COHOMOLOGICAL INTERLUDE: THE BDJ CONJECTURE

London N.T. Study Group - 3 June 2020
The weight part of Serre's Conj - week 7

Recall a continuous irreducible $\bar\rho: G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is <u>modular</u> of weight k (≥ 2) and level N (prime to p) if $\bar\rho \simeq \bar\rho_f$ for some eigenform $f \in S_k(N,\mathbb{C}) = S_k(\Gamma_1(N))$, i.e.

$$\mathrm{tr}(\bar\rho(\mathrm{Frob}_v)) = \overline{a_v(f)} \quad \& \quad \det(\bar\rho(\mathrm{Frob}_v)) = \overline{v^{k-1}\chi_f(v)}$$

for all (but finitely many) primes $v \nmid Np$.

$$\left(\text{Fix: } \begin{array}{ccccccc} & & \overline{\mathbb{Q}} & \supset & \overline{\mathbb{Z}} & a \mapsto \bar a & \\ & \swarrow & & \searrow & & \searrow & \overline{\mathbb{F}}_p \\ \mathbb{C} & & & \overline{\mathbb{Q}}_p & \supset & \overline{\mathbb{Z}}_p & \nearrow \end{array} \right)$$

Equivalently, $\exists$ eigenform $f \in S_k(N, \overline{\mathbb{F}}_p)$ s.t.

$$\mathrm{tr}(\bar\rho(\mathrm{Frob}_v)) = a_v(f) \quad \& \quad \det(\bar\rho(\mathrm{Frob}_v)) = v^{k-1}\chi_f(v)$$

for all but finitely many $v \nmid Np$.

Proof: $\Rightarrow$ easy

$\Leftarrow$: Recall $S_k(N, \overline{\mathbb{Z}}_p) \twoheadrightarrow S_k(N, \overline{\mathbb{F}}_p)$ for $k \geq 2$

$$\tilde{\mathbb{T}} \twoheadrightarrow \mathbb{T} \twoheadrightarrow \overline{\mathbb{T}}$$

polyn. alg$/\overline{\mathbb{Z}}$ gen by $T_v, \langle v\rangle$ — endom. algebras gen. by $T_v, \langle v \rangle$

char 0 eigenforms $\longleftrightarrow$ minimal primes in $\mathbb{T}$

Going Down Thm

char p eigenforms $\longleftrightarrow$ max'l ideals in $\mathbb{T}$ or $\overline{\mathbb{T}}$ $\square$

Equivalently, $m_{\bar\rho} = \ker(\tilde{\mathbb{T}} \to \overline{\mathbb{F}}_p) \in \mathrm{Supp}(S_k(N, \overline{\mathbb{Z}}_p))$.

For $k \geq 2$, can also give a more algebraic / cohomological interpretation of modularity, using:

Theorem (Eichler-Shimura):
$\exists$ Hecke-equivariant $S_k(N,\mathbb{C})^2 \cong H^1_p(Y_1(N), \mathcal{L}_{k-2}(\mathbb{C}))$
$\cong H^1_p(\Gamma_1(N), L_{k-2}(\mathbb{C}))$

where $H^1_p = \mathrm{im}(H^1_c \to H^1)$,
$L_n(R) = \mathrm{Sym}^n R^2 \circlearrowleft SL_2(\mathbb{Z})$, and
$\mathcal{L}_n(R) = \Gamma_1(N)\backslash(\mathfrak{h} \times L_n(R))$ over $Y_1(N) = \Gamma_1(N)\backslash\mathfrak{h}$

So: ρ is modular of wt k, level N
$\Leftrightarrow$ m_ρ is in the support of $H^1_p(\Gamma_1(N), L_{k-2}(\mathbb{C}))$
$\Leftrightarrow$ " " $H^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$

(since $\mathrm{coker}(H^1_c \to H^1) \leftrightarrow$ Eisenstein series,
so ρ irred $\Rightarrow$ $m_\rho \notin$ support)
$\Leftrightarrow$ m_ρ is in the support of $H^1(\Gamma_1(N), L_{k-2}(\mathbb{Z}_p))$
(since torsion $\leftrightarrow$ $H^0(\Gamma_1(N), L_{k-2}(\mathbb{F}_p))$ is "Eisenstein")
$\Leftrightarrow$ m_ρ is in the support of $H^1(\Gamma_1(N), L_{k-2}(\mathbb{F}_p))$
$\Leftrightarrow$ " " of $H^1(\Gamma_1(N), V_{m,n}) \cong H^1(Y_1(N), \mathcal{V}_{m,n})$
for some JH-factor $V_{m,n} = \det{}^m \mathrm{Sym}^n \mathbb{F}_p^2$ $(0 \leq n \leq p-1)$
of $\mathrm{Sym}^{k-2}\mathbb{F}_p^2$ as a repn of $\Gamma_1(N) \twoheadrightarrow GL_2(\mathbb{F}_p)$

Furthermore $H^1(\Gamma_1(N), V_{m,n}) \cong H^1(\Gamma_1(N), V_{0,n})$

$v^m T_v \longleftrightarrow T_v$

$v^{2m}\langle v\rangle \longleftrightarrow \langle v\rangle$

Conclusion: ρ is modular of wt k, level N

$\Longleftrightarrow$ $\mathfrak{m}_\rho$ is in the support of $H^1(\Gamma_1(N), V_{m,n})$
for some $V_{m,n} \in JH(\mathrm{Sym}^{k-2}\mathbb{F}_p^2)$

$\Longleftrightarrow$ $\mathfrak{m}_{\rho\otimes\omega^{-m}} \in \mathrm{Supp}(H^1(\Gamma_1(N), V_{m,n}))$ for such a $V_{m,n}$

$\Longleftrightarrow$ $\rho\otimes\omega^{-m}$ is modular of level N
and wt $n+2$ for such a $V_{m,n}$

This proves that if ρ is modular of some weight k (≥ 2) & level N prime to p,

then $\{0\leq m\leq p-2,\ 0\leq n\leq p-1 \mid \rho\otimes\omega^{-m}$ is modular of level N, wt $n+2\}$

is non-empty, and determines

$\{k\geq 2 \mid \rho$ is modular of wt k, level $N\}$

For such (m,n), say ρ is modular of wt $V_{m,n}$
if $\mathfrak{m}_\rho$ is in the support of $H^1(Y_1(N), \mathcal{V}_{m,n})$

Instead of the minimal weight,
ask for: $\{V_{m,n} \mid \rho$ is modular of wt $V_{m,n}\}$

Generalize this formulation... given ρ, automorphic w.r.t. G, describe its "Serre weights" as a set of irred. repns of $G(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$

- Ash et al: GL_d, $d>2$
- Buzzard-D-Jarvis: $\mathrm{Res}_{F/\mathbb{Q}} GL_2$ (Hilbert modular setting)
- Gee-Herzig-Savitt: more general reductive G

Rest of today's talk:

F totally real, $\Sigma = \{F \hookrightarrow \mathbb{R}\}$

$d = [F:\mathbb{Q}]$ $\quad = \{F \hookrightarrow \overline{\mathbb{Q}}_p\} = \coprod_{v|p} \Sigma_v$

$\vec{k}, \vec{l} \in \mathbb{Z}^\Sigma$ s.t. $k_\theta + 2l_\theta$ indep. of θ

Theorem: If $f \in M_{\vec{k},\vec{l}}(U_1(n))$ is a Hecke eigenform with $T_v f = a_v f$ and $S_v f = d_v f$ $\forall v \nmid np$, then $\exists!$ semisimple (irred $\Leftrightarrow$ f cuspidal) $\rho_f : G_F \to GL_2(\overline{\mathbb{Q}}_p)$ s.t. $\forall v \nmid np$, ρ_f is unram. at v & $\rho_f(\mathrm{Frob}_v)$ has char poly: $X^2 - a_v X + d_v N_{F/\mathbb{Q}} v$.

Furthermore if $k_\theta \geq 2$ $\forall \theta$ and $v \mid p$, then $\rho_f|_{G_{F_v}}$ is de Rham (crystalline $\Leftarrow$ $v \nmid n$) with labelled HT weights $(l_\theta, k_\theta + l_\theta - 1)_{\theta \in \Sigma_v}$.

$\left(i \text{ s.t. } \mathrm{gr}^i D_{HT}(\rho_f|_{G_{F_v}}) = \bigoplus_{\theta \in \Sigma_v} \mathrm{gr}^i D_{HT}(\rho_f|_{G_{F_v}})_\theta \right)$

$F_v \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = \bigoplus_{\theta \in \Sigma_v} \overline{\mathbb{Q}}_p$

Fontaine-Mazur-Langlands Conjecture: Every totally odd, irreducible, geometric $\rho : G_F \to GL_2(\overline{\mathbb{Q}}_p)$ arises this way.

Folklore Conjecture: Every totally odd, irreducible $\rho : G_F \to GL_2(\overline{\mathbb{F}}_p)$ arises this way (i.e. $\cong \bar{\rho}_f$)

What can we say about the possible n & $\vec{k}$?

Minimal prime-to-p part of $\mathfrak{n}$ = Artin conductor of ρ
(determined by $\rho|_{I_{F_v}}$ for $v \nmid p$)

$\vec{k}$? Again expect this to be determined by $\rho|_{I_{F_v}}$ for $v|p$, but unlike $F=\mathbb{Q}$, not all ρ can arise from level prime to p
(would $\Rightarrow$ $\det \rho|_{I_{F_v}} = \omega_{cyc}^{n}$, $n = k_\theta + \ell_\theta - 1$).

Instead take the alg. p.o.v. & ask if the corresp. Hecke eigensystems arise in suitable cohomology groups. Direct analogue would be:

$H^d(Y_1(\mathfrak{n}), \mathcal{V})$ for $(\mathfrak{n}, p) = 1$ & suitable local systems $\mathcal{V} \leftrightarrow$ irred. repns of $GL_2(\mathcal{O}_F/p)$ over $\overline{\mathbb{F}}_p$.
$d > 1$ introduces complications (e.g. torsion), and this also isn't where the ρ_f come from.

Instead use Jacquet-Langlands to switch setting from $GL_2(F)$ to a suitable quat. alg. D/F.

For simplicity, assume for now d odd, p inert in F, and let D be a quat. alg. $/F$ split at exactly one real place and all finite places, so $(D \otimes \mathbb{A}_{\mathbf{f}})^\times \cong GL_2(\mathbb{A}_{F,\mathbf{f}})$

$\rightarrow$ Shimura curve $Y = Y_1^D(\mathfrak{n})_{/F}$ with

$$Y(\mathbb{C}) = D_{>0}^\times \backslash \mathfrak{h} \times GL_2(\mathbb{A}_{F,\mathbf{f}}) / U_1(\mathfrak{n}) \cong \coprod_t \Gamma_t \backslash \mathfrak{h},$$

with locally const. $\mathcal{V}_{\vec{m},\vec{n}} = \coprod_t \Gamma_t \backslash (\mathfrak{h} \times V_{\vec{m},\vec{n}})$

$$V_{\vec{m},\vec{n}} = \bigotimes_{\theta \in \Sigma} \left(\det^{m_\theta} \otimes \operatorname{Sym}^{n_\theta} \overline{\mathbb{F}}_p^2 \right) \underset{\theta}{\circlearrowleft} GL_2(\mathcal{O}_F/p),$$

irred $\Longleftrightarrow$ $n_\theta \le p-1 \ \forall \theta$

Say ρ is modular of weight $V_{\vec{m},\vec{n}}$ (and level n)
if m_ρ $(= \ker(\tilde{\mathbb{T}} \to \bar{\mathbb{F}}_p))$ is in the support of $H^1(Y(\mathbb{C}), \mathcal{V}_{\vec{m},\vec{n}})$
($\Leftrightarrow$) ρ arises in $H^1_{ét}(Y_{\bar{F}}, \mathcal{V}^{ét}_{\vec{m},\vec{n}})$).

As in the classical case, if ρ is modular, then

$$W_{mod}(\rho) := \{ \text{irred } V \mid \rho \text{ is modular of wt } V \text{ \& level prime to } p \}$$

is non-empty, and determines all possible wts (& level structures at $v \mid p$) of eigenforms f with all $k_\theta \geq 2$ giving rise to ρ (i.e. $\rho \simeq \bar{\rho}_f$).

Conjecture: $\quad W_{mod}(\rho) = W_{cris}(\rho)$, where

$$W_{cris}(\rho) := \{ V_{\vec{m},\vec{n}} \mid \rho|_{G_{F_v}} \text{ has a crys. lift w/ labelled HT wts } (m_\theta, m_\theta + n_\theta + 1)_{\theta \in \Sigma} \}.$$

This formulation is due to Gee, based on a more explicit $W_{BDJ}(\rho) \subset W_{cris}(\rho)$, later proved $=$ by G-Liu-Savitt, and generalizes to arbitrary totally real F.

Example: $\quad d=2$, p inert, fix $\theta = \theta_0$,

$$\rho|_{G_{F_v}} = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}, \quad \text{with} \quad \psi|_{I_{F_v}} = \omega_\theta^{a_0 + a_1 p},$$

$$1 \leq a_0, a_1 \leq p \quad (\text{not both } 1)$$

where ω_θ is the fund. char:

$$I_{F_v} \twoheadrightarrow \mathcal{O}_{F_v}^\times \twoheadrightarrow (\mathcal{O}_F/p)^\times \xrightarrow[\theta]{\sim} \bar{\mathbb{F}}_p^\times$$

$$g \longmapsto g(\varpi)/\varpi \bmod \varpi, \quad \varpi^{p^2-1} = p$$

Then $V_{\vec{0},\vec{n}} \in W_{cris}(\rho)$, where $\vec{n} = (n_0 - 1, n_1 - 1)$,
but there may be more ...

Assume for simplicity $\psi|_{I_{F_v}} \neq 1$ or $\omega_{cyc} = \omega_\theta^{p+1}$ $(= \omega_\theta^{p+p^2})$.

$\Updownarrow$ $a_0 = a_1 = p-1$ $\quad$ $\searrow$ $a_0 = a_1 = p$

Then $* \rightsquigarrow c_\rho \in H^1(G_{F_v}, \bar{\mathbb{F}}_p(\psi))$, which has dim 2,

and

$$\rho|_{I_{F_v}} \simeq \omega_\theta^{-a_0'} \otimes \begin{pmatrix} \omega_\theta^{pa_1'} & * \\ 0 & \omega_\theta^{a_0'} \end{pmatrix}$$

for some $\vec{n}' = (a_0', a_1')$ with $1 \leq a_0', a_1' \leq p$,

$(= (p-a_0, a_1+1)$ if $a_0, a_1 < p)$

& get $V_{\vec{m}', \vec{n}'} \in W_{cris}(\rho) \iff c_\rho \in L'$,

where L' is a one-dim'l subsp. of $H^1(G_{F_v}, \bar{\mathbb{F}}_p(\psi))$.

Similarly get a $V_{\vec{m}'', \vec{n}''} \in W_{cris}(\rho) \iff c_\rho \in L''$

for some one-dim'l L'' $(= L' \iff a_0$ or $a_1 = p)$,

and yet another weight if $\rho|_{G_{F_v}}$ splits (i.e. $c_\rho = 0$).

Theorem (Barnet-Lamb-G-Geraghty, G-Liu-Savitt, G-Kisin, Newton): If ρ satisfies a T-W hypothesis, then

$$W_{cris}(\rho) = W_{mod}(\rho).$$

Strategy initiated by Toby:

- Use automorphy lifting theorems to prove existence (!) and automorphy of lifts with prescribed local behavior at $v|p$.

- Play off the relation between the weights and level structures at p.

Example (related to DKS): Suppose p inert &
$\psi : (\mathcal{O}_F/p)^\times \to \overline{\mathbb{Z}}^\times$. Then $\rho \simeq \bar{\rho}_f$ for some f of
wt $\vec{k} = (2, \ldots, 2)$, level np, char ψ (i.e. inertial type $1 \oplus \psi$) at p
$\Leftrightarrow$ ρ is modular of some wt $\in$ $JH(\mathrm{Ind}_B^{GL_2(\mathcal{O}_F/p)}(1 \otimes \bar{\psi}))$.

Applying the approach with this example in the classical setting already leads to a proof of the Companion Forms Theorem: Suppose
$\rho|_{I_p} \sim \begin{pmatrix} \omega^{k-1} & 0 \\ 0 & 1 \end{pmatrix}$, $3 \le k \le p-1$.

Then $W_{crys}(\rho) = \{ V_{0,k-2}, V_{k-1,p-1-k} \}$ (& $V_{p-2,p-1}$ if $k = p-1$)

Show ρ is modular of wt 2 & type $1 \oplus \omega^{k-2}$.
Therefore ρ is modular of some weight
$\in JH(\mathrm{Ind}_B^{GL_2(\mathbb{F}_p)}(1 \otimes \omega^{k-2})) = \{ V_{0,k-2}, \cancel{V_{k-2,p+1-k}} \}$
$\Rightarrow$ modular of wt k, level prime to p.

In general, knowing the set of such ψ, or even all inertial types τ, s.t. ρ is modular of wt $(2, 2, \ldots, 2)$ & type τ, doesn't determine $W_{mod}(\rho)$, but more refined info - provided by the Breuil-Mézard Conjecture - does:

Gee et al prove that for $\sigma : G_{F_v} \to GL_2(\overline{\mathbb{F}}_p)$,
$\exists\ \mu_{\vec{m},\vec{n}}(\sigma)$ such that $\forall$ inertial types τ:

$$e_{HS}(\bar{R}_{\sigma,\tau}) = \sum_{\vec{m},\vec{n}} \mu_{\vec{m},\vec{n}}(\sigma) \cdot \nu_\tau(V_{\vec{m},\vec{n}})$$

Furthermore (for ρ satisf. TW-hypothesis)

$$V_{\vec{m},\vec{n}} \in W_{mod}(\rho) \Leftrightarrow \mu_{\vec{m},\vec{n}}(\rho|_{G_{F_v}}) > 0 \Leftrightarrow V_{\vec{m},\vec{n}} \in W_{crys}(\rho).$$

Here $R_{\sigma,\tau}$ = wt $(2,2,\ldots,2)$, type τ deformation ring.

e_{HS} = Hilbert-Samuel multiplicity

$\nu_\tau(V_{\vec{m},\vec{n}})$ = mult of $V_{\vec{m},\vec{n}}$ in the reduction of a representation of $GL_2(\mathcal{O}_{F,p})$ determined by τ (via local Langlands + theory of types).

Everything above assumes $k_\theta \geq 2 \ \forall \theta \in \Sigma$.

Next week - back to geometric setting to be able to study:

- (partial) weight one
- minimal weights
- θ-cycles