The weight part of Serre's conjecture

Introduction

$f \in S_k(N, \chi)$ $\quad \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

normalised Hecke eigenform

$E = E_f \subset \mathbb{C}$ number field generated by Hecke eigenvalues of f.

$\lambda | \ell$ prime of E.

$\to$ $\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) =: G_{\mathbb{Q}} \to GL_2(E_\lambda)$

(abs. irred.) $\quad$ $*$ unramified at $p \nmid N\ell$,

$\det(X - \rho_f(\mathrm{Frob}_p)) = X^2 - a_p(f)X + \chi(p)p^{k-1}$

compactness argument: $\rho_{f,\lambda}(G_{\mathbb{Q}})$ stabilizes an $\mathcal{O}_\lambda$-lattice in E_λ^2

$\to$ reduce mod λ and semi-simplify to get $\overline{\rho}_{f,\lambda} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_\ell)$

$k(\lambda) \hookrightarrow \overline{\mathbb{F}}_\ell$

"$\circledast$ mod ℓ" characterises $\overline{\rho}_{f,\lambda}$ in terms of $\overline{a_p(f)} \in \mathcal{O}_E/\lambda$.

"mod ℓ modular form" can be defined as "reductions mod ℓ" of char 0 mod forms.

Questions about $f \rightsquigarrow \overline{\rho}_{f,\lambda}$

1. Which $\overline{\rho} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_\ell)$ do we obtain? i.e. which $\overline{\rho}$ are modular?

2. $\overline{\rho}$ modular. Can we describe the possible values of k, N, χ such that $\overline{\rho} \cong \overline{\rho}_{f,\lambda}$ $f \in S_k(N, \chi)$.

(i) Necessary conditions:

(*) $\bar{\rho}$ unramified away from finitely many primes.

(*) $\chi(-1)(-1)^k = 1 \Rightarrow \det \bar{\rho}(\text{cx conj}) = -1$ "odd"

"weak" form of Serre's conjecture:

these conditions are sufficient.

Thm of Khare-Wintenberger.

Examples: a) $S_k(SL_2(\mathbb{Z})) = 0 \quad k < 12$

+ Serre's conjecture: $\Rightarrow$

no irreducible ors. $\bar{\rho}: G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_\ell)$

unramified outside ℓ for $\ell < 11$.

Serre-Tate verified for $\ell = 2, 3$.

b) $\Delta \in S_{12}(SL_2(\mathbb{Z})) \rightsquigarrow \bar{\rho}_{\Delta,11} \cong \bar{\rho}_{X_0(11),11}$

$\Delta \equiv F \bmod 11, \quad \langle F \rangle = S_2(\Gamma_0(11))$.

In general can find congruences mod ℓ to wt. two modular forms with a power of ℓ in the level.

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(2) Question: fix $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_\ell)$, modular what is the minimal weight k such that
$\bar{\rho} \cong \bar{\rho}_{f,\lambda}$ for $f \in S_k(N, \chi)$ with $\ell \nmid N$.
Minimal value of $k =: k(\bar{\rho})$.

Weight part of Serre's conj answers this question.

Idea: $k(\bar{\rho})$ only depends on $\bar{\rho}|_{I_\ell}$ ← inertia subgp of $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$

example: $\bar{\rho}|_{I_\ell} \cong \begin{pmatrix} \bar{\chi}_{cyc}^{a} & * \\ 0 & 1 \end{pmatrix}$ $\quad 1 \le a \le \ell - 1$

Assume $\bar\rho|_{I_\ell}$ non-trivial

Then $k(\bar\rho) = \begin{cases} 1+a & \text{otherwise} \\ \ell+1 & \text{if } a=1 \text{ and } \bar\rho \text{ is not } \ell\text{-finite} \end{cases}$

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e.g. $\bar\rho_{\Delta,11} \simeq \bar\rho_{X_0(11),11}$ $\quad$ $\underline{k(\bar\rho)=12}$.

Weight one. $f \in S_1(N,\chi)$

Deligne-Serre $\rightsquigarrow$ $\exists\ \rho_f : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r] \arrow[dr] & GL_2(\mathbb{C}) \arrow[d, leftrightarrow] \\ & GL_2(E_f) \end{tikzcd}$$

unramified at $p \nmid N$

"condition $\circledast$".

Can still reduce mod λ $\rightsquigarrow$ $\bar\rho_{f,\lambda} : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_\ell)$

If $\ell \nmid N$, $\bar\rho_{f,\lambda}$ is unramified at ℓ.

So if $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_\ell)$ is unramified when restricted to $G_{\mathbb{Q}_\ell}$, might want to predict $k(\bar{\rho}) = 1$.

Turns out that this is not the right prediction.

<u>example</u> (Buzzard) $\ell = 199$, $\exists\, \bar{\rho}$ unramified outside $2, 41$ which doesn't come from $f \in S_1(82, \chi)$.

Issue is we're thinking of mod ℓ modular forms as reductions of the usual modular forms.

To handle weight 1, need more intrinsic definition of a mod ℓ modular form.

$N \geq 5$

Geometric interpretation of modular forms:

$$S_k(N, \mathbb{C}) = H^0(X_1(N)_{\mathbb{C}}, \omega^{\otimes k}(-c))$$

level $\Gamma_1(N)$

cusps

extend $X_1(N)_{\mathbb{C}}$ to a smooth relative curve $X_1(N)/\mathbb{Z}[\frac{1}{N}]$

If $p \nmid N$, can define

$$S_k(N, \mathbb{F}_p) = H^0(X_1(N)_{\mathbb{F}_p}, \omega^{\otimes k}(-c))$$

$, \mathbb{Z}_p$

$, \mathbb{Q}_p$

Understanding "reduction mod p" map:

$$S_k(N, \mathbb{Z}_p) / pS_k(N, \mathbb{Z}_p) \xrightarrow{\pi} S_k(N, \mathbb{F}_p)$$

π is surjective if $k \geq 2$

π is not in general surjective if $k=1$.

<u>Idea</u>: consider LES from the SES of sheaves on $X_1(N)_{\mathbb{Z}_p}$: $\quad i : X_1(N)_{\mathbb{F}_p} \hookrightarrow X_1(N)_{\mathbb{Z}_p}$

$$0 \to \omega^{\otimes k}(-C) \xrightarrow{\times p} \omega^{\otimes k}(-C) \to i_* \omega^{\otimes k}(-C) \to 0$$

$$S_k(N, \mathbb{Z}_p) \xrightarrow[\sim]{\times p} S_k(N, \mathbb{Z}_p) \to S_k(N, \mathbb{F}_p) \to \boxed{H^1(X_{\mathbb{Z}_p}, \omega^{\otimes k}(-C))[p]} \to 0$$

use Serre duality.
vanishes if $k \geq 2$.

<u>Remark</u>: possible to side-step these delicate issues with weight 1:

Multiplication by Hasse invariant relates weight 1 and weight p mod p modular forms.

So we can formulate a version of the weight part of Serre's conjecture where the weight is always assumed to be ≥ 2.