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Moduli of Galois representations and generalized class groups

James Newton

Abstract. This lecture describes some recent progress on establishing the conjectural relationships between Galois representations and automorphic forms, together with the number theoretic motivation for these results. This includes equidistribution theorems like the Sato–Tate conjecture concerning the number of points on reductions of elliptic curves modulo primes. The connection to these equidistribution statements goes back to Dirichlet's work on primes in arithmetic progressions, in which the class number formula for real quadratic fields played an important role. We discuss how results on 'generalized class groups' defined using Galois cohomology have been crucial to more recent developments.

1. Dirichlet's theorem

1.1. Primes in arithmetic progressions

A seminal result in number theory is Dirichlet's theorem on primes in arithmetic progressions:

Theorem 1.1 ([9]). *There are infinitely many primes of the form $a + kN$, with a and N fixed coprime integers, and k varying over $\mathbf{Z}$.*

This theorem (and its refinement, the Chebotarev density theorem) tells us about the distribution of the images of the primes in the multiplicative group $(\mathbf{Z}/N\mathbf{Z})^\times$. Dirichlet's proof relied on using the character theory of the group $(\mathbf{Z}/N\mathbf{Z})^\times$.

Indeed, suppose we have a homomorphism $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$.[1] Orthogonality of characters tells us that we can express the indicator function of the subset $\{a\} \subset (\mathbf{Z}/N\mathbf{Z})^\times$ as a linear combination of characters:

$$\mathbf{1}_{\{a\}} = \frac{1}{\phi(N)} \sum_{\chi} \overline{\chi}(a)\chi.$$

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[1] As is standard, we extend by zero to view χ as a function on $\mathbf{Z}/N\mathbf{Z}$ when convenient.

Informally, to show that the primes distribute evenly over $(\mathbf{Z}/N\mathbf{Z})^\times$, it suffices to show that the expectation $\mathbf{E}(\chi(p)) = 0$ for every non-trivial character χ, which gives $\mathbf{P}(p \equiv a \bmod N) = \frac{1}{\phi(N)}$.

More precisely, if one shows that

$$\frac{\sum_{p \le x} \chi(p)}{\#\{p \le x\}} \to 0 \text{ as } x \to \infty$$

for every non-trivial χ, then one deduces the equidistribution statement:

$$\frac{\#\{p \le x : p \equiv a \bmod N\}}{\#\{p \le x\}} \to \frac{1}{\phi(N)} \text{ as } x \to \infty. \tag{1.1}$$

In fact, Dirichlet proved something a little weaker than this, showing that the infinite sum over primes

$$\sum_p \frac{\chi(p)}{p^s}$$

(which converges absolutely for $\Re(s) > 1$) remains bounded as s approaches 1 from above. In contrast, the sum

$$\sum_p \frac{1}{p^s}$$

diverges as s approaches 1.

The behaviour of these sums is closely related to, respectively, the behaviour of the Dirichlet L-function $L(\chi, s)$ and the Riemann zeta function $\zeta(s)$ at $s = 1$. Recall that

$$L(\chi, s) = \sum_{n \ge 1} \frac{\chi(n)}{n}^s = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

(again, these expressions converge absolutely for $\Re(s) > 1$).

A formal manipulation shows that $\sum_p \frac{\chi(p)}{p^s}$ contributes to the principal branch of the logarithm of $L(\chi, s)$:

$$\log L(\chi, s) = \sum_p \sum_{n \ge 1} \frac{\chi(p)^n}{np^{ns}},$$

and the key input to Dirichlet's theorem is to show that $\log L(\chi, s)$ remains bounded as s approaches 1. This comes down to showing that $L(\chi, 1) \neq 0$ for our non-trivial character χ. Next, we will relate the non-vanishing of this L-value to one of the fundamental objects in algebraic number theory, the ideal class group of a number field.

1.2. Class groups and Dirichlet L-functions

We let $\zeta_N = e^{2i\pi/N}$. We may identify the multiplicative group $(\mathbf{Z}/N\mathbf{Z})^\times$ with the Galois group $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ using the cyclotomic character:

$$\chi_N : \mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q}) \to (\mathbf{Z}/N\mathbf{Z})^\times$$
$$\sigma \mapsto i, \text{ where } \sigma(\zeta_N) = \zeta_N^i.$$

From this point of view, the characters $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ are *Artin representations*, i.e. complex representations of a Galois group. A special case of the Artin formalism for L-functions of Artin representations gives the product formula:[2]

$$\zeta_{\mathbf{Q}(\zeta_N)}(s) = \prod_\chi L(\chi, s),$$

where the product is over all characters χ of $(\mathbf{Z}/N\mathbf{Z})^\times$ and

$$\zeta_{\mathbf{Q}(\zeta_N)}(s) = \sum_{\mathfrak{n} \triangleleft \mathbf{Z}[\zeta_N]} \frac{1}{\mathrm{Nm}(\mathfrak{n})^s}$$

is the Dedekind zeta function of the number field $\mathbf{Q}(\zeta_N)$.

The analytic class number formula of Dedekind gives a precise formula for the residue $\lim_{s\to 1^+}(s-1)\zeta_{\mathbf{Q}(\zeta_N)}(s)$ in terms of the size of the class group of $\mathbf{Z}[\zeta_N]$ (and other arithmetic data). In particular this residue is non-zero, so $\zeta_{\mathbf{Q}(\zeta_N)}(s)$ has a simple pole at $s = 1$.[3] We can deduce immediately from this that the $L(\chi, s)$ are non-vanishing at $s = 1$ for non-trivial χ, since the trivial character already contributes a simple pole.

2. Distribution of Frobenius elements

We can re-interpret Dirichlet's theorem, or more precisely the equidistribution result (1.1), as a statement about the distribution of certain elements in the Galois group $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$.

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[2] Strictly speaking, the product formula is only correct up to some simple factors indexed by primes dividing N. This can be fixed by replacing each χ with its associated *primitive* character: a character of $(\mathbf{Z}/M\mathbf{Z})^\times$ for the smallest divisor M of N such that χ factors through the quotient $(\mathbf{Z}/N\mathbf{Z})^\times \to (\mathbf{Z}/M\mathbf{Z})^\times$.

[3] To make sense of this, we rely on the fact that our L- and ζ-functions extend to meromorphic functions for $\Re(s) > 0$.

2.1. Frobenius elements

Let L/F be a Galois extension of number fields and suppose $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_F$ which is unramified in $\mathcal{O}_L$ (i.e. $\mathfrak{p}\mathcal{O}_L$ factors as a product of distinct prime ideals in $\mathcal{O}_L$). Then for each prime $\mathfrak{q}|\mathfrak{p}\mathcal{O}_L$ there is a unique element $\sigma_{\mathfrak{q}|\mathfrak{p}} \in \mathrm{Gal}(L/F)$ satisfying

$$\sigma_{\mathfrak{q}|\mathfrak{p}}(x) \equiv x^{\mathrm{Nm}(\mathfrak{p})} \bmod \mathfrak{q} \text{ for all } x \in \mathcal{O}_L.$$

The conjugacy class of $\sigma_{\mathfrak{q}|\mathfrak{p}}$ depends only on $\mathfrak{p}$, so we often denote one of these elements by $\sigma_{\mathfrak{p}}$.

In our cyclotomic example $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$, the unramified primes p are those coprime to N and the Frobenius element σ_p corresponds under the cyclotomic character to the class of p in $(\mathbf{Z}/N\mathbf{Z})^\times$. So the equidistribution statement (1.1) says that the σ_p distribute uniformly over the $\phi(N)$ elements of $\mathrm{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ as p varies.

2.2. Sato–Tate conjecture

We now move on to a more complicated equidistribution result, which is similar in spirit. Suppose we have an elliptic curve E defined over $\mathbf{Q}$, given by the projective planar curve with (affine) equation

$$y^2 = x^3 + Ax + B \tag{2.1}$$

for some coefficients $A, B \in \mathbf{Z}$ satisfying $4A^3 + 27B^2 \neq 0$ (so the curve is nonsingular). Consider odd primes p with $p \nmid 4A^3 + 27B^2$, so that the reduction of the equation (2.1) defines an elliptic curve over $\mathbf{F}_p$.[4]

The Sato–Tate conjecture describes the distribution of the integers

$$a_p(E) = p - \#\{(x, y) \in \mathbf{F}_p^2 : y^2 = x^3 + Ax + B\} = p + 1 - \#E(\mathbf{F}_p),$$

or, more precisely, the real numbers

$$\frac{a_p(E)}{2\sqrt{p}}$$

which lie in the interval $[-1, 1]$ by a theorem of Hasse.

See [22] for examples of histograms generated from the values $\frac{a_p(E)}{2\sqrt{p}}$ for many primes p.

We state the conjecture for the generic case of elliptic curves E with no complex multiplication:

[4] From this point on, we'll ignore the finitely many primes p dividing the discriminant $4A^3 + 27B^2$. Omitting these primes doesn't effect our subsequent analysis.

Conjecture 2.1. *For any subinterval* $[a, b] \subset [-1, 1]$, *the limit*

$$\lim_{X\to\infty} \frac{\#\{p \le X : \frac{a_p(E)}{2\sqrt{p}} \in [a, b]\}}{\#\{p \le X\}}$$

exists and is given by the integral

$$\int_a^b \frac{2}{\pi}\sqrt{1 - t^2}dt.$$

This says that a histogram plotting the values of $\frac{a_p(E)}{2\sqrt{p}}$ for primes p up to X will approach a (rescaled) semicircle as X gets large.

This conjecture was proved in a series of works published between 2008 and 2011, using *automorphic forms* [3, 7, 11, 23]. The strategy is modelled on Dirichlet's, as we will now describe.

2.3. Sato–Tate and Frobenius elements

As in the case of Dirichlet's theorem, we can rephrase the Sato–Tate conjecture as a question about the distribution of Frobenius elements.

For each p, the integer $a_p(E)$ can be interpreted as the trace of a 2x2 matrix with determinant p. One way to construct this matrix is by l-adic approximation, for a prime $l \neq p$. Firstly we look mod l. The l-torsion points $E[l]$ of E form a two dimensional $\mathbf{F}_l$-vector space with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. One can show that the trace of the Frobenius element σ_p under this representation is equal to $a_p(E)$ mod l [21, Thm. V.2.3.1]. Extending this to the l^n-torsion points $E[l^n]$ for each $n \ge 1$ and taking the limit identifies $a_p(E)$ with the trace of σ_p under a Galois representation

$$\rho_{E,l} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{Z}_l).$$

The Hasse bound implies that

$$x^2 - \frac{a_p(E)}{\sqrt{p}}x + 1$$

is the characteristic polynomial of a matrix $u_p(E) \in SU(2)$, which is well-defined up to conjugation.

A reformulation of Conjecture 2.1 is that the conjugacy classes of the $u_p(E)$ are equidistributed with respect to the Haar (i.e. translation-invariant) measure on the Lie group $SU(2)$.

3. Symmetric power L-functions and functoriality

The proof of Dirichlet's theorem on primes relied on a non-vanishing result for certain (Dirichlet) L-functions. The same is true for the proof of the Sato–Tate conjecture. To prove the equidistribution of the conjugacy classes of the $u_p(E)$ in $SU(2)$, we must show that

$$\lim_{X\to\infty} \frac{\sum_{p\le X} f(u_p(E))}{\#\{p \le X\}} = \int_{SU(2)} f(u)du,$$

with du the Haar probability measure on $SU(2)$, for any continuous class function f on $SU(2)$.

The Peter–Weyl theorem implies that it suffices to check this equality for f the characters of irreducible representations of $SU(2)$. These irreducible representations are given by $V_n = \mathrm{Sym}^n \mathbf{C}^2$ for $n \ge 0$, where the action of $SU(2)$ on $\mathbf{C}^2$ is the standard action by matrix multiplication. Orthogonality of characters tells us that integrating the character χ_n of V_n over $SU(2)$ gives 0 if $n \ge 1$ and 1 if $n = 0$.

We are left needing to prove that

$$\lim_{X\to\infty} \frac{\sum_{p\le X} \chi_n(u_p(E))}{\#\{p \le X\}} = 0 \tag{3.1}$$

for $n \ge 1$.

As in §1.1, the sum $\sum_p \frac{\chi_n(u_p(E))}{p^s}$ contributes the main term to the log of an L-function, the nth symmetric power L-function defined by

$$L(E, \chi_n, s) := \prod_p \det(1 - u_p(E)p^{-s}|V_n).$$

To show (3.1), it suffices to prove that, for each $n \ge 1$, $L(E, \chi_n, s)$ extends to a meromorphic function on an open neighbourhood of the region $\Re(s) \ge 1$, and is non-zero on this half-plane. See, for example, [20, Appendix to Ch. I] for a proof using the Wiener-Ikehara theorem.

Example 3.1. The first symmetric power L-function we need to understand is for $n = 1$. In this case, $L(E, \chi_1, s)$ is essentially the Hasse–Weil L-function of the elliptic curve E. Its non-vanishing for $\Re(s) \ge 1$ implies that (informally) the expected value $\mathbf{E}\left(\frac{a_p(E)}{2\sqrt{p}}\right)$ is zero.

The proof of this non-vanishing combines the modularity of elliptic curves with a non-vanishing result for Hecke L-functions of modular forms.

Theorem 3.2 ([4, 24, 28]). *There is a modular form f (a cuspidal Hecke eigenform of weight 2) with*

$$L(E, \chi_1, s) = L(f, s + 1/2).$$

The Hecke L-function $L(f, s)$ can be identified (up to some explicit factors) with the Mellin transform $\int_0^\infty f(iy)y^{s-1}dy$.

Theorem 3.3 ([18, 19]). *Suppose f is a cuspidal Hecke eigenform of weight 2. Then $L(f, s) \neq 0$ for $\Re(s) \geq 3/2$.*

3.1. Symmetric power functoriality

In order to prove the Sato–Tate conjecture, the natural way to proceed is to generalize Example 3.1 to handle the higher ($n > 1$) symmetric power L-functions. The generalization of the modularity theorem follows from special cases of Langlands's functoriality conjectures for automorphic forms. They are formulated in terms of the modular form f, and predict the existence of a related automorphic form $\mathrm{Sym}^n f$ (a special function on $\mathrm{GL}_{n+1}(\mathbf{R})$). We were able to establish this existence, in joint work with Thorne:

Theorem 3.4 ([14–16]). *For each $n \geq 1$, there exists an automorphic form $\mathrm{Sym}^n f$ with L-function $L(E, \chi_n, s)$.*

Importantly, whilst the L-function $L(E, \chi_n, s)$ is very difficult to analyse directly, the automorphic L-function $L(\mathrm{Sym}^n f, s)$ has an integral representation and the appropriate generalization of Theorem 3.3 was proved by Jacquet and Shalika [12].

With these ingredients, we can deduce the Sato–Tate conjecture. The original proof of the Sato–Tate conjecture instead proved something weaker about the L-functions $L(E, \chi_n, s)$: that they can be represented as an alternating product of automorphic L-functions. The more precise statement of Theorem 3.4 can be used to prove an effective enhancement of the Sato–Tate conjecture [26].

4. Hints at proofs of symmetric power functoriality

We have given two different proofs of symmetric power functoriality for modular forms f. The second ([16]) applies more generally, to Hilbert modular forms. The two proofs follow different strategies, but both work with the Galois representations

$$\mathrm{Sym}^n \rho_{E,l} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_{n+1}(\mathbf{Z}_l)$$

given by composing the nth symmetric power of the standard representation of GL_2 with $\rho_{E,l}$. From this point of view, we want to show that these Galois representations are *automorphic*, which amounts to constructing the associated automorphic form $\mathrm{Sym}^n f$.

Both strategies also build on the *modularity lifting theorems* introduced by Wiles in the proof of Fermat's last theorem, which propagate automorphy via congruences between Galois representations. However, the basic structure of the two strategies differs:

- [14, 15] A key ingredient is a result due to Buzzard and Kilford about 2-adic modular forms [5]. It is closely related to the observation of Tate that if a Galois representation
$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{SL}_2(\overline{\mathbf{F}}_2)$$
is unramified at every odd prime, then its semisimplification is trivial. The proof of our main theorem is an induction on the level of the modular form (equivalently, on the conductor of the elliptic curve or associated Galois representation).
- [16] Our later work follows a strategy due to Clozel and Thorne [8] which relies on reducibility of the representation $\mathrm{Sym}^n(\overline{\mathbf{F}}_l^2)$ of GL_2 in characteristic $l \leq n$. That enables an induction on n, since the irreducible constituents of $\mathrm{Sym}^n(\overline{\mathbf{F}}_l^2)$ involve smaller symmetric powers.

On the other hand, a key ingredient in both proofs is to show that we don't have too many Galois representations.

Example 4.1. Let $F \subset \mathbf{C}$ be a number field. There are finitely many Galois extensions L/F (in $\mathbf{C}$) such that

- $\mathrm{Gal}(L/F)$ is Abelian
- all primes of $\mathcal{O}_F$ are unramified in L.

In fact, the maximal such L is the Hilbert class field, whose Galois group over F is isomorphic to the ideal class group of $\mathcal{O}_F$.

Conjecture 4.2 ([10]). *There are finitely many isomorphism classes of representations* $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\mathbf{Q}_l)$ *with bounded ramification*[5].

We prove a weak form of this finiteness conjecture in [17], which is a crucial input to the methods of [14] and [16]. This weaker statement is best interpreted in terms of a moduli space of Galois representations, so we will end the lecture by giving this interpretation.

[5] Describing the ramification condition at l involves p-adic Hodge theory

5. Geometry of group representations

To give an idea of how to think about these moduli spaces of Galois representations, we introduce the representation and character variety of a finitely presented group

$$G = \langle g_1, \ldots, g_d \rangle / \langle r_1, \ldots, r_k \rangle.$$

A homomorphism $G \to \mathrm{GL}_n(\mathbf{C})$ corresponds to a d-tuple of matrices $A_1, \ldots, A_d \in \mathrm{GL}_n(\mathbf{C})$ satisfying the relations $r_1, \ldots, r_k$. These can be identified with the **C**-points of an algebraic variety $R_{G,n}$, a closed subset of $(\mathrm{GL}_n)^d$. This is called the *representation variety*, whilst the quotient $A_{G,n} = R_{G,n}/\mathrm{PGL_n}(\mathbf{C})$ by the adjoint action is the *character variety*, whose points parameterize isomorphism classes of semisimple representations of G.

If we fix an irreducible representation $\rho : G \to \mathrm{GL}_n(\mathbf{C})$, the tangent space $T_\rho A_{G,n}$ can be identified with a cohomology group $H^1(G, M_n(\mathbf{C}))$, where G acts on $M_n(\mathbf{C})$ by conjugation via ρ.

Constructing an analogue $A^l_{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}),n}$ of $A_{G,n}$ parameterizing l-adic Galois representations with bounded ramification is more involved, but this has been done by Wake and Wang-Erickson [27] building on work of Mazur [13] and Chenevier [6].

The tangent space at an irreducible representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\mathbf{Q}_l)$ is a subspace of $H^1(\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), M_n(\mathbf{Q}_l)$, where the group action on the coefficient space is again conjugation via ρ. The subspace corresponds to the ramification conditions. If Conjecture 4.2 holds, the Galois moduli space is just a finite set of points, and so the tangent spaces would vanish[6]. Our weak finiteness result is the vanishing of these tangent spaces:

Theorem 5.1. [1, 2, 17, 25] *Under some technical assumptions, the tangent spaces to $A^l_{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}),n}$ at points corresponding to automorphic Galois representations vanish.*

We view this vanishing result as finiteness of (the l-primary part of) a generalized class group. In fact, for $n = 1$, it follows from finiteness of the usual class group.

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[6]More precisely, finiteness implies that the underlying reduced of the Galois moduli space is a finite set of points, so we are implicitly enhancing the finiteness conjecture to include reducedness of the moduli space

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James Newton
Mathematical Institute, University of Oxford, Woodstock Road, OX2 6GG Oxford, United Kingdom; newton@maths.ox.ac.uk