Modular curves & Θ operator

§ 0 References

Here are some references:

[DR]: "Les schémas de modules des courbes elliptiques", P. Deligne, M. Rapoport.

[KM]: "Arithmetic moduli of elliptic curves" N. Katz, B. Mazur.

[MFMS]: "p-Adic properties of modular forms and modular schemes", N. Katz.

[AV]: "Abelian varieties", D. Mumford

[Oda]: "The first de Rham cohomology group and Dieudonné Modules", T. Oda.

[KO]: "On the differentiation of De Rham cohomology classes with respect to parameters", N. Katz, T. Oda.

→ [Katz]: "A result on modular forms in characteristic p", N. Katz.

§ 1 Modular curves $Y_1(N) \subseteq X_1(N)$

Let us fix $\mathbb{F} = \overline{\mathbb{F}}$, $\operatorname{char} \mathbb{F} = p > 0$.
Also take $N \geq 4$, $p \nmid N$, a <u>true level</u>.
We can consider the moduli problem

$$Y_1(N) : \underline{\mathrm{Alg}}_{\mathbb{F}} \subseteq \underline{\mathrm{Sch}}_{\mathbb{F}}^{op} \longrightarrow \underline{\mathrm{Set}}$$

$$A \longmapsto \{ (E/A, \mu : \underline{\mathbb{Z}/N\mathbb{Z}}_A \hookrightarrow E[N]) \}/_{\simeq}$$

Theorem ([DR], [KM]): The functor $Y_1(N)$ is represented by an affine, smooth, irreducible $\mathbb{F}$-curve. $\square$

Fix a rep. $Y = Y_1(N) \longrightarrow \operatorname{Spec} \mathbb{F}$.
We have an universal obj. $/ Y$

$$\mathcal{E} = \mathcal{E}_{un} \xrightarrow{f_{un} = f} Y \qquad \left(\begin{array}{c}\text{universal} \\ \text{ell. curve}\end{array}\right)$$

w/ a canonical lvl $\Gamma_1(N)$-str

$$\mu = \mu_{un} : \underline{\mathbb{Z}/N\mathbb{Z}}_Y \hookrightarrow \mathcal{E}[N].$$

Write $e \in \mathcal{E}(Y)$ for the unit section of $\mathcal{E}$.
Then we can consider

$$\underline{\omega} := f_* \Omega^1_{\mathcal{E}/Y} \cong e^* \Omega^1_{\mathcal{E}/Y}$$

the Hodge bundle. This is an invertible $\mathcal{O}_Y$-shf $\rightsquigarrow$ we want to use it def. modular forms.

Pb: $Y_1(N)$ is not proper!

One can go around this in many ways:
(1) "generalise the notion of ell. curve" $\rightsquigarrow$ extending the moduli prob.
$\rightsquigarrow$ compactifying $Y_1(N) \overset{\circ}{\subseteq} X_1(N)$ [DR]

(2) [KM]

$$\begin{tikzcd} Y_1(N) \arrow[r, hook, "\circ"] \arrow[d, "j"] & X_1(N) \arrow[d] \\ \mathbb{A}^1_{\mathbb{F}} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{F}} \end{tikzcd}$$

j-invariant

the "normalisation of $\mathbb{P}^1$ in $Y_1(N)$".

Either way we a compactified $X_1(N)$ which is proper, smooth, irreducible $/\mathbb{F}$. We can embed $Y_1(N) \overset{\circ}{\subseteq} X_1(N)$ as a dense open, $C = X_1(N) \setminus Y_1(N)$ is the divisor of <u>cusps</u>.

* We can extend $\underline{\omega}$ to an inv. shf on $X_1(N)$

<u>Def</u>. (*) A <u>modular form</u> of <u>wt k</u>, <u>level $\Gamma_1(N)$</u> <u>with coeff.s in $\mathbb{F}$</u> is

$$f \in H^0(X_1(N), \underline{\omega}^{\otimes k} =: \underline{\omega}^k) =: M_k(N).$$

(*) A <u>cusp form</u> (w/ wt k, lvl ...) is a

$$f \in H^0(X_1(N), \underline{\omega}^k(-C)) =: S_k(N).$$

<u>Lemma</u>: The sum $M(N) = \bigoplus_{k \in \mathbb{Z}} M_k(N)$ is a finitely generated graded $\mathbb{F}$-alg. and $S(N) = \bigoplus S_k(N)$ is a graded

ideal of $M(N)$.

§ 1.1 q-Expansions

One can define the Tate curve

$$\mathrm{Tate}(q) := \mathbb{G}_{m,K} / q^{\mathbb{Z}} \qquad K = \mathbb{F}((q^{1/N}))$$

Choose coord.: $\mathbb{G}_{m,K} = \mathrm{Spec}(K[T^{\pm 1}])$, we have a canonical nowhere vanishing inv. diff.

$$\omega_{can} := \frac{dT}{T}$$

Fixing a $\Gamma_1(N)$-level str. $\mu : \mathbb{Z}/N\mathbb{Z} \hookrightarrow \mathrm{Tate}(q)$ we get a classifying map

$$g : \mathrm{Spec}\, K \longrightarrow Y$$

corresponding to $(\mathrm{Tate}(q), \mu)$.
For any $f \in M_k(N)$ we can take the pullback

$$g^*(f) \in H^0(\mathrm{Tate}(q), (\Omega^1_{\mathrm{Tate}(q)/K})^{k})$$
$$\| $$
$$f_0(q) \cdot \omega_{can}^k \qquad f_0(q) \in K$$

Def. The power series $f_0(q) \in \mathbb{F}((q^{1/N}))$ is the q-expansion of f at the cusp corresponding to μ.

We have the following:

Lemma (q-Expansion principle):
The $\mathbb{F}$-linear map

$$\begin{aligned} M_k(N) &\longrightarrow \mathbb{F}((q^{1/N})) \\ f &\longmapsto f_0(q) \end{aligned}$$

is injective for any μ, has image in $\mathbb{F}[[q^{1/N}]]$ and sends $S_k(N)$ to $q^{1/N}\mathbb{F}[[q^{1/N}]]$.

Pf. [MFMS]. □

§ 2 De Rham Cohom. & 1-root splitting

In general, we can consider

$$X \xrightarrow[\substack{\text{proper}\\ \text{smooth}}]{f} S \xrightarrow[\text{sm.}]{} \operatorname{Spec} k. \quad \leftarrow \text{any field}$$

The we can define the de Rham cpx

$$\Omega^\bullet_{X/S} : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \longrightarrow \cdots \longrightarrow \Omega^{n = \dim X/S}_{X/S} \longrightarrow 0 \longrightarrow \cdots$$

w/ $\Omega^i_{X/S} := \Lambda^{i^{\text{th}}}_{\mathcal{O}_X} \Omega^1_{X/S}$.

Def. The i^{th} de Rham cohomology shf of X/S is

$$H^i_{dR}(X/S) := R^i f_* \Omega^\bullet_{X/S}.$$

We mostly care about $X = A/S$ ab. sch.

Lemma: The $\mathcal{O}_S$-sheaves $H^i_{dR}(A/S)$ are locally free of finite rank. □

Rmk: In the case $S = \operatorname{Spec} K$, then

$$f_* \cong \Gamma(A = X, \cdot) \quad \text{as fctrs}$$

$$\underline{\mathrm{Coh}}_X \longrightarrow \underline{K\text{ v.sp.}}$$

"Morally" we are doing the same for "smooth S-families"; underlying this there are statements of "cohomology & base chg", see [AV].

For A/S ab. sch. we have the s.e.s.

$$0 \to \underline{\omega}_{A/S} := f_* \Omega^1_{A/S} \to H^1_{dR}(A/S) \to R^1 f_* \mathcal{O}_A \to 0$$

$$\begin{matrix} R^1 f_* \mathcal{O}_A \\ \Vert\wr \longleftarrow [AV] \\ \underline{\omega}^\vee_{A^\vee/S} \end{matrix}$$

When $A = E$ an ell. curve $E \cong E^\vee$ canonically and we get

$$0 \to \underline{\omega}_{E/S} \to H^1_{dR}(E/S) \to \underline{\omega}^\vee_{E/S} \to 0 \qquad \text{(HF)}$$

(Hodge filtration)

Rmk: One can consider the "analytic" $H^1_{dR}(A^{an}/\mathbb{C})$, $A/\mathbb{C}$ ab. var., and by classical Hodge thy one has a canonical way to decompose

$$H^i_{dR}(A^{an}/\mathbb{C}) := \bigoplus_{p+q=i} H^q(A, \Omega^p)$$

$\hookrightarrow$ this is fundamental in defining Maass - Shimura op.s in char 0.

Q: can we find a canonical, alg.c splitting of (HF) in char $p>0$?

A: On some opens of S, yes.

Let us focus on $E/S/\mathbb{F}$. We can define

$$\begin{tikzcd} E^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d] \\ S \arrow[r, "F_{abs}"] & S \end{tikzcd}$$

where $F_{abs}: S \to S$ is the Absolute Frobenius: identity on $|S|$ and

$$\mathcal{O}_S \longrightarrow \mathcal{O}_S$$
$$s \longmapsto s^p \quad \text{on loc. sections.}$$

One can also def. $F_{abs}: E \to E$ and by the UP of $E^{(p)}$ we get:

$$\begin{tikzcd}
E \arrow[dr, dashed, "\exists ! F"] \arrow[drr, bend left, "F_{abs}"] \arrow[ddr, bend right] & & \\
& E^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] \\
& S \arrow[r, "F_{abs}"] & S
\end{tikzcd}$$

where $F : E \longrightarrow E^{(p)}$ is the Relative Frobenius.

$$\begin{tikzcd}
& & H \arrow[d, equal] \\
\text{We can consider } F^* : & H^1_{dR}(E^{(p)}/S) \arrow[r] \arrow[d, "\cong"] & H^1_{dR}(E/S) \\
& H^{(p)} \arrow[ur] &
\end{tikzcd}$$

where we write $\mathcal{F}^{(p)} = F^*_{abs,S}\, \mathcal{F}$ for $\mathcal{F} \in \underline{QCoh}_S$.

Q: What does F^* do to (HF) ?

In general $\mathcal{L}^{(p)} \cong \mathcal{L}^p$ for $\mathcal{L}$ inv. shf. $\qquad \cong \underline{\omega}^p_{E/S}$

$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}^{(p)}_{E/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & H^{(p)} \arrow[r] \arrow[d, "F^*"] \arrow[dr, phantom, "\circ"] & \underline{\omega}^{\vee,p}_{E/S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\omega}_{E/S} \arrow[r] & H \arrow[r] & \underline{\omega}^{\vee}_{E/S} \arrow[r] & 0
\end{tikzcd}$$

(*) One can see that F^* kills $\underline{\omega}^p_{E/S}$.
[boils down "$F^*(dt) = d\,F(t) = dt^p = 0$"]

(*) locally on S we can fix $\mathcal{B} = \{\omega, \omega'\}$ a basis of H "adapted to HF" : ω is a basis of $\underline{\omega}$, ω' reduces to a basis of $\underline{\omega}^{\vee}$ dual to ω.
Wrt $\mathcal{B}^{(p)}$ & $\mathcal{B}$ F^* has matrix

$$\begin{pmatrix} 0 & b_0 \\ 0 & h_0 \end{pmatrix} \qquad b_0, h_0 \text{ sec.}^s \text{ of } \mathcal{O}_S.$$

Theorem (Oda): For $S = \operatorname{Spec} K$ perfect there is an iso.

$$(H, F^*_{abs}) \cong (\mathbb{D}(E[p]), F)$$

← Dieudonné fctr

s.t. HF becomes iso. to

$$0 \longrightarrow \mathbb{D}(E[F])^{(p^{-1})} \longrightarrow \mathbb{D}(E[p]) \longrightarrow \mathbb{D}(E[V]) \to 0.$$

Let us assume $S = \operatorname{Spec} K$ perfect for a moment: we have two cases

(1) E/K ordinary: $h_0 \in K^\times$ is invertible.

(2) E/K supersingular: $F \cong F^*$ is nilpotent of order 2 $\rightsquigarrow$ $h_0 = 0$, $b_0 \neq 0$

$$F \sim \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

To sum up: in (1) $\operatorname{Im} F^*$ gives a splitting of HF, while in (2) $\operatorname{Im} F^* = \underline{\omega}$ (so no splitting).

Back to $S \longrightarrow \operatorname{Spec} \mathbb{F}$.

Lemma (Unit-root splitting): Set $\mathcal{U} := \operatorname{Im} F^*$. Then $\mathcal{U}$ is an inv. shf. which gives a splitting of HF over

$$S^h := \{ x \in S \mid \text{"} h_0(x) \neq 0 \text{"} \}$$
$$\overset{\circ}{\subseteq} S \qquad \left(\begin{smallmatrix}\text{ordinary}\\ \text{locus}\end{smallmatrix}\right)$$

which cannot be extended (canonically) to

$$S^{ss} := \{ x \in S \mid \text{"} h_0(x) = 0 \text{"} \}$$
$$\left(\begin{smallmatrix}\text{supersingular}\\ \text{locus}\end{smallmatrix}\right)$$

because there $\mathcal{U} = \underline{\omega}$.

In defining S^h we used h_0, we are considering

$$F^* : \underline{\omega}^{\vee,p} \longrightarrow \underline{\omega}^{\vee}$$
$$(\overline{\omega'})^p \longmapsto h_0 \cdot \overline{\omega'}$$
$$\in \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}^{\vee,p}, \underline{\omega}^{\vee}) \cong H^0(S, \underline{\omega}^{p-1})$$

__Def / lemma__: this gives a modular form $h \in M_{p-1}(N)$, called the __Hasse invariant__.

__Lemma__: (1) $h_0(q) \equiv 1$ at all cusps. [MFMS]

(2) h vanishes w/ simple zeroes [Igusa] over Y^{ss}. $\square$

__Def.__ Let $f \in M_k(N)$, we can write

$$f = g \cdot h^r \qquad \begin{array}{l} r \geq 0 \\ g \in M_{k'}(N) \end{array}$$

where $h \nmid g$ (in $M(N)$)

we define the (weight) filtration of f as $w(f) = k'$.

§ 3 GM connection & KS map

* One can consider $X/S/K$ as before. In [KO] the authors define a flat connection

$\boxed{\text{Gauß}}$ (Gauss-Manin connection)
$$\nabla : H^1_{dR}(X/S) \longrightarrow H^1_{dR}(X/S) \otimes_{\mathcal{O}_S} \Omega^1_{S/k}$$

* One can define a perfect, alternating $\mathcal{O}_S$-bilinear pairing (for $E/S/\mathbb{F}$)

$$\langle \cdot , \cdot \rangle : \underbrace{H^1_{dR}(E/S)}_{=H} \times H^1_{dR}(E/S) \longrightarrow \mathcal{O}_S$$

s.t. $\underline{\omega} \subseteq H$ is maximal isotropic, set in duality by $\langle \cdot,\cdot \rangle$ with $\underline{\omega}^\vee$, via HF (compatibly w/ the canonical duality $\underline{\omega} \leftrightarrow \underline{\omega}^\vee$).

* With this pairing and ∇ we can give one definition of the Kodaira-Spencer morphism

$$\underline{KS}_{E/S} : \underline{\omega}_{E/S}^{\otimes 2} \longrightarrow \Omega^1_{S/\mathbb{F}}$$
$$(\omega, \omega') \longmapsto \langle \omega, \nabla(\omega') \rangle$$

Prop: Consider the classifying map of $(E/S, \mu^{\ell})$ any lvl str,

$$g : S \longrightarrow Y_1(N).$$

Then $\underline{KS}_{E/S}$ is cotangent map of g. In particular for g étale (such as $Y_1(N) \xrightarrow{id} Y_1(N)$) $\underline{KS}_{E/S}$ is an iso.

Pf. Def. thy [MFMS]. $\square$

§ 4 Θ & its properties

Write Y^{h} for ordinary locus, there we have the 1-root splitting

$$H^1_{dR} := H^1_{dR}(\mathcal{E}/Y^{h}) \cong \underline{\omega} \oplus \mathcal{U}$$

We can use this to mimic the constr. of Maass-Shimura op.^{s}

$$\Theta_3 := \begin{cases} \underline{\omega}^k \xrightarrow{S^k(\nabla)} S^k(H^1_{dR}) \otimes \Omega^1_{Y^h/\mathbb{F}} \\ \xrightarrow{id \otimes KS^{-1}} S^k(H^1_{dR}) \otimes \underline{\omega}^2 \\ \xrightarrow{\text{projection on } \underline{\omega}^k /\!/ \mathcal{U}} \underline{\omega}^k \otimes \underline{\omega}^2 = \underline{\omega}^{k+2} \end{cases}$$

Prop. θ_0 acts on q-expansions as $q\frac{d}{dq}$.

Pf. [Kat7]. $\square$

One can actually extend θ_0 from Y^h to $X_1(N)$ by considering

$$\Theta := h \cdot \theta_0 \qquad \text{[Kat7]}$$

Def The Θ-operator is the degree $p+1$ derivation

$$\begin{array}{ccc} M(N) & \longrightarrow & M(N) \\ \cup\!\! \mid & & \cup\!\! \mid \\ M_k(N) & \longrightarrow & M_{k+p+1}(N) \\ f & \longmapsto & h\,\theta_0(f). \end{array}$$

$\square$

$\rightsquigarrow$ "A tameness criterion ..."