamond–Kassaei–Sasaki part 2

The aim of this talk is to discuss a generalization the following result due to Serre and Gross:

1 Introduction

Let p be a prime. There is a Hecke equivariant short exact sequence

$$0 \to H^0(\overline{X}_1(N), \delta^m \omega^{p+1-m}(-C)) \to H^0(\overline{X}_1(Np), \overline{\mathcal{K}})^{\chi_m} \to H^0(\overline{X}_1(N), \omega^{m+2}(-C)) \to 0$$

where $\overline{X}$ is the special fiber of the compactified modular $X_1(N)$ of tame level N, ω the usual Hodge bundle, C the divisor of cusps, δ a trivial sheaf which twist the action of T_q by q, and finally, $\overline{\mathcal{K}}$, the dualizing sheaf of $\overline{X}_1(Np)$. This can be seen as a geometric counterpart to the sequence of local systems on $\Gamma_1(N)\backslash\mathcal{H}$:

$$0 \to \det^m \otimes \mathrm{Sym}^{p-1-m}\mathbb{F}_p^2 \to Ind_P^{GL_2(\mathbb{F}_p)}(1 \otimes \chi_m) \to \mathrm{Sym}^m \mathbb{F}_p^2 \to 0.$$

where P denotes the upper triangular matrices and $(1 \otimes \chi_m)$ is the character

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto d^m.$$

The work of DS17 suggests that there should be a generalization to Hilbert modular forms. For a totally real field F in which p is unramified, there should be a link between mod p Hilbert modular forms of parallel weight 2, with Iwahori level $U_1(p)$ and character $\chi : (\mathcal{O}_F/p)^\times \to \overline{\mathbb{F}}_p^\times$, to mod p quaternionic modular forms of weights given by Jordan-Holder factors of $V_\chi = Ind_P^{GL_2(\mathcal{O}_F/p)}(1 \otimes \chi)$. where P denotes the upper triangular matrices.

Recall that the irreducible representations of $GL_2(\mathcal{O}_F/p)$ are of the form

$$V_{m,n} = \bigotimes_{\theta \in \overline{\Theta}_p} \det^{m_\theta} \otimes \mathrm{Sym}^{n_\theta} V_\theta$$

where $(m, n) \in \mathbb{Z}^\Theta \times \mathbb{Z}^\Theta$ satisfy $0 \le m_\theta, n_\theta \le p-1$ with at least one $m_\theta < p-1$ and $V_\theta = \overline{\mathbb{F}}_p^2$ with $GL_2(\mathcal{O}_F/p)$ acting via θ.

Recall the following:

Theorem 1.1. *There is a decreasing filtration*

$$V_\chi = Fil^0 V_\chi \supset Fil^1 V_\chi \supset \cdots \supset Fil^{d+1} V_\chi = 0$$

on $V_\chi = Ind_P^{GL_2(\mathcal{O}_F/p)}(1 \otimes \chi)$ *such that* $gr^j V_\chi \simeq \oplus_{|J|=j} V_{\chi,J}$ *where the* $V_{\chi,J}$ *are irreducible or* 0.

We will explain the following main results from DKS20:

Theorem 1.2 (Theorem B)**.** *For sufficiently small $U \subset GL_2(\mathbb{A}_{F,f})$ containing $GL_2(\mathcal{O}_{F,p})$ there is a Hecke equivariant spectral sequence*

$$E_1^{j,i} = \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J}) \Rightarrow H^{i+j}(\overline{Y}_1(p), \mathcal{K})^{\chi}.$$

Corollary 1.3 (Corollary C)**.** *In particular there is a length $d+1$, Hecke equivariant filtration on $H^0(\overline{Y}_1(p), \mathcal{K})^{\chi}$ with a Hecke equivariant embedding*

$$gr^j \left(H^0(\overline{Y}_1(p), \mathcal{K})^{\chi}\right) \hookrightarrow \bigoplus_{|J|=j} H^0(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J}).$$

I will refrain from defining these objects for now. The space $H^0(\overline{Y}_1(p), \mathcal{K})^{\chi}$ is our definition of parallel weight 2 mod p Hilbert modular forms of Iwahori level $U_1(p)$ and the $H^0(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J})$ are the mod p quaternionic modular forms of weight $V_{\chi,J}$ as introduced above.

2 Models of Shimura varieties

We start by introducing the necessary notation and recapping the contents of last week.

Let F be a totally real field of degree d and p be a prime which is unramified in F. Let Θ denote the set of embeddings $\tau : F \to \overline{\mathbb{Q}}$ which we identify with $\Theta_p = \mathrm{Hom}(F, \overline{\mathbb{Q}}_p)$ via a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and $\overline{\Theta}_p = \mathrm{Hom}(\mathcal{O}_F/p, \overline{\mathbb{F}}_p)$. We write $\mathbb{A}_{F,f}$ and $\mathbb{A}_{F,f}^{(p)}$ for the finite adèles over F and prime to p finite adèles over F respectively. For any ring R, we write $R_p = R \otimes \mathbb{Z}_p$. For any even subset $\Sigma \subset \Theta$ we let $B = B_\Sigma$ be the unique (up to isomorphism) quaternion algebra over F which is ramified exactly at the places in Σ. We let G_Σ be the algebraic group over $\mathbb{Q}$ whose R-points are given by

$$G_\Sigma(R) = (B \otimes_{\mathbb{Q}} R)^{\times}.$$

We also let $G = \mathrm{Res}_{F/\mathbb{Q}}(GL_2)$.

2.1 Hilbert modular varieties

For a sufficiently small open compact subgroup $U \subset GL_2(\mathbb{A}_f)$ which contains $GL_2(\mathcal{O}_{F,p})$ we have a canonical integral model over $\mathbb{Z}_p$ for the Shimura variety of level U for G which we denote $Y_U(G)$. It is obtained as an étale quotient of a (union of) PEL Shimura variety $\widetilde{Y}_U(G)$. This variety is given as the moduli space which classifies isomorphism classes of tuples $(A/S, \iota, \lambda, \eta)$ where:

- S is a $\mathbb{Z}_{(p)}$-scheme and A/S an abelian scheme of dimension d with a ring embedding $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}_S(A)$ such that $\mathrm{Lie}(A/S)$ is locally free of rank one over $\mathcal{O}_F \otimes \mathcal{O}_S$,
- λ is a prime to p quasi-polarization whose associated Rosati involution fixes ι,
- η is a level structure for U^p structure on A.

Both $\tilde{Y}_U(G)$ and $Y_U(G)$ are smooth varieties of relative dimension d over $\mathbb{Z}_p$.

Last week's result involved Shimura varieties for G of Iwahori level $U_0(p)$. We will also need Iwahori level $U_1(p)$ which we recall now:

Let

$$U_1(p) = \{g \in U \,|\, g_p \cong \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod p\} \subset U_0(p) = \{g \in U \,|\, g_p \cong \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p\}.$$

We define $Y_{U_0(p)}(G)$ as a quotient of the moduli space $\tilde{Y}_{U_0(p)}(G)$ of isomorphism classes of tuples $(\underline{A_1}, \underline{A_2}, f)$ over S where

- $\underline{A_i}$ define points of $\widetilde{Y}_U(G)$.
- $f : A_1 \to A_2$ is a degree p^d $\mathcal{O}_F$-linear isogeny such that f respects the level structures, $f^\vee \circ \lambda_2 \circ f = p\lambda_1$ and $H = \ker f$ decomposes as $H = \prod_{v|p} H_v$ where $H_v \subset A_1[v]$ has rank $p^{[F_v:\mathbb{Q}_p]}$.

It is a local complete intersection of dimension d and so Cohen-Macaulay.

We also define $Y_{U_1(p)}(G)$ as a quotient of the moduli space $\tilde{Y}_{U_1(p)}(G)$ of isomorphism classes of tuples (f, P) over S where

- f defines a point of $\widetilde{Y}_{U_0(p)}(G)$.
- $P \in H(S)$ is a $\mathcal{O}_F/p$ Drinfeld generator for H in the sense of Katz-Mazur.

The forgetful map $h : Y_{U_1(p)}(G) \to Y_{U_0(p)}(G)$ is finite flat so in particular $Y_{U_1(p)}(G)/\mathcal{O}$ is Cohen-Macaulay as well.

2.2 Quaternionic Shimura varieties

Similarly, for a sufficiently small open compact subgroup $U \subset G_\Sigma(\mathbb{A}_f)$ which contains $\mathcal{O}_{B,p}^\times \simeq GL_2(\mathcal{O}_{F,p})$, we have a canonical integral model over $\mathbb{Z}_p$ for the Shimura variety of level U for G_Σ which we denote $Y_U(G)$. Similarly, this obtained as a quotient of an auxiliary Unitary Shimura variety. It is smooth of relative dimension $d - \#\Sigma$.

3 Automorphic vector bundles

3.1 Building blocks

We keep the same notation and assumptions as the previous section.

Over $\overline{S} = \widetilde{Y}_U(G) \times \overline{\mathbb{F}}_p$ we have the universal abelian scheme $s : A \to \overline{S}$ with its Hodge filtration

$$0 \to s_*\Omega^1_{A/S} \to \mathcal{H}^1_{dR}(A/S) \to R^1s_*\mathcal{O}_A \to 0$$

which yields for each embedding $\tau : \mathcal{O}_F \to \overline{\mathbb{Z}}_p$, by taking isotypic components, a sequence of locally free sheaves

$$0 \to \tilde{\omega}_\tau \to \tilde{\mathcal{V}}_\tau \to \tilde{v}_\tau \to 0$$

of ranks one, two and one respectively. We also define $\tilde{\delta}_\tau = \wedge^2\tilde{\mathcal{V}}_\tau$. If we take U to be "p-neat", that is $\alpha - 1 \in p\mathcal{O}_F$ for all $\alpha \in U \cap \mathcal{O}_F^\times$, we can descend these sheaves to $\overline{Y} = Y_U(G) \times \overline{\mathbb{F}}_p$ which we denote by removing the tilde.

Similarly, only under the assumption again that $U \subset G_\Sigma(\mathbb{A}_f)$ is has level prime to p, we can define analogous line bundles ω_θ for $\theta \notin \Sigma$, and rank two vector bundles $\mathcal{V}_\theta$ for all $\theta \in \Theta$. We also write $\delta_\theta = \wedge^2\mathcal{V}_\theta$.

On the geometric special fiber $\overline{Y}_0(p) = Y_0(p) \times \overline{\mathbb{F}}_p$, we have a decomposition into smooth closed subschemes of pure dimension d, $i_J : \overline{Y}_J \to \overline{Y}_0(p)$ indexed by subsets $J \subset \Theta$ given as the descent of

$$V(\{\mathrm{Lie}(f)_\theta\}_{\theta \notin J} \cup \{\mathrm{Lie}(f^\vee)_\theta\}_{\theta \in J}) \subset \overline{S}.$$

Theorem A of DKS20 then states that there is a Hecke equivariant isomorphism

$$\Xi_J : \overline{Y}_J \to \prod_{\theta \in \Sigma_J} \mathbb{P}_{\overline{Y}_{\Sigma_J}}(\mathcal{V}_\theta)$$

for $\Sigma_J = \{\theta \in J \,|\, \phi(\theta) \notin J\} \cup \{\theta \notin J \,|\, \phi(\theta) \in J\}$. We explain their relation to the analogous Hilbert ones later.

3.2 Hilbert modular forms

For a pair $(k,\ell) \in \mathbb{Z}^\Theta \times \mathbb{Z}^\Theta$ which is paritious (ie $k_\theta + 2\ell_\theta$ is independent of θ), we define the sheaf $\mathcal{A}_{k,\ell}$ on $\overline{Y}$ as the descent of the sheaf

$$\widetilde{\mathcal{A}}_{k,\ell} := \bigotimes_{\theta\in\Theta} \tilde{\delta}_\theta^{\ell_\theta} \tilde{\omega}_\theta^{k_\theta}.$$

We define Hilbert modular forms of level U and weight (k,ℓ) over $\overline{\mathbb{F}}_p$ by

$$M_{k,\ell}(U, \overline{\mathbb{F}}_p) = H^0(\overline{Y}, \mathcal{A}_{k,\ell}.)$$

There is a Kodaira-Spencer isomorphism

$$\widetilde{\mathcal{A}}_{2,-1} = \bigotimes_{\theta} \tilde{\delta}_\theta^{-1} \tilde{\omega}^2 \simeq \mathcal{K}_{S/\mathcal{O}}$$

where $\mathcal{K}_{S/\mathcal{O}} \simeq \wedge^d \Omega^1_{S/\mathcal{O}}$ is the dualizing sheaf of S which descends to $\overline{Y}$. This can be seen via an explicit analysis of the deformation theory of S by using Grothendieck-Messing theory. There is furthermore a canonical trivialization of $\widetilde{\mathcal{A}}_{0,-1} = \wedge^2_{\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_S} \mathcal{H}^1_{dR}(A/S)$ which also descends to Y. This combines to give an isomorphism

$$M_{2,0}(U,R) \simeq H^0(Y, \mathcal{K}_{Y/R}).$$

This leads to the definition of Hilbert modular forms of parallel weight 2 of level $U_0(p)$ and level $U_1(p)$ as

$$M_{2,0}(U_0(p), R) = H^0(Y_0(p), \mathcal{K}_{Y_0(p)/R}) \quad \text{and} \quad M_{2,0}(U_1(p), R) = H^0(Y_1(p), \mathcal{K}_{Y_1(p)/R})$$

respectively. The natural action of $(\mathcal{O}_F/p)^\times \simeq U_0(p)/U_1(p)$ on $Y_1(p)$ by acting on the generator P yields a decomposition

$$M_{2,0}(U_1(p), \mathcal{O}) = \bigoplus_{\chi:(\mathcal{O}_F/p)^\times \to \mathcal{O}^\times} M_{2,0}(U_1(p), \mathcal{O})^\chi$$

into χ components.

Remark 3.1. All of these constructions above make sense over a suitable localization at a prime above p of the ring of integers $\mathcal{O}_L$ of a large enough field $L \subset \overline{\mathbb{Q}}$. For such a field, the Kodaira-Spencer isomorphism holds on $Y_U(G) \times L$ for any U without any hypothesis on the level at p. This (among other details) justifies our definition.

4 The main result

The main input in the proof of Theorem B is a technique dubbed dicing by the authors in which they use the stratification of $\overline{Y}_0(p)$ to define a filtration on its dualizing sheaf such that the graded pieces are supported on appropriate maximal strata.

Preliminaries: Let $h : Y_1(p) \to Y_0(p)$ be the natural projection, it is finite flat. Hence we have the relation $h_*\mathcal{K}_{Y_1(p)/\overline{\mathbb{F}}_p} \simeq \mathcal{H}om(h_*(\mathcal{O}_{Y_1(p)}, \mathcal{K}_{Y_0(p)/\overline{\mathbb{F}}_p})$ and $R^j h_* \mathcal{K}_{Y_1(p)/\overline{\mathbb{F}}_p} = 0$ for $j > 0$. Furthermore since $Y_1(p)$ can be described as a closed subscheme of the universal kernel $H \to Y_0(p)$ which is a Raynaud $\mathcal{O}_F/p$-scheme, we have a decomposition

$$h_*\mathcal{O}_{Y_1(p)} = \bigoplus_{\chi} \mathcal{L}_\chi$$

where the χ are characters of $(\mathcal{O}_F/p)^\times$, and $\mathcal{L}_\chi$ are line bundles on which $(\mathcal{O}_F/p)^\times$ acts via χ. For example Hence, by the above discussion (and the Leray Spectral Sequence), it follows that $H^i(Y_1(p), \mathcal{K}_{Y_1(p)/\overline{\mathbb{F}}_p})^\chi = H^i(Y_0(p), \mathcal{L}_{\chi^{-1}}^{-1}\mathcal{K}_{Y_0(p)/\overline{\mathbb{F}}_p})$ and in particular

$$M_{2,0}(U_1(p), \mathbb{F})^\chi = H^0(\overline{Y}_0(p), \mathcal{L}_{\chi^{-1}}^{-1}\mathcal{K}_{Y_0(p)/\overline{\mathbb{F}}_p}).$$

We now define a filtration on $\mathcal{O}_{\overline{Y}_0(p)}$ by the ideal sheaves $\mathcal{I}_j$ which are given locally by (the descent of)

$$\mathcal{I}_j = \langle \prod_{\theta\in J} \mathrm{Lie}(f)_\theta \rangle_{|J|=j}.$$

Then $\mathcal{I}_0 = \mathcal{O}_{\overline{Y}_0(p)}$ and $\mathcal{I}_{d+1} = 0$. Writing $\mathcal{I}_J$ for the analogous ideal sheaf on $\overline{Y}_J$, ie generated by $\prod_{\theta\in J} \mathrm{Lie}(f)_\theta$, one can check that the natural map

$$gr^j(\mathcal{O}_{\overline{Y}_0(p)}) \hookrightarrow \bigoplus i_{J,*}\mathcal{I}_J$$

is an isomorphism. Now, write $\mathcal{J}_J$ for the invertible ideal sheaf on $\overline{Y}_J$ generated by $\prod_{\theta\notin J} \mathrm{Lie}(f^\vee)_\theta$. Some general Yoga yields a canonical isomorphism

$$i_{J,*}\mathcal{K}_{\overline{Y}_J} \xrightarrow{\sim} i_{J,*}(\mathcal{I}_J\mathcal{J}_J) \otimes \mathcal{K}_{\overline{Y}_0(p)}.$$

Now, we can use this to define a filtration on $\mathcal{K}_{\overline{Y}_0(p)}$ by

$$\mathrm{Fil}^j = \mathcal{I}_j\mathcal{K}_{\overline{Y}_0(p)} \simeq \mathcal{I}_j \otimes \mathcal{K}_{\overline{Y}_0(p)}.$$

Then combining the discussion above yields an isomorphism

$$gr^j(\mathcal{K}_{\overline{Y}_0(p)}) \xrightarrow{\sim} \bigoplus_{|J|=j} i_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_J}).$$

Repeating the same process after tensoring by $\mathcal{L}_{\chi^{-1}}^{-1}$ combined with the machinery of spectral sequences yields a Hecke equivariant spectral sequence

$$E_1^{j,i} = H^{i+j}(\overline{Y}_0(p), gr^j(\mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1}) \simeq \bigoplus_{|J|=j} H^{i+j}(\overline{Y}, \mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_J} i_J^*\mathcal{L}_{\chi^{-1}}^{-1}) \Rightarrow H^{i+j}(\overline{Y}_0(p), \mathcal{K}_{\overline{Y}_0(p)}\mathcal{L}_{\chi^{-1}}^{-1}).$$

We finish the proof of our result by computing the sheaves $\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_J} i_J^*\mathcal{L}_{\chi^{-1}}^{-1})$ on $\prod \mathbb{P}(\mathcal{V}_\theta)$ and their projection to$\overline{Y}_{\Sigma_J}$.

We start by remarking that there is a canonical isomorphism

$$\mathcal{J}_J^{-1} \xrightarrow{\sim} \bigotimes_{\theta\notin J} i_J^*(\mathcal{L}_\theta^{-1}\mathcal{L}_{\phi^{-1}\circ\theta}^p).$$

Roughly speaking, this identification comes in two parts, the first is to relate the sections $\mathrm{Lie}(f^\vee)_\theta$ to $\mathrm{Lie}(g)_\theta$ where $g : A_2 \to A_1$ is the unique map such that $g \circ f = p$. The second is to exploit the properties of the stratification coupled with an explicit analysis of the description of H in terms of Raynaud bundles. As an example of such a property, we have isomorphisms $Lie(H)_\theta \simeq Lie(A_1)_\theta$ and so $i_J^*\mathcal{L}_\theta \simeq i_J^*\omega_{1,\theta}$ for $\theta \notin J$.

If we write $\psi_J : X = \prod \mathbb{P}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta) \to \overline{Y}_\Sigma$ for the natural projection, it is a fact that the dualizing sheaf $\mathcal{K}_X$ is isomorphic to

$$\psi_J^*\mathcal{K}_{\overline{Y}_\Sigma} \bigotimes \left(\bigotimes_{\theta\in\Sigma} \psi_J^*(\delta_\theta)(-2)_\theta \right),$$

where the $(-2)_\theta$ denotes a twist by $\mathcal{O}(-2)_\theta = \mathcal{O}(1)_\theta^{-2}$ where $\mathcal{O}(1)_\theta$ is the pullback of $\mathcal{O}(1)$ to the θ component of the product bundle. Combining this with the Kodaira-Spencer isomorphism and the fact that $\Xi_{J,*}\mathcal{K}_{\overline{Y}_0(p)} \simeq \mathcal{K}_X$ yields

$$\mathcal{K}_X \simeq \left(\bigotimes_{\theta\notin\Sigma} \delta_\theta^{-1}\omega_\theta^2\right)\left(\bigotimes_{\theta\in\Sigma}\delta_\theta(-2)_\theta\right).$$

Finally, we remark that the line bundles $\Xi_{J*}\mathcal{L}_\chi$ have explicit descriptions in terms of the sheaves ω_θ, δ_θ and $\mathcal{O}(1)_\theta$, which come from an explicit analysis of the moduli theoretic construction of the map Ξ_J.

Now, Recalling that in the situation $\psi : \mathbb{P}_S(\mathcal{V}) \to S$, we have an isomorphism $\psi_*\mathcal{O}(n) \simeq \mathrm{Sym}^n_S(\mathcal{V})$ We now have a description of $\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_J} i_J^*\mathcal{L}^{-1}_{\chi^{-1}})$ in terms of the sheaves ω_θ, δ_θ, and $\mathrm{Sym}^n_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)$. In particular, we find that

$$R^i\psi_{J,*}\left(\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_J} i_J^*\mathcal{L}^{-1}_{\chi^{-1}})\right) = 0$$

for $i \geq 1$ and

$$\psi_{J,*}\left(\Xi_{J,*}(\mathcal{J}_J^{-1}\mathcal{K}_{\overline{Y}_J} i_J^*\mathcal{L}^{-1}_{\chi^{-1}})\right) \simeq \left(\bigotimes_{\theta\notin\Sigma}\delta_\theta^{\ell_{J,\theta}-1}\omega_\theta^{n_{J,\theta}+2}\right)\left(\bigotimes_{\theta\in\Sigma}\delta_\theta^{\ell_{J,\theta}}\mathrm{Sym}^{n_{J,\theta}}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)\right).$$

Where the ℓ_J and n_J are given by $V_{\chi,J} \simeq V_{n_J,\ell_J}$. This can be rewritten as $\delta^{-1}\mathcal{A}_{\chi,J}$ for $\delta = \bigotimes \delta_\theta$ a trivial sheaf and

$$\mathcal{A}_{\chi,J} = \mathcal{A}_{n_J+2,\ell_J} = \left(\bigotimes_{\theta\notin\Sigma}\delta_\theta^{\ell_{J,\theta}}\omega_\theta^{n_{J,\theta}+2}\right)\left(\bigotimes_{\theta\in\Sigma}\delta_\theta^{\ell_{J,\theta}+1}\mathrm{Sym}^{n_{J,\theta}}_{\overline{Y}_\Sigma}(\mathcal{V}_\theta)\right)$$

One last application of the Leray Spectral sequence gives us Theorem B:

$$E_1^{j,i} = \bigoplus_{|J|=j} H^{i+j}(\overline{Y}_{\Sigma_J}, \mathcal{A}_{\chi,J}) \Rightarrow H^{i+j}(\overline{Y}_1(p), \mathcal{K})^\chi.$$

The corollary follows readily from the shape of the spectral sequence.