Hilbert modular forms

Reference: A SERRE WEIGHT CONJECTURE FOR GEOMETRIC HILBERT MODULAR FORMS IN CHARACTERISTIC p

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Plan:

1. Complex Hilbert modular forms
2. Hilbert modular varieties
3. General Hilbert modular forms
4. Partial Hasse Invariants
5. Theta operators.

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§1 Complex HMFs

Notation: • Let F be a totally real field of degree $g > 1$. $(F \neq \mathbb{Q})$

• Let Σ be the set of embedding of $F \hookrightarrow \overline{\mathbb{Q}}$

$$F \hookrightarrow \overline{\mathbb{Q}}_p \to \overline{\mathbb{F}}_p$$

- $k \in \mathbb{Z}^{\Sigma}$ with k_σ all have the same parity
- Let $\mathfrak{b}, \mathfrak{n}$ be fractional ideals in F

$\mathfrak{d}_F$ = different ideal of F.

Let
$$\Gamma_1(\mathfrak{b},\mathfrak{n}) = GL_2(F)^+ \cap \begin{Bmatrix} \mathcal{O}_F & \mathfrak{b}^{-1}\mathfrak{d}_F^{-1} \\ \mathfrak{b}\mathfrak{n}\mathfrak{d}_F & 1+\mathfrak{n} \end{Bmatrix}$$

<u>Defn</u>: A complex Hilbert modular form of wt. k and level $\Gamma_1(\mathfrak{b},\mathfrak{n})$ is a holomorphic function $f : \mathcal{H}^{\Sigma} \to \mathbb{C}$ s.t.

$$f\left(\prod_{\sigma\in\Sigma} \frac{a_\sigma z_\sigma + b_\sigma}{c_\sigma z_\sigma + d_\sigma}\right) = \left(\prod_{\sigma\in\Sigma} (c_\sigma z_\sigma + d_\sigma)^{k} \det(\sigma(\gamma))^{-k/2}\right) f(\underline{z})$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(\mathfrak{b},\mathfrak{n})$ $\quad a_\sigma = \sigma(a)$

$\underline{z} = (z_\sigma)_\sigma$. Denoted $M_k(\Gamma_1(\mathfrak{b},\mathfrak{n}))$

Let $U_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathcal{O}}_F) \mid c \in \mathfrak{n},\ d \equiv 1 \bmod \mathfrak{n} \right\}$

Let J_t be a set of representatives for classes in the narrow class group of F

(assume coprime to $\mathfrak{n}$ and lies to a fixed prime p)

$$M_k(U_1(\mathfrak{n})) \cong \bigoplus_{t \in Cl(F)^+} M_k(\Gamma_1^t(J_t, \mathfrak{n}))$$

these have "nice" q-expansions

this has an action by Hecke operators, which in general permute the spaces on RHS

§2) Hilbert modular varieties

Notation: $\cdot\ G = \mathrm{Res}_{F/\mathbb{Q}} GL_2$

$\cdot\ N \in \mathbb{Z}_{\geq 3}$, p a prime unramified in F $(N, p) = 1$.

$\cdot\ L$ be an ext^n of $\mathbb{Q}_p$ containing the images of F under all $\sigma \in \Sigma$.

$\mathcal{O} = \mathcal{O}_L$, π is a uniformizer, $\mathbb{F} = \mathcal{O}/\pi$.

Let J be a fractional ideal of F and we will consider the functor $\mathcal{M}_{J,N}$ sending an $\mathcal{O}$-scheme S to the iso. classes of the data (A, i, λ, η) where

- A is an abelian scheme over S of relative dimension $g = [F:\mathbb{Q}]$
- $i : \mathcal{O}_F \hookrightarrow \mathrm{End}(A/S)$ (the real multiplication)
- λ is a $\mathfrak{J}$-polarization satisfying the Deligne-Pappas condition.

i.e. $\lambda : A \to A^\vee$ inducing isomorphisms

$$A \otimes_{\mathcal{O}_F} \mathfrak{J} \xrightarrow{\sim} A^\vee \quad \text{(Deligne-Pappas)}$$

$$\text{and iso} \quad (\mathfrak{J}, \mathfrak{J}_+) \xrightarrow{\sim} (\mathrm{Sym}(A/S), \mathrm{Pol}(A/S))$$

where $\mathrm{Sym}(A/S) = \{ \lambda : A \to A^\vee \mid \lambda = \lambda^\vee \text{ and } \lambda \circ i = i^\vee \circ \lambda \}$

$\mathrm{Pol}(A/S) =$ cone of polarizations

- (Level structure) η is a $\mathcal{O}_F$-linear isomorphism

$$\eta : \left(\mathcal{O}_F/N\right)^2 \xrightarrow{\sim} A[N].$$

this is called a $\mathfrak{J}$-polarized Hilbert-Blumenthal (HBAV) abelian variety with level η.

$$(A, i, \lambda, \eta) \cong (A', i', \lambda', \eta')$$

if $\exists$ an $\mathcal{O}_F$-lin iso $A \to A'$
s.t. $\lambda = \alpha^\vee \circ \lambda' \circ \alpha$
and sends η to η'.

Fact: $\mathcal{M}_{J,N}$ is representable by a smooth $\mathcal{O}$-scheme denoted $Y_{J,N}$. Moreover, it is quasi-projective over $\mathcal{O}$.

Action on points of $Y_{J,N}$:

- let $\mathcal{O}_{F,+}^\times$ denote the tot. pos. units in F.

  then $v \in \mathcal{O}_{F,+}^\times$ acts on $(A, \iota, \lambda, \eta) \in Y_{J,N}(S)$
  by sending it to $(A, \iota, v\lambda, \eta) \in Y_{J,N}(S)$.

- Similarly if $u \in GL_2(\mathcal{O}_F/N)$, this acts on points via the level structure
  sending $(A, \iota, \lambda, \eta) \longmapsto (A, \iota, \lambda, \eta \circ r_{u^{-1}})$
  where $r_{u^{-1}}$ is mult on right by u^{-1}!

Composing with reduction mod N you get an action by $GL_2(\hat{\mathcal{O}}_F)$.

The thing to note is that if $\mu \in \mathcal{O}_F^\times$
then the action of μ^2 on pol.

coincides with the action of $\begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ on the level.

Let U be a open compact subgroup of $G(\hat{\mathbb{Z}})$ containing $GL_2(\mathcal{O}_F \otimes \mathbb{Z}_p)$ and contains

$$U(N) = \operatorname{Ker}\left(GL_2(\hat{\mathcal{O}_F}) \to GL_2\left(\mathcal{O}_F/N\right)\right)$$

then we get an induced action on $Y_{J,N}$ by the finite group

$$G_{U,N} = \left(\mathcal{O}_{F,+}^{+} \times U\right) \Big/ \left\{ \mu \in \mathcal{O}_F^{*}, u \in U \mid u \equiv \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \bmod N \right\}$$

Fact: One can choose U small enough so that $G_{U,N}$ acts freely on $Y_{J,N}$.

Using this we form the quotient $Y_{J,N}/G_{U,N}$ which is representable by a scheme over $\mathcal{O}$.

Define: $$Y_U = \coprod_{t \in Cl(F)^+} Y_{J t,N} \Big/ G_{U,N}$$

$$= \left(\text{" Union of fact of Shim. Var } G^{*} = G \times_{\operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_a} \text{"} \right)$$

Fact: Y_U is smooth over $\mathcal{O}$ and

$\coprod_{t} Y_{\mathfrak{J}_{t},N} \to Y_U$ is Galois and étale with Galois group $G_{U,N}$.

Moreover, Y_U is defined over $\mathcal{O} \cap \overline{\mathbb{Q}}$

and is independent of N and $\{\mathfrak{J}_t\}$.

On $\mathbb{C}$-points we have

$$Y_U(\mathbb{C}) \cong GL_2(F)^+ \backslash GL_2(\mathbb{A}_{F,f}) \times \mathcal{H}^{\Sigma} / U.$$

(3) General HMFs

Since we have assumed p is unramified and the Deligne-Pappas Condition is assumed to hold.

This means the Rapoport Condition holds which says that $\mathrm{Lie}(A/S)$ is locally on S, free of rank 1 over $\mathcal{O}_F \otimes \mathcal{O}_S$.

Therefore, ω is $s_* \Omega^1_{A/S}$ where $s: A \to S$.

Now, as $\mathcal{O}_F \otimes \mathcal{O}_S \simeq \bigoplus_{\theta \in \Sigma} \mathcal{O}_S$ as coherent sheaves of $\mathcal{O}_S$-algs.

We can use this to decompose $s_* \Omega^1_{A/S}$.

In particular, applying this to $A_{J,N}$ the universal HBAV over $S = Y_{J,N}$, we get

$$s_* \Omega^1_{A_{J,N}/S} \cong \bigoplus_{\sigma \in \Sigma} \omega_\sigma \quad \text{, with } \omega_\sigma \text{ a line bundle.}$$

Using this, for $k \in \mathbb{Z}^\Sigma$, we let

$$\omega^k = \bigotimes_{\sigma \in \Sigma} \omega_\sigma^{\otimes k_\sigma} \quad \text{this is again a line Bundle}$$

if we stopped here we would only get HMFs for G^*
so we need to do a little more work to go to G.

To do this, recall, we have a s.e.s

$$0 \to s_* \Omega^1_{A/S} \to H^1_{dR}(A/S) \to R^1 s_* \mathcal{O}_A \to 0$$

of loc. free modules over $\mathcal{O}_F \otimes \mathcal{O}_S$

if we take Λ^2 of middle term

$$\Lambda^2_{\mathcal{O}_F \otimes \mathcal{O}_S} H^1_{dR}(A/S) \cong s_* \Omega^1_{A/S} \otimes R^1_{s_*} \mathcal{O}_A$$

is loc. free of rank 1 over $\mathcal{O}_F \otimes \mathcal{O}_S$
so we can again decompose this into a product of line bundles

$$\Lambda^2_{\mathcal{O}_F\otimes\mathcal{O}_S} H^1_{dR}(A/S) \cong \bigoplus_{\sigma\in\Sigma} \delta_\sigma$$

where δ_σ are line bundles.

For $l \in \mathbb{Z}_{?}^{\Sigma}$ let $\delta^l = \bigotimes_{\sigma\in\Sigma} \delta_\sigma^{\otimes l_\sigma}$

Using this we define $\mathcal{L}^{\kappa,l}_{\mathfrak{J},N}$ to be $\omega^{\otimes\kappa}\otimes\delta^{\otimes l}$

The action of $\mathcal{O}^\times_{F,+} \cap U$ on $Y_{\mathfrak{J},N}$ induces an

action on $\mathcal{L}^{\kappa,l}_{\mathfrak{J},N}$

One finds that if $\mu \in \mathcal{O}_F^\times \cap U$ it acts

on $\mathcal{L}^{\kappa,l}_{\mathfrak{J},N}$ by multiplication by

$$\mu^{\kappa+2l} = \prod_{\sigma\in\Sigma} \mu_\sigma^{\kappa_\sigma+2l_\sigma}$$

If we assume $\kappa_\sigma + 2l_\sigma = w$ for all σ, then

$$\mu^{\kappa+2l} = N_{F/\mathbb{Q}}(\mu)^w.$$

So in order to descend to Y_U, we need to kill

this action

If assume that $\kappa+2l = (w, \ldots, w)$

and U is suff. small and s.t

$N_{F/\mathbb{Q}}(\mu) = 1 \quad \forall\, \mu \in \mathcal{O}_F^\times \cap U$ if w is odd

Then this line bundle descends to Y_U and we call it $\mathcal{L}_U^{k,l}$.

(HMFC for G)

<u>Def^n</u>: An element of $H^0(Y_U, \mathcal{L}_U^{k,l})$ is a HMF of wt (k,l) and level U. Denote this $M_{k,l}(U)$

("this is the analogue of the space $M_k(U_1(n))$")

Let $\overline{Y}_{J,N}$ denote the special fibre of $Y_{J,N}$ and $\overline{\mathcal{L}}_{J,N}^{k,l}$ the pull-back, then if we

$$\mu^{k+2l} \equiv 1 \mod \pi \quad \forall\, \mu \in \mathcal{O}_F^\times \cap U$$

the sheaf will descen to $\overline{Y}_U$ giving a sheaf $\overline{\mathcal{L}}_U^{k,l}$.

Its sections are mod p Hilbert modular form.

In general if R is an $\mathcal{O}$-alg s.t. $\mu^{k+2l} = 1$ in R we get a line bundle $\mathcal{L}_{U,R}^{k,l}$ on

$$Y_{U,R} = Y_U \times R.$$

whose sections are HMFs of wt (k,l), level U over R. Denote these by $M_{k,l}(U,R)$

<u>Fact</u>: For R Noetherian, then $M_{k,l}(U,R)$ is a fin. gen. R-module.

4) Partial Hasse invariants.

Recall we have the Verschiebung $Ver_A : A^{(p)} \to A$ an isogeny on A/S of char p.

this induces a $\mathcal{O}_F \otimes \mathcal{O}_S$-lin homomorphism

$$s_* \Omega^1_{A/S} \longrightarrow s_* \Omega^1_{A^{(p)}/S} =: Fr_S^* s_* \Omega^1_{A/S}$$

So if we take $A = A_{J,N}$ the universal HBAV

and $S = \overline{Y}_{J,N}$.

Then if we decompose these spaces on each side we have

$$\bigoplus_{\theta \in \Sigma} \overline{\omega}_\theta \longrightarrow \bigoplus_{\theta \in \Sigma} \overline{\omega}^{\otimes p}_{Fr^{-1} \circ \theta}$$ here Fr is Frob on $\overline{\mathbb{F}}_p$.

So projecting to one component we have a map

$$\overline{\omega}_\theta \longrightarrow \overline{\omega}^{\otimes p}_{Fr^{-1} \circ \theta}$$

Which we view as a section of

$$\overline{\omega}_{Fr^{-1}\sigma}^{\otimes p} \otimes \omega_\sigma^{-1} =: \overline{\mathcal{L}}_{J,N}^{(K_{Ha_\sigma},0)} \quad \text{denoted } Ha_{J,N,\sigma}$$

where K_{Ha_σ} is $\begin{cases} K_\sigma = p-1, \ K_{\sigma'} = 0 \ \sigma' \neq \sigma \text{ if } Fr^{-1}\sigma = \sigma \\ K_\sigma = -1 \quad K_{Fr^{-1}\sigma} = p, \ K_{\sigma'} = 0 \text{ else} \\ \qquad \text{if } Fr^{-1}\sigma \neq \sigma \end{cases}$

As before, since $\mu^{K \in \mathbb{Z}\ell} \equiv 1 \mod \pi \ \forall\, \mu \in \mathcal{O}_F^\times \cap U$

$\overline{\mathcal{L}}_U^{(K_{Ha_\sigma},0)}$ is defined on $\overline{Y}_U$

and since $V_{K,\ell}$ is compatible with base change

the section on $\coprod \overline{Y}_{J,N}$ descend to $\overline{Y}_U$
and we denote them $Ha_{U,\sigma}$.

Looking at the Hecke action one finds that

$$[U_1 g U_2]\, Ha_{U_2,\sigma} = N_{K/\mathbb{Q}}(g)^{-1}\, Ha_{U_1,\sigma}$$

So in $Ha_{U,\sigma} \in M_{K_{Ha_\sigma},0}(E) = \varinjlim_U M_{K,\ell}(E,U)$

This is indep of U and we denote it by $H_{a\sigma}$.

Call it σ-th partial Hasse invariant.

<u>Prop</u>: For any (κ, ℓ), multiplication by $H_{a\sigma}$ gives maps

- $M_{\kappa,\ell}(\mathcal{E}) \to M_{\kappa + \kappa_{H_{a\sigma}}, \ell}(\mathcal{E})$

- $M_{\kappa,\ell}(U,\mathcal{E}) \longrightarrow M_{(\kappa + \kappa_{H_{a\sigma}}, \ell)}(U,\mathcal{E})$

Commuting with Hecke operators away from p.

5| Theta operators

To define them we will make use of Igusa level structures.

We have a scheme $Y_U^{Ig} \xrightarrow{\pi_U} \overline{Y_U}$

which is finite flat and has an action by $(\mathcal{O}_{F/p})^\times$ which identifies

$$\overline{Y_U} \text{ with } Y_U^{Ig} / (\mathcal{O}_{F/p})^\times$$

To briefly see what this is one has

$$Y^{Ig}_{J,\mathcal{N}} \to \overline{Y_{J,\mathcal{N}}}$$ which relatively parametrizes

a choice of closed emersion of $\mathcal{O}_{F/p} \hookrightarrow A[p]$

"then Y_U^{Ig} is the quotient of the union over J_E of $Y^{Ig}_{J_{E},N}$ by the action of $G_{0,N}$"

This construction comes with a injective morphism

$$\overline{\mathcal{L}}_U^{K,0} =: \overline{\omega}_\sigma \quad \text{where } K = (0_c, \underset{\sigma}{1}, 0, 0)$$

$$\overline{\omega}_\sigma^{-1} \to \pi_{U,*}\mathcal{O}_{Y_U^{Ig}}$$

which we view as a section of $\pi_U^*\overline{\omega}_\sigma$.

we call it the fundamental Hasse invariant, h_σ

Fact: $h^{p}_{F^{-1}\sigma} = h_\sigma \, \pi_U^*(Ha_\sigma)$

- let $\mathcal{F}_U$ be the sheaf of total fractions on $\overline{Y_U}$

and $F_U = H^0(\overline{Y_U}, \mathcal{F}_U)$ = Product of function fields of components of $\overline{Y_U}$

Similarly for Y_U^{Ig}, $\mathcal{F}_U^{Ig}$, F_U^{Ig}.

- We have a Kodaira-Spencer iso on $Y_{S,N}$ whose restriction over $\mathfrak{I}$ descends to an iso.

$$\Omega^1_{\overline{Y}_U/\mathcal{E}} \xrightarrow{\sim} \bigoplus_{\sigma} \left(\overline{\omega}_\sigma^{\,2} \otimes \overline{\delta}_\sigma^{\,-1}\right)$$

giving a map $KS_\sigma : \Omega^1_{\overline{Y}_U/\mathcal{E}} \to \overline{\omega}_\sigma^{\,2} \otimes \overline{\delta}_\sigma^{\,-1}$

- We also have an iso

$$\pi_U^*\left(\Omega^1_{\overline{Y}_U/\mathcal{E}} \otimes F_U\right) \xrightarrow{\sim} \Omega^1_{Y_U^{Ig}/\mathcal{E}} \otimes F_U^{Ig}$$

<u>Defn</u>: Let $f \in M_{k,\ell}(U,\mathcal{E})$, $h^k = \prod h_\sigma^{k_\sigma}$, $g^\ell = \prod g_\sigma^{\ell_\sigma}$ where g_σ is a triv. of $\overline{\delta}_\sigma$

$$\Theta_\sigma^{Ig}(f) \quad \begin{array}{c} f \\ \big\downarrow \\ h^{-k}\pi_U^*(g^{-\ell}f) \in F_U^{Ig} \\ \big\downarrow d \\ d\left(h^{-k}\pi_U^*(g^{-\ell}f)\right) \in \Omega^1_{F_U^{Ig}/\mathcal{E}} \\ \big\downarrow \end{array}$$

$$KS_\theta^{I_g}\left(d\left(h^{-k}\pi_U^*(g^{-\ell}f)\right)\right) \in \pi_U^*\left(\overline{\omega}_\theta^{2}\otimes\delta_\theta^{-1}\right)\otimes\mathcal{F}_U^{I_g}$$

$$\downarrow \qquad\qquad \vee$$

$$h^{k}\pi_U^*(g^{\ell}Ha_\theta)\, KS_\theta^{I_g}\left(d\left(h^{-k}\pi_U^*(g^{-\ell}f)\right)\right)$$

$$\wedge$$

$$H^0\left(Y_U^{I_g},\ \pi_U^*\left(\overline{\mathcal{L}}_U^{k',\ell'}\right)\otimes\mathcal{F}_U^{I_g}\right)$$

Thm: if $f \in M_{k,\ell}(U,E)$ then $\Theta_\theta^{I_g}$ induces a map $\Theta_\theta(f)$ to $M_{k',\ell'}(U,E)$ which commutes with Hecke operators away from p

and moreover $\Theta_\theta(f)$ is divisible by Ha_θ iff f is divisible by Ha_θ or k_θ is divisible by p.

where

- if $Fr\circ\theta = \theta$ $\quad k'_\theta = k_\theta + p + 1$, $k'_{\theta'} = k_{\theta'}$ $\theta' \neq \theta$
- if $Fr\circ\theta \neq \theta$ $\quad k'_\theta = k_\theta + 1$, $k'_{Fr^{-1}\circ\theta} = k_{Fr^{-1}\circ\theta} + p$
  $k'_{\theta'} = k_{\theta'}$ else
- $\ell'_\theta = \ell_\theta - 1$ and $\ell'_{\theta'} = \ell_{\theta'}$ if $\theta' \neq \theta$.