A geometric Jacquet–Langlands for mod p Hilbert modular forms, following Diamond–Kassaei–Sasaki

1. Statement of the main theorem

Let F be a totally real field, and let Θ be the set of embeddings $F \to \overline{\mathbb{Q}}$. Fix embeddings $\overline{\mathbb{Q}} \to \mathbb{C}$, $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Then we identify Θ with $\Theta_\infty := \mathrm{Hom}(F, \mathbb{R})$, $\Theta_p := \mathrm{Hom}(F, \overline{\mathbb{Q}}_p)$, and $\overline{\Theta}_p := \mathrm{Hom}(\mathcal{O}_F/p, \overline{\mathbb{F}}_p)$.

For any set $\Sigma \subset \Theta$ of infinite places of F of even cardinality, let B_Σ be the quaternion algebra over F ramified precisely at Σ. Let G_Σ be the algebraic group over $\mathbb{Q}$ defined by

$$G_\Sigma(R) = B \otimes_{\mathbb{Q}} R\,.$$

We will write G for $G_{\{\}}$.

The rough idea of the Jacquet–Langlands correspondence is that for any automorphic form for G_Σ, we should be able to find a corresponding automorphic form for G. Diamond–Kassei–Sasaki recently proved a Jacquet–Langlands correspondence for geometric mod p Hilbert modular forms. They construct an isomorphism between pieces of a mod p Hilbert modular variety and products of $\mathbb{P}^1$ bundles over quaternionic Shimura varieties. We will state their result more precisely.

Let p be a prime unramified in F. Fix a sufficiently small compact open subgroup $U \subset \mathrm{GL}_2(\mathbb{A}_f)$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$. Note that $\mathrm{GL}_2(\mathbb{A}_f) \cong G_\Sigma(\mathbb{A}_F)$ for all Σ. Let $Y_U(G_\Sigma)$ be the canonical integral model for the Hilbert modular variety of level U for G_Σ over $\mathbb{Z}_{(p)}$. Let

$$\overline{Y}_\Sigma := Y_U(G_\Sigma) \times_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p\,.$$

Let

$$U_0(p) := \left\{ g \in U \,\middle|\, g_p \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p\mathcal{O}_{F,p} \right\}.$$

Similarly, define $Y_{U_0}(G)$ to be the integral model of Pappas for G at level U_0, and let $\overline{Y}_0(p) := Y_{U_0}(G) \times_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$.

There is a decomposition

$$\overline{Y}_0(p) = \bigcup_{J \subset \overline{\Theta}_p} \overline{Y}_0(p)_J\,.$$

Each $\overline{Y}_0(p)_J$ is a smooth.

For each Σ and for each $\theta \in \Theta$, one can define a rank two automorphic vector bundle $\mathcal{V}_\theta$ on $\overline{Y}_\Sigma$.

Theorem 1. *For each J, there is a Hecke-equivariant isomorphism*

$$\overline{Y}_0(p)_J \xrightarrow{\sim} \prod_{\theta \in \Sigma} \operatorname{Proj} \mathcal{V}_\theta$$

where the product is a fiber product over $\overline{Y}_\Sigma$ and

$$\Sigma := \{\theta \in \Theta | \text{exactly one of } \theta, \phi \circ \theta \in J\}\,.$$

2. PEL Shimura varieties

A Hilbert modular variety of abelian type, which means that it does not have a nice moduli interpretation, but each connected component admits a finite étale surjection from a Shimura variety that does have a moduli interpretation.

In this talk, I will explain how to prove an analogue of Theorem 1 for the Shimura varieties with a moduli interpretation. One can then use a descent argument to prove Theorem 1.

A PEL Shimura variety is a Shimura variety that parameterizes abelian varieties with polarization, endomophism, and level structure. A PEL datum consists of the following.

- A finite-dimensional semisimple algebra B over $\mathbb{Q}$ with a positive involution * (for any nonzero $x \in B$, $\operatorname{Tr}_{B/\mathbb{Q}} xx^* > 0$).
- An order $\mathcal{O}$ in B mapped to itself under $*$;
- A finite free $\mathbb{Z}$-module L with an $\mathcal{O}$-action;
- A symplectic pairing $\langle \cdot, \cdot \rangle$ on L satisfying $\langle b^* x, y \rangle = \langle u, by \rangle$ for $x, y \in L \otimes_{\mathbb{Z}} \mathbb{Q}$, $b \in B$;
- An $\mathbb{R}$-algebra homomorphism $h \colon \mathbb{C} \to \operatorname{End}_{\mathbb{R}}(L \otimes_{\mathbb{Z}} \mathbb{R})$ satisfying $\langle h(z)x, y \rangle = \langle x, h(z^*)y \rangle$ for all $x, y \in L \otimes_{\mathbb{Z}} \mathbb{R}$, $z \in \mathbb{C}$, such that the pairing $(x, y) \mapsto \langle x, h(i)y \rangle$ is symmetric and postive definite.

Given a PEL datum, we can construct a Shimura datum (G, X) as follows. The group G is group of endomorphisms of L commuting with B and preserving the polarization up to scalars, and X is the conjugacy class of the cocharacter induced by h.

Decompose $L \otimes_{\mathbb{Z}} \mathbb{C} = V_0 \oplus V_0^c$, where V_0 (resp. V_0^c) is the i (resp. $-i$)-eigenspace of $h(i)$. For any $b \in B$, the eigenvalues of the action of b on V_0 are algebraic number. We define the *reflex field* K to be the number field generated by the traces of the elements of B acting on V_0.

Definition 2 (Hilbert PEL datum)**.**

$$B = F, \qquad ^* = \mathrm{id}, \qquad \mathcal{O} = \mathcal{O}_F, \qquad L = \mathcal{O}_F^2$$
$$\langle x, y \rangle = \operatorname{Tr}_{F/\mathbb{Q}} x \wedge y$$
$$h(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

Then G is the subgroup of $\operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ consisting of those matrices with rational determinant. The trace of $f \in F$ acting on V_0 is just $\operatorname{Tr}_{F/\mathbb{Q}} f$. So the reflex field is $\mathbb{Q}$.

Definition 3 (PEL moduli problem at hyperspecial level at p)**.** Let U^p be an open compact subgroup of $G(\mathbb{A}_f^p)$. Consider the functor that sends a $(\mathcal{O}_K)_{(p)}$-scheme S to the set of isomorphism classes of tuples $\underline{A} = (A, \iota, \lambda, \eta)$, where

- A is an abelian scheme over S of dimension d.
- λ is a prime-to-p quasi-polarization.
- ι is an embedding $\mathcal{O}_B \to \operatorname{End}_S A$ such that:
  - The Rosati involution on $\operatorname{End}_S(A)$ determined by λ fixes ι.
  - The determinant condition is satisfied: for any $b \in \mathcal{O}$, the determinant of the action of b on V_0 must be equal to the determinant of the action of b on $\operatorname{Lie}_{A/S}$. (More precisely, $\det(b|V_0)$ can be considered as an element of $\mathcal{O}_K$, and its image in $\mathcal{O}_S$ must equal $\det(b|\operatorname{Lie}_{A/S})$).
- η is a level U^p-structure on A.

For sufficiently small level structures, this moduli problem is representable by a smooth scheme over $\mathcal{O}_{K_{(p)}}$. Moreover, the generic fiber of this scheme is a disjoint union of Shimura varieties for G. (In the examples that we will consider today, the generic fiber is a single Shimura variety.)

We will fix a prime-to-p level U^p for the Hilbert PEL datum and denote the corresponding hyperspecial level PEL moduli functor by Y.

Example 4. In the case of Example 2, we get a Shimura variety for $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2 \times_{\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m} \mathbb{G}_m$. Then we can get connected components of Shimura varieties for $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2$ by taking quotients.

For simplicity, we will only define the Iwahori level moduli problem for the Hilbert PEL datum.

Definition 5 (Hilbert PEL moduli problem at Iwahori level)**.** Consider the functor $Y_0(p)$ that sends a $\mathbb{Z}_{(p)}$-scheme S to the set of isomorphism classes of triples $(\underline{A}_1, \underline{A}_2, f)$, where:

- $\underline{A}_j$ for $j = 1, 2$ represent S-points of Y.
- $f\colon A_1 \to A_2$ is an $\mathcal{O}$-equivariant isogeny of degree p^d such that $p\lambda_1 = f^\vee \circ \lambda_2 \circ f$, $\eta_2 = f \circ \eta_1$, and $H = \ker f$ decomposes as $\prod_{v|p} H_v$, where H_v is the v-isotypic part of $\ker f$, and H_v has rank $p^{[F_v:\mathbb{Q}_p]}$.

Let $\overline{Y}_0(p) = Y_0(p) \times_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$. The stratification on $\overline{Y}_0(p)$ is defined as follows. There are maps of vector bundles $f_*\colon \mathrm{Lie}_{A_1/S} \to \mathrm{Lie}_{A_2/S}$ and $f_*^\vee\colon \mathrm{Lie}_{A_2^\vee/S} \to \mathrm{Lie}_{A_1^\vee/S}$ on $\overline{Y}_0$. For any $\theta\colon \mathcal{O}_F \to \overline{\mathbb{F}}_p$, the θ-isotypic components of $\mathrm{Lie}_{A_1/S}$, $\mathrm{Lie}_{A_2/S}$, $\mathrm{Lie}_{A_1^\vee/S}$, $\mathrm{Lie}_{A_2^\vee/S}$ are line bundles.

We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Lie}_{A_2/S})^\vee \arrow[r] \arrow[d, "f_*^\vee"] & H^1_{\mathrm{dR}}(A_2/S) \arrow[r] \arrow[d] & \mathrm{Lie}_{A_2^\vee/S} \arrow[r] \arrow[d, "(f_*)^\vee"] & 0 \\
0 \arrow[r] & (\mathrm{Lie}_{A_1/S})^\vee \arrow[r] & H^1_{\mathrm{dR}}(A_1/S) \arrow[r] & \mathrm{Lie}^\vee_{A_1/S} \arrow[r] & 0\,.
\end{tikzcd}$$

The condition $p\lambda_1 = f^\vee \circ \lambda_2 \circ f$ implies that the θ-isotypic components of the middle vertical arrow cannot be isomorphisms. So for any irreducible component of $\overline{Y}_0$ and any θ, the θ-isotypic component of one of the outside vertical arrows must be zero. Therefore, we can write

$$\overline{Y}_0(p) = \bigcup_{J \subset \Theta} \overline{Y}_0(p)_J$$

where $\overline{Y}_0(p)_J$ is the locus where

$$\begin{cases} (f_*)_\theta = 0, & \theta \in J \\ (f_*^\vee)_\theta = 0, & \theta \in \Theta \setminus J\,. \end{cases}$$

Definition 6 (Quaternionic PEL datum)**.** Let $\Sigma \subset \Theta$. Choose $\delta \in \mathcal{O}_{D,p}^\times$ satisfying $\mathrm{Tr}\,\delta = 0$.

$$\begin{gathered}
B = B_\Sigma \otimes_F E \cong M_2(E) \\
\mathcal{O} = \mathcal{O}_\Sigma \otimes_{\mathcal{O}_F} \mathcal{O}_E \cong M_2(\mathcal{O}_E) \\
(b \otimes e)^* = (\delta^{-1}\bar{b}\delta) \otimes e^c \\
L = \mathcal{O} \\
\langle x, y \rangle = \mathrm{Tr}_{B/\mathbb{Q}}\, x(\delta \otimes 1)y^*
\end{gathered}$$

Here $b \mapsto \bar{b} = \mathrm{Tr}(b) - b$ is the standard involution on B (Tr is the reduced trace), and ^{c} is complex conjugation on E.

It is always possible to choose δ, h so that $(x,y) \mapsto \mathrm{Tr}_{B/\mathbb{Q}}\, xh(i)(\delta \otimes 1)y^*$ is positive definite. One can check that for such a choice of h, we have

$$\dim_{\mathbb{C}}(V_0)_{\tilde{\theta}} = \begin{cases} 2, & \theta \notin \Sigma \\ 0 \text{ or } 4, & \theta \in \Sigma \end{cases}$$

Then G is the group defined by

$$G(R) = \left\{(b,\lambda) \middle| b \in B \otimes_{\mathbb{Q}} R, \lambda \in R^\times, bb^* = 1 \otimes \lambda\right\} .$$

Let Y'_Σ be the moduli problem corresponding to the quaternionic PEL datum (again with the prime-to-p level U^p fixed earlier). The moduli problem $Y'_{\{\}}$ is actually equivalent to Y, via $A \mapsto A \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$, with the polarization given by the tensor product of the polarization on A with the trace pairing on $\mathcal{O}_E$. (Note that for $b \in M_2(F)$, $b\bar{b} = (\det b)\,\mathrm{id}$.) Let $\overline{Y}'_\Sigma := Y'_\Sigma \times_{(\mathcal{O}_K)_{(p)}} \overline{\mathbb{F}}_p$.

Definition 7. Let $\mathcal{V}$ be the vector bundle $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot H^1_{\mathrm{dR}}(A/S)$ on $\overline{Y}'_\Sigma$. For each $\tau \colon \mathcal{O}_E \to \overline{\mathbb{F}}_p$, let $\mathcal{V}_\tau$ be the θ-isotypic part of $\mathcal{V}$.

Each $\mathcal{V}_\tau$ has rank two.

For any $\theta \in \Theta$, let $\tilde{\theta}$ be one of the homomorphisms $E \to \overline{\mathbb{Q}}$ lying over θ, and let $\tilde{\theta}^c$ be the other. If $\theta \notin \Sigma$, then the determinant condition implies that $\mathrm{Fil}^1 \mathcal{V}_{\tilde{\theta}}$ and $\mathrm{Fil}^1 \mathcal{V}_{\tilde{\theta}^c}$ have rank one. Otherwise, one of them has rank zero and the other has rank two.

Now that we have described all of the moduli problems, we can finally describe the isomorphisms. In the interest of time, I will only describe the isomorphims at the level of $\overline{\mathbb{F}}_p$-points.

Definition 8. Let $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$. The *Dieudonné ring* $D_{\overline{\mathbb{F}}_p}$ is the (noncommutative) ring over $\breve{\mathbb{Z}}_p$ generated by element F and V subject to the relation $FV = VF = p$, $Fc = \phi(c)F$, $cV = V\phi(c)$ for $c \in \breve{\mathbb{Z}}_p$. Here ϕ is the Frobenius endomorphism of $\breve{\mathbb{Z}}_p$.

The Dieudonné module functor D is an anti-equivalence of categories between p-divisible groups over $\overline{\mathbb{F}}_p$ and $D_{\overline{\mathbb{F}}_p}$-modules that are finite free as $\breve{\mathbb{Z}}_p$-modules. If A is an abelian variety, then its Dieuonné module $A[p^\infty]$ has rank $2\dim A$ over $\breve{\mathbb{Z}}_p$. Moreover, there is an identification

$$(\mathrm{Lie}\, A^{(p)})^\vee \cong D(A[p^\infty])/FD(A[p^\infty])\,.$$

Now suppose we have a map $f \colon A_1 \to A_2$ corresponding to a point on the Hilbert PEL moduli problem at Iwahori level. Let $M_i = D(A_i[p^\infty])$. For $\theta \in \Theta$, let $(M_i)_\theta$ denote the part of M_i on which F acts by θ. Each $(M_i)_\theta$ is two-dimensional. The map $f^*_\theta \colon (M_2)_\theta \to (M_1)_\theta$ is injective with cokernel isomorphic as a $\breve{\mathbb{Z}}_p$-module to $\overline{\mathbb{F}}_p$. We have

$$\mathrm{im}\, f^*_\theta = \begin{cases} F(M_1)_{\phi\circ\theta}, & \theta \in J \\ V(M_1)_{\phi^{-1}\circ\theta}, & \theta \notin J \end{cases}$$

It is possible to find an abelian variety A_J isogenous to $A_1 \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$ and $A_2 \otimes_{\mathcal{O}_F} \mathcal{O}_E^2$ so that $M_J := D(A_J[p^\infty])$ satisfies

$$(M_J)_{\tilde\theta} \cong \begin{cases} (M_1)_\theta^2, & \theta \notin J \\ (M_2)_\theta^2, & \theta \in J \end{cases}$$

$$(M_J)_{\tilde\theta^c} \cong \begin{cases} (M_1)_\theta^2, & \theta \notin J \\ p^{-1}(M_2)_\theta^2, & \theta \in J \end{cases}$$

It then follows that

$$\dim_{\overline{\mathbb{F}}_p}(\operatorname{Lie} A_J)_{\tilde\theta} = 4 - \dim_{\overline{\mathbb{F}}_p}(\operatorname{Lie} A_J)_{\tilde\theta^c} = \begin{cases} 4, & \theta \in J \text{ and } \phi\circ\theta \notin J \\ 0, & \theta \notin J \text{ and } \phi\circ\theta \in J \\ 2, & \text{otherwise} \end{cases}$$

In particular, A_J satisfies the determinant condition for Y'_Σ, where

$$\Sigma := \{\theta \in \Theta | \text{exactly one of } \theta, \phi\circ\theta \in J\}\ .$$

One can then check that the construction $(A_1 \to A_2) \mapsto A_J$ extends to a map of moduli problems $\overline{Y}_0(p)_J \to \overline{Y}'_\Sigma$. Moreover, if $\theta \in J$ and $\phi\circ\theta \notin J$, then we can identify $(\operatorname{Lie} A_1)_\theta$ with a one-dimensional quotient of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (\operatorname{Lie} A_J)_{\tilde\theta} \cong \mathcal{V}^*_{\tilde\theta}$. Similarly, if $\theta \notin J$ and $\phi\circ\theta \in J$, then we can identify $(\operatorname{Lie} A_1)_\theta$ with a one-dimensional subspace of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (\operatorname{Lie} A_J)_{\tilde\theta^c} \cong \mathcal{V}_{\tilde\theta}$. In either case, we get a map $\overline{Y}_0(p)_J \to \operatorname{Proj} \mathcal{V}_{\tilde\theta}$ for each $\theta \in \Sigma$. Putting these together, we get the desired map

$$\overline{Y}_0(p)_J \to \prod_{\theta\in\Sigma} \operatorname{Proj} \mathcal{V}_{\tilde\theta}\ .$$

DKS define this map (for the full moduli problem, not just on $\overline{\mathbb{F}}_p$-points). They then use Diedonné module techniques to show that it induces a bijection $\overline{\mathbb{F}}_p$-points. They also show that the map is étale, which then implies that it must be an isomorphism.