Minimal weights of mod p Hilbert modular forms

(joint w/ F. Diamond).

Introduction: $N \geq 3$, $p \nmid N$ prime

Recall from Lorenzo's talk:

- $\overline{X}_1(N)$ Complete modular curve$/\overline{\mathbb{F}}_p$
- ω the usual sheaf
- $H^0(\overline{X}_1(N), \omega^k) = M_k(\Gamma_1(N); \overline{\mathbb{F}}_p)$ Space of mod-p modular forms.
- $h \in M_{p-1}(\Gamma_1(N), \overline{\mathbb{F}}_p)$ the Hasse invariant.

Unlike complex modular forms, mod-p modular forms of different weight can have the same q-expansion. Let $0 \neq f \in M_k(\Gamma_1(N); \overline{\mathbb{F}}_p)$. The <u>filtration of f</u> is the minimal weight of a modular form with the same q-expansion as f. If $f = h^r \cdot g$ where $r \geq 0$, and g is not divisible by h, then the filtration of f equals the weight of g.

Since ω is an ample line bundle on $\overline{X_1(N)}$, the filtration of f is always ≥ 0.

Now let F be a quadratic totally real field in which p is inert. Recall from Chris's talk the notion of mod-p HMF for F. In particular, we have the partial Hasse invariants

$$h_1 \text{ of weight} = (-1, p)$$
$$h_2 \text{ of weight} = (p, -1)$$

Unlike the case of modular forms, it is possible for mod p HMFs to have weights with negative components.

Andreatta-Goren asked if the phenomenon of negative weights for HMFs can be accounted for entirely by partial Hasse invariants

Let us explain. Let $f \neq 0$ be a mod p HMF.

Write $f = h_1^r h_2^s g$, $r, s \geq 0$ & g is not divisible by h_1, h_2.

We define the filtration of

F, $\Phi(f)$, to be the weight of g.

AG asked if it is true that $\Phi(f)$ lies in the cone of non-negative weights. In particular, this would imply that if

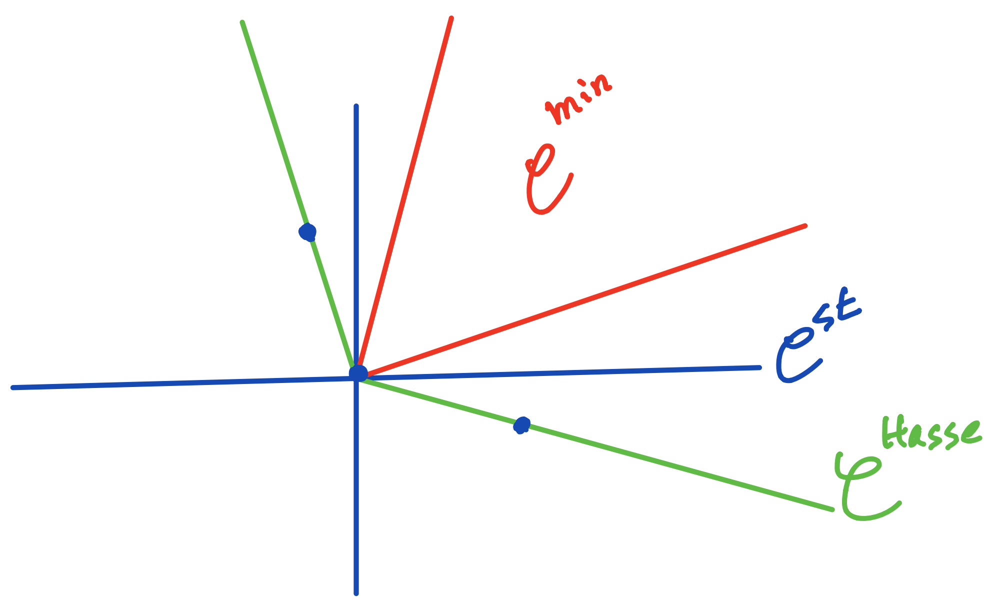
$\mathcal{C}^{\text{Hasse}}$ = Cone of weights spanned by the weights of the partial Hasse invariants

then there are no nonzero mod p HMFs of weight outside $\mathcal{C}^{\text{Hasse}}$.

We answer this question in the positive. In fact, we prove a stronger result.

Let $\mathcal{C}^{\min} = \left\{ (x,y) \in \mathbb{Q}^2 \;\middle|\; \begin{array}{l} px \geq y \\ py \geq x \end{array} \right\}$

We show that for every nonzero mod p HMF f, we have $\Phi(f) \in \mathcal{C}^{\min}$.



Notation

- p prime
- F totally real field of degree $d > 1$
- $\mathcal{O}_F$ ring of integers
- $\forall\, \mathfrak{p} \mid p$: $F_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}}, \mathbb{F}_{\mathfrak{p}} = \mathcal{O}_F/\mathfrak{p}$
- $\mathbb{F}$ a finite field containing all $\mathbb{F}_{\mathfrak{p}}$ for $\mathfrak{p} \mid p$
- $\Sigma = \mathrm{Hom}(F, \overline{\mathbb{Q}})$
- $$\begin{aligned} \Sigma_{\mathfrak{p}} &= \mathrm{Hom}(F_{\mathfrak{p}}, \overline{\mathbb{Q}}_p) \\ &= \mathrm{Hom}(F_{\mathfrak{p}}, W(\mathbb{F})) \\ &= \mathrm{Hom}(\mathbb{F}_{\mathfrak{p}}, \mathbb{F}) \circlearrowleft \sigma \end{aligned}$$

$\sigma = \text{Frob.}$

$$\Rightarrow \Sigma = \coprod_{\mathfrak{p} \mid p} \Sigma_{\mathfrak{p}}$$

Hilbert modular varieties:

Recall from Chris's talk the definition of HMVs.

$N \geq 3$, $p \nmid N$, $J \subset F$ fractional ideal

$$\boxed{M_{J,N} = X_J / W(\mathbb{F})}$$

the scheme representing the functor

$$\left(\left(\begin{matrix}\text{loc. noeth.}\\ W(\mathbb{F})\text{-sc}\end{matrix}\right)\right) \longrightarrow ((\text{Sets}))$$

$$S \longmapsto \{\underline{A} = (A, i, \lambda, \alpha)/S\}/\simeq$$

- A/S abelian scheme of $\dim = d$
- $i : \mathcal{O}_F \hookrightarrow \mathrm{End}_S(A)$
- $\lambda : A^\vee \xrightarrow{\sim} A \otimes J$ polarization
- $\alpha : (\mathcal{O}_F / N\mathcal{O}_F)^2 \xrightarrow{\sim} A[N]$ $\mathcal{O}_F$-equiv.

J-polarized HBAS.

X_J is a smooth quasi-projective scheme over $W(\mathbb{F})$.

Let $J_1, \ldots, J_r$ be representatives for $Cl^+(F)$.

<u>Def</u> $X = \coprod_{i=1}^{r} X_{J_i}$

<u>Def</u> $\overline{X} = X \otimes_{W(\mathbb{F})} \mathbb{F}$

$$\overline{X}_{\overline{\mathbb{F}}_p} = \overline{X} \otimes_{\mathbb{F}} \overline{\mathbb{F}}_p$$

We also need the HMV of Iwahori-level. Let $Y_J/W(\mathbb{F})$ be the HMV classifying

$$\left\{ (\underline{A}, H) = (A, i, \lambda, \alpha, H) \right\} / \cong$$

- $\underline{A}$ is a J-polarized HBAS as before.
- $H \subset A[p]$ is an $\mathcal{O}_F$-stable fin. flat subgp scheme of rk p^d <u>isotropic</u> w.r.to any/all $\lambda \in \mathrm{Hom}^{\mathrm{Symm}}_{\mathcal{O}_F}(A, A^\vee)$ (i.e. $\lambda(H) \subset \left(\frac{A[p]}{H}\right)^\vee \subset A^\vee[p]$).

Def $\quad Y := \coprod_{i=1}^{r} Y_{J_2}.$

$$\overline{Y} = Y \otimes_{W(\mathbb{F})} \mathbb{F}$$

$$\exists \quad \pi, \pi' : Y \longrightarrow X$$

$$\pi(\underline{A}, H) = \underline{A}$$

$$\pi'(\underline{A}, H) = \underline{A}/H$$

<u>Note</u> If R is a $W(\mathbb{F})$-algebra

$$\Rightarrow \mathcal{O}_F \otimes_{\mathbb{Z}} R = \prod_{\tau \in \Sigma} R_\tau$$

where R_τ is R with $\mathcal{O}_F$-action coming from $\mathcal{O}_F \xrightarrow[\tau]{} W(\mathbb{F}) \longrightarrow R$

$\Rightarrow \forall$ $(\mathcal{O}_F \otimes_{\mathbb{Z}} R)$-module Λ decomposes as

$$\boxed{\Lambda = \bigoplus_{\tau \in \Sigma} \Lambda_\tau}$$

where

$$\Lambda_\tau = \{x \in \Lambda \mid ax = \tau(a)x, \ \forall a \in \mathcal{O}_F\}$$

<u>Def</u> Let $A \xrightarrow{\varepsilon} X$ be the universal HBAS. Define

$$\mathbb{H} = R^1 \varepsilon_* \Omega^\bullet_{A/X}$$

$$\cup$$

$$\underline{\omega} = \varepsilon_* \Omega^1_{A/X}$$

- Then
$$\mathbb{H} = \bigoplus_{\tau\in\Sigma} \mathbb{H}_\tau$$
$$\omega = \bigoplus_{\tau\in\Sigma} \omega_\tau$$
and $\omega_\tau \subset \mathbb{H}_\tau$ are locally free sheaves of rk 1, 2 on X respectively.

- We denote the pullback of $\mathbb{H}_\tau, \omega_\tau$ under $\pi : Y \to X$ by the same notation.

- Let $(\underline{B}, H)$ be the universal object over Y. We define
$$\bigoplus_{\tau\in\Sigma} \omega'_\tau = \omega' = \pi_* \Omega^1_{(B/H)/Y}$$
Each ω'_τ is a line bundle on Y.

Mod p Hilbert modular forms

- Let $\vec{k} = \sum k_\tau \vec{e}_\tau \in \mathbb{Z}^\Sigma$

define $\omega^{\vec{k}} := \bigotimes_{\tau \in \Sigma} \omega_\tau^{k_\tau}$

- Let R be an $\mathbb{F}$-algebra. The space of mod-p HMFs of weight k over R is defined to be

$$M_{\vec{k}}(N;R) := H^0(\overline{X} \otimes_{\mathbb{F}} R, \omega^{\vec{k}})$$

- Example: $\forall \tau \in \Sigma$, the partial Hasse invariant $h_\tau \in M_{\vec{h}_\tau}(N, \mathbb{F})$ where

$$\vec{h}_\tau = p\vec{e}_{\sigma^{-1}\tau} - \vec{e}_\tau,$$

defined by Chris last time:

$$h_\tau = \mathrm{Ver}^*_\tau \in \mathrm{Hom}(\omega_\tau, \omega^p_{\sigma^{-1}\tau})$$

Filtration: $0 \neq f \in M_{\vec{k}}(N, \mathbb{F}_p)$

The filtration of f, $\Phi(f)$, is the weight of the corresponding g for the unique max'l element in

$$\left\{ \sum x_\tau \vec{e}_\tau \in \mathbb{Z}^{\Sigma}_{\geq 0} \;\middle|\; \underset{\text{for some } g}{f = g \prod h_\tau^{x_\tau}} \right\}.$$

<u>Def</u>

$$\mathcal{C}^{st} = \left\{ \sum_\tau x_\tau \vec{e}_\tau \in \mathbb{Q}^\Sigma \;\middle|\; x_\tau \geq 0 \quad \forall \tau \in \Sigma \right\}$$

$$\mathcal{C}^{min} = \left\{ \sum x_\tau \vec{e}_\tau \in \mathbb{Q}^\Sigma \;\middle|\; p x_\tau \geq x_{\sigma^{-1}\tau} \quad \forall \tau \in \Sigma \right\}$$

$$\mathcal{C}^{Hasse} = \left\{ \sum y_\tau \vec{h}_\tau \in \mathbb{Q}^\Sigma \;\middle|\; y_\tau \geq 0 \quad \forall \tau \in \Sigma \right\}$$

$$\mathcal{C}^{min} \subset \mathcal{C}^{st} \subset \mathcal{C}^{Hasse}$$

Each inclusion is an equality iff p splits completely in F.

Theorem (Diamond–K)

Suppose $\vec{k} = \sum k_\tau \vec{e}_\tau \in \mathbb{Z}^\Sigma$, and $\tau \in \Sigma$ is such that $p k_\tau < k_{\sigma^{-1}\tau}$

Then multiplication by h_τ induces an isomorphism

$$M_{\vec{k}-\vec{h}_\tau}(N;\overline{\mathbb{F}}_p) \xrightarrow{\sim} M_{\vec{k}}(N,\overline{\mathbb{F}}_p)$$

Cor: Let $0 \neq f \in M_{\vec{k}}(N,\overline{\mathbb{F}}_p)$. Then $\Phi(f) \in \mathcal{C}^{\min}$.

Cor: If $M_{\vec{k}}(N,\overline{\mathbb{F}}_p) \neq 0$, then $\vec{k} \in \mathcal{C}^{\mathrm{Hasse}}$

(Cor. proved indep. by Goldring–Koskivirta)

We prove the analogues of these results in the case p is ramified in F in a recent work.

Stratifications on $\overline{X}, \overline{Y}$

$\overline{X}$: $\forall\ T \subset \Sigma$, we define

(Goren–Oort) $Z_T = V(h_\tau : \tau \in T)$

$$W_T = Z_T - \bigcup_{T' \supsetneq T} Z_{T'}$$

Then

- Z_T, W_T are nonsingular of pure dimension $d - |T|$
- Z_T projective if $T \neq \emptyset$
  W_T quasi-affine
- The collection $\{W_T\}_{T \in \Sigma}$ gives a stratification of $\overline{X}$.

$\overline{\underline{\Upsilon}}$ (Goren-K).

Let $(\underline{A}, H)$ be the universal object over $\overline{\Upsilon}$. Let $f: A \to A/_H$ be the natural projection and $g: A/_H \to A$ induced by $[p]$.

then $g \circ f = [p]_A$. Consider for $\tau \in \Sigma$:

$$f^*_\tau : \omega'_\tau \longrightarrow \omega_\tau$$

$$g^*_\tau : \omega_\tau \longrightarrow \omega'_\tau$$

In particular, $f^*_\tau \circ g^*_\tau = 0$ $\forall \tau \in \Sigma$.

Def Let $\varphi, \eta \subset \Sigma$ s.t. $\sigma^{-1}(\varphi) \cup \eta = \Sigma$

Define

$$Z_{\varphi,\eta} = \bigcap_{\tau \in \sigma^{-1}\varphi} V(f_\tau^*) \cap \bigcap_{\tau \in \eta} V(g_\tau^*)$$

$$W_{\varphi,\eta} = Z_{\varphi,\eta} - \bigcup_{(\varphi',\eta') \supsetneq (\varphi,\eta)} Z_{\varphi',\eta'}$$

Then

- $\{W_{\varphi,\eta}\}_{\varphi,\eta}$ forms a stratification of $\overline{Y}$.
- $Z_{\varphi,\eta}, W_{\varphi,\eta}$ are nonsingular of pure dimension $2d - (|\varphi| + |\eta|)$.
- $W_{\varphi,\eta}$ is quasi-affine.
- The irreducible components of $\overline{Y}$ are the irreducible components

of $Z_{\sigma(\eta)^c,\eta}$ for $\eta \subset \Sigma$.

Relationship between stratifications

- $\pi(Z_{\varphi,\eta}) = Z_{\varphi\cap\eta}$
- $\pi'(Z_{\varphi,\eta}) = Z_{\sigma^{-1}\varphi\,\cap\,\sigma\eta}$
- We have a commutative diagram

$$\begin{tikzcd}
Z_{\varphi,\eta} \arrow[r, "g"] \arrow[dr, "\pi"'] & \mathbb{P}\Big(\bigoplus_{\tau\in\varphi^c\cap\eta^c} \mathbb{H}_\tau\Big) \arrow[d, "pr"] \\
 & Z_{\varphi\cap\eta}
\end{tikzcd}$$

where pr is the natural projection and g is a Frobenius factor, i.e.,

$\exists \quad \mathbb{P}\big(\bigoplus_{\tau\in\varphi^c\cap\eta^c} \mathbb{H}_\tau\big) \xrightarrow{g'} Z_{\varphi,\eta}$

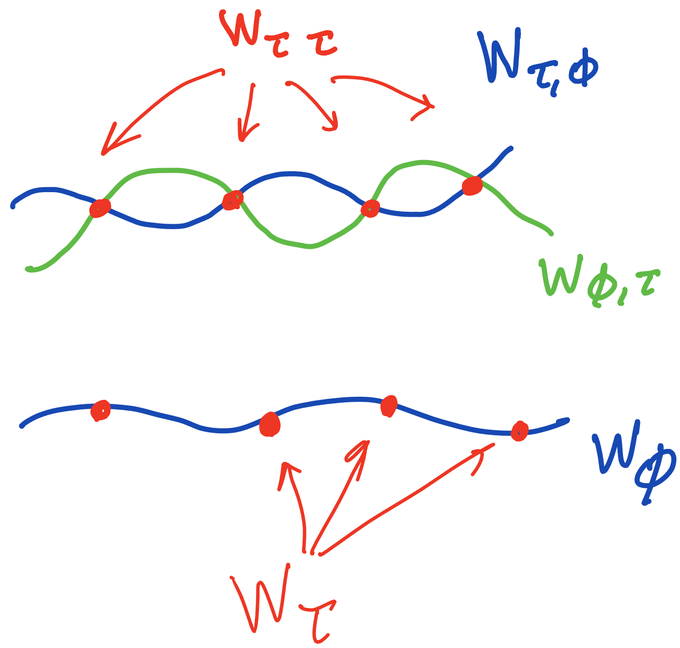
s.t. $gg' = F_{abs}^n$ for some $n>0$.

Some Pictures:

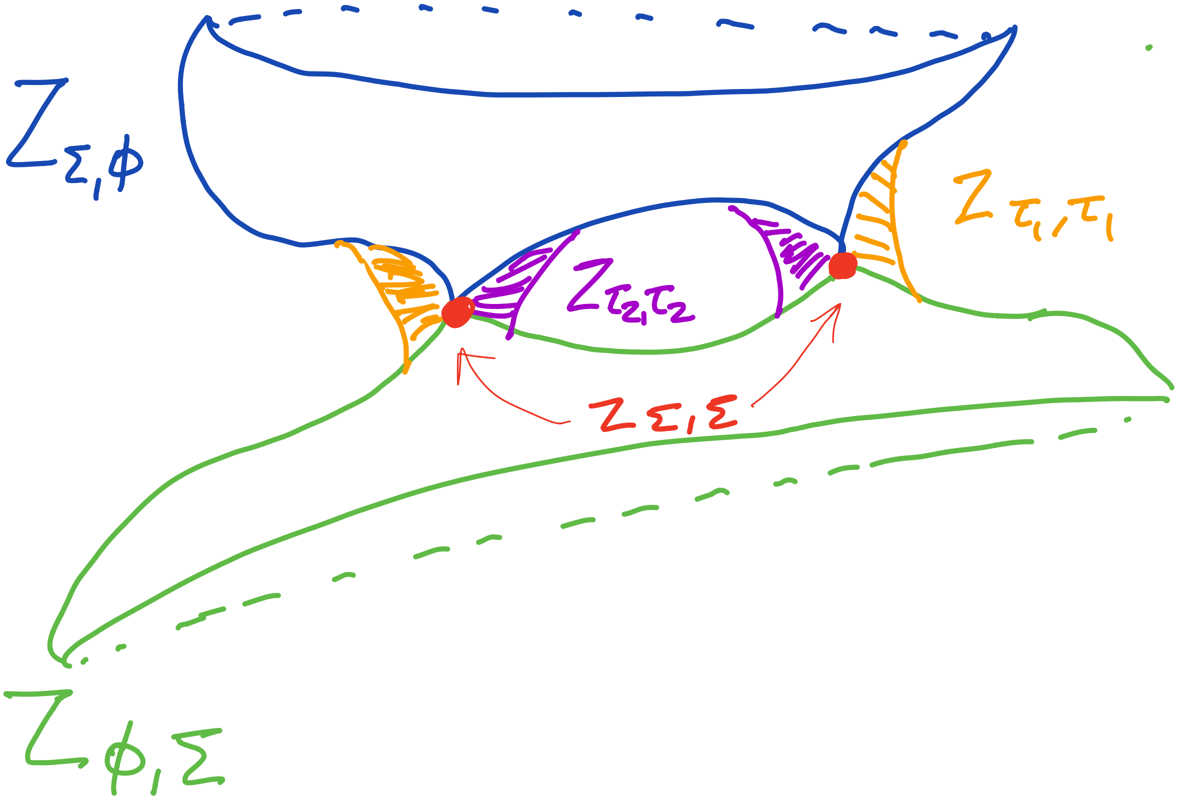
$d=1$ $\qquad$ Y

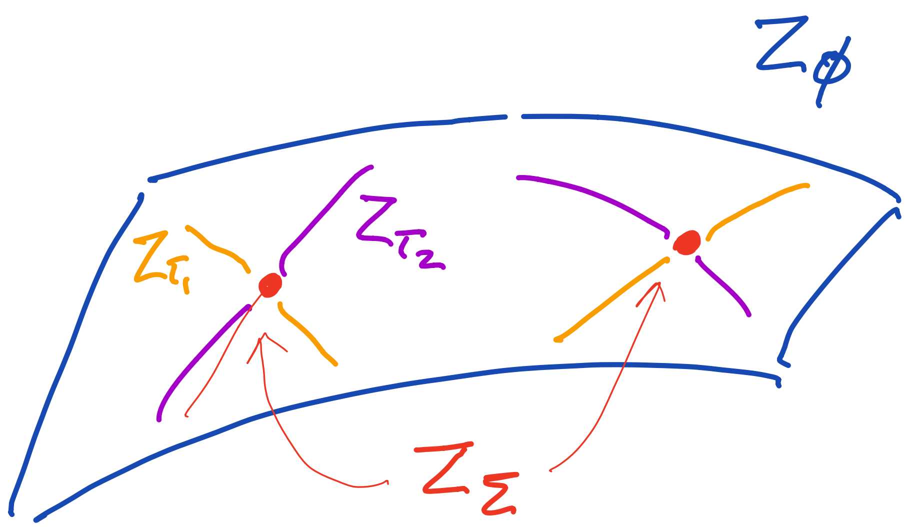
$\Sigma=\{\tau\}$ $\qquad$ $\downarrow$

$\qquad\qquad\quad X$



$d=2$, p inert, $\Sigma = \{\tau_1, \tau_2\}$





Global geometry of strata

(à la Helm)

<u>Tian-Xiao</u> Each $Z_T \subset \overline{X}$ is (Hecke) isomorphic to a $(\mathbb{P}^1)^r_-$bundle over some quaternionic Shimura variety (precise recipe)

<u>Example</u>: assume $\mathbb{F}_{\mathfrak{p}} \neq \mathbb{F}_p$, and $\tau_0 \in \Sigma_{\mathfrak{p}}$. Then Z_{τ_0} is isom. to a $\mathbb{P}^1_-$bundle over a quaternionic Shim. variety associated to $B^\times$ where B/F ramifies exactly at $\{\sigma^{-1}\tau_0, \tau_0\}$.

(placed at infinity)

Diamond–K.–Sasaki

Each $Z_{\sigma(\eta)^c,\eta}$ is (Hecke) isomorphic to a $(\mathbb{P}^1)^r$-bundle over a quaternionic Shimura variety associated to the quaternion alg. B/F ramified exactly at $(\sigma^{-1}\eta - \eta) \cup (\eta - \sigma^{-1}\eta)$. (placed at ∞)

Note

$$\begin{tikzcd}
Z_{r(\eta)^c,\eta} \arrow[d, "\pi"', "\text{Frob. } (\mathbb{P}^1)^s\text{-bundle}"] \arrow[dr, dotted, "??"] & \\
Z_{\eta - r(\eta)} \arrow[r, "(\mathbb{P}^1)^r\text{-bundle}"'] & X_B
\end{tikzcd}$$

Warning

This doesn't give correct Quat. algebras!

We initially proved our main result in the case p unramified in F using the above result of TX. which is as yet unavailable in the ramified case. We Later proved the general case using instead properties of the stratification on $\overline{Y}$ that generalize to the ramified case.

2 cases $\quad \mathfrak{q} \mid p$, $\tau_0 \in \Sigma_{\mathfrak{q}}$

① $\mathbb{F}_{\mathfrak{q}} \neq \mathbb{F}_p$. Assume $p k_{\tau_0} < k_{\sigma^{-1}\tau_0}$

We have an exact sequence:

$$0 \to H^0(\overline{X}, \omega^{\vec{k}-\vec{h}_{\tau_0}}) \xrightarrow{h_{\tau_0}} H^0(\overline{X}, \omega^{\vec{k}}) \to H^0(Z_{\tau_0}, \omega^{\vec{k}})$$

Enough to show $H^0(Z_{\tau_0}, \omega^{\vec{k}}) = 0$

$Z_{\tau_0} \cong \mathbb{P}^1$-bundle over a quaternionic shimura variety X_B.

Let $x \in X_B$ & $\mathbb{P}^1_x$ its fibre in Z_{τ_0}

Prop $\quad \vec{\omega}^{k}\big|_{\mathbb{P}^1_x} \cong \mathcal{O}_{\mathbb{P}^1_x}(p k_{\tau_0} - k_{\sigma^{-1}\tau_0})$

$\Rightarrow \quad H^0(Z_{\tau_0}, \omega^{\vec{k}}) = 0$

② $\mathbb{F}_q = \mathbb{F}_p$

We prove a stronger result:

If $k_{\tau_0} < 0$, then

$H^0(\overline{X}, \omega^{\vec{k}}) = 0$.

This uses very closely the geometric properties of the Goren-Oort strata including the quasi-affineness of the open strata.

(which we prove in the general case).

The new proof

- Assume we are in case 1 above: $\mathbb{F}_{\mathfrak{p}} \neq \mathbb{F}_p$.

  $\tau_0 \in \Sigma_{\mathfrak{p}}$, $k_{\sigma^{-1}\tau_0} > p k_{\tau_0}$

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "g'"] \arrow[d, "\pi' g'"'] & Z_{\{\sigma\tau_0\}, \{\tau_0\}^c} \arrow[r, "g"] \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \mathbb{P} \arrow[d] \\
Z_{\{\tau_0\}} & & Z_{\{\sigma\tau_0\}}
\end{tikzcd}$$

Enough to show $H^0(Z_{\tau_0}, \omega^{\vec{k}}) = 0$

OR $H^0(\mathbb{P}, (\pi' g')^* \omega^{\vec{k}}) = 0$

Prop: $(\pi' g')^* \omega^{\vec{K}'}\big|_{\mathbb{P}^1_x} \simeq$

$$\mathcal{O}_{\mathbb{P}^1_x}\left(\left(p\,K_{\tau_0} - K_{\sigma^{-1}\tau_0}\right) p^{r-1}\right)$$

From which the result follows.

Question asked by Pol on relationship between Goren-Oort strata & Newton Polygon strata.

Assume p is inert in F.

The possible p-div. groups up to isogeny are given by

$$G_{i/d} + G_{(d-i)/d} \quad \text{for } 0 \leq i \leq d$$

and $dG_{1/2}$.

Let $\beta_0 < \beta_1 < \cdots < \beta_{\left[\frac{d+1}{2}\right]}$ be the Newton polygons

Let $T \subset \Sigma$, and recall

W_T = open stratum where $h_\tau = 0 \iff \tau \in T$.

Let $\lambda(T)$ = the size of a maximal spaced subset S of T (i.e., $\forall \tau$: $\{\tau, \sigma\tau\} \not\subset S$)

[exception: d odd, $\lambda(\{1,\ldots,d\}) = \frac{d+1}{2}$]

Then the generic point of every component of W_T has $NP = \beta_{\lambda(T)}$

For example when $d=4$, p_{inert} the dimension of the supersingular locus is 2. And the above gives the following generic description of NPs on

the Goren-Oort Strata:

T in blue, $\lambda(T)$ in red, $\dim W_T$ green

ϕ 0	$\{1\}$ 1	$\{1,2\}$ 1	$\{1,2,3\}$ 2	$\{1,2,3,4\}$ 2
	$\{2\}$ 1	$\{1,3\}$ 2	$\{1,2,4\}$ 2	
	$\{3\}$ 1	$\{1,4\}$ 1	$\{1,3,4\}$ 2	
	$\{4\}$ 1	$\{2,3\}$ 1	$\{2,3,4\}$ 2	
		$\{2,4\}$ 2		
		$\{3,4\}$ 1		
4	3	2	1	0