SERRE WEIGHTS AND SHIMURA CURVES

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Abstract. We compare the cohomology of Shimura curves constructed using quaternion algebras over a totally real field with the cohomology of related unitary Shimura curves. This allows us to pass information about generalised Serre weight conjectures from unitary groups to indefinite quaternion algebras.

1. Introduction

In this paper, we investigate the relationship between generalised Serre weight conjectures in two different contexts. On the one hand we consider the setting of [BDJ10]. On the other hand we replace the quaternionic Shimura curves of [BDJ10] with Shimura curves associated to unitary similitude groups[1].

In the unitary situation, the approach of [BLGG] allows us to obtain a lot of information about the possible weights of modular Galois representations. The crucial difference between the quaternionic and unitary cases, which enables this approach, is that the mod p local systems on unitary Shimura curves (with level prime to p) attached to Serre weights arise as the reduction mod p of p-adic local systems. In the quaternionic case (as for classical Hilbert modular forms) conditions on the parity of weights and central characters mean that many of the relevant mod p local systems do not lift to characteristic zero.

The close relationship between quaternionic and unitary Shimura curves, as explained by Deligne [Del71] and Carayol [Car86a], allows us to transfer information about the weights of modular Galois representations between the unitary and quaternionic settings. As a result, we are able to deduce results similar to those of [BLGG] for quaternion algebras over totally real fields (see Theorem 7.1.3). We now develop some notation to allows us to outline the statement of this Theorem.

Let F^+ be a totally real number field of degree $d > 1$. For a place $v|p$ of F^+ denote the residue field by k_v. Let D denote a quaternion algebra with centre F^+, such that D is split at precisely one infinite place, and is split at every place above p. For any compact open subgroup K of $(D \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$ we have an associated Shimura curve S_K/F^+. Supposing K to be maximal compact at places over p (and sufficiently small), any irreducible mod p representation of $\otimes_{v|p}\mathrm{GL}_2(k_v)$ (which we call a 'weight') defines a mod p étale local system on S_K. We define an irreducible two-dimensional representation ρ of G_{F^+} (the absolute Galois group) on an $\overline{\mathbb{F}}_p$-vector space to be modular of weight W if it appears in the étale cohomology of S_K with coefficients in the local system attached to W for some choice of K. The representation ρ is said to be modular if it is modular of any weight. Note that these notions depend on our fixed quaternion algebra D. We define a set of weights $W_{pd}(\rho)$ (see Definition 7.1.1) comprising those W such that ρ has, locally at

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[1]More precisely, similitude groups with rational similitude factor. The associated Shimura curves are then PEL Shimura varieties.

places $v|p$, crystalline and potentially diagonalisable lifts of appropriate Hodge-Tate weights. We then prove

Theorem. *Suppose $p > 2$, that the continuous irreducible representation $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular, and that $\rho(G_{F^+(\zeta_p)})$ is adequate*[2]. *Then for any $W \in W_{pd}(\rho)$, ρ is modular of weight W.*

By a crucial local result in [GLSa], our results are sufficient to completely determine the weights of irreducible modular representations

$$\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

when p is unramified in the totally real field F^+ (see Corollary 7.2.1), under the same technical assumptions as the above Theorem. This essentially proves the generalised Serre weight conjectures made in [BDJ10], although in a few cases it is still only conjectured that the 'explicit weight sets' defined in [GLSa] coincide with those defined in [BDJ10].

In recent work [GK] (announced whilst the author was preparing this paper), Gee and Kisin have also, under similar technical assumptions[3], proven the Serre weight conjectures of [BDJ10]. Their work again transfers information from the unitary case to the quaternionic case, but uses modularity lifting theorems and the Breuil-Mézard conjecture to do so. They additionally treat the case of definite quaternion algebras.

By contrast, the way we transfer information from the unitary to the quaternionic setting is geometric (using the appropriate Shimura curves). We make no use of modularity lifting theorems in this part of the paper. This means that we prove in complete generality (see Theorem 7.1.2), that if an irreducible representation

$$\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is modular of some weight, then the local representations $\rho|_{G_{F_v^+}}$ for places $v|p$ have crystalline lifts with the expected Hodge–Tate weights[4].

In a very recent preprint [GLSb], Gee, Liu and Savitt have removed the assumption that p is unramified from the main local theorem of [GLSa], and so (using the results of [GK]) prove the weight part of Serre's conjecture for GL_2 over arbitrary totally real fields (with $p > 2$ and a condition on the image of $\rho|_{G_{F^+(\zeta_p)}}$). The main local theorem of [GLSb] can also be combined with the results of this paper to remove the assumption that p is unramified in F^+ from Corollary 7.2.1.

Remark. The case of definite quaternion algebras could probably also be studied geometrically, since at a finite place v of F where our indefinite quaternion algebra D is non-split, there is a semistable model for S_K (K maximal compact at v) over $\mathscr{O}_{F_v}$ whose special fibre is described by the definite quaternion algebra with the same non-split places as D, except it is non-split at every infinite place and split at v. The author has not pursued this.

In what remains of the introduction, we summarise the contents of this paper. In Section 2 we define the Shimura curves we will be working with. We have to define four different Shimura data, with three different underlying algebraic groups. This entails some cumbersome notation, for which we apologise to the reader. However,

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[2]by [BLGG, Proposition A.2.1], this is equivalent to assuming $\rho|_{F^+(\zeta_p)}$ irreducible and if $p = 3$ or 5 that the projective image of $\rho|_{G_{F^+(\zeta_p)}}$ is not conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$ or $\mathrm{PSL}_2(\mathbb{F}_5)$ respectively.

[3]More precisely, they relax the adequacy condition for $p = 3$ to irreducibility of $\rho|_{G_{F^+(\zeta_p)}}$.

[4]If we assume that $p > 2$ and $\rho|_{G_{F^+(\zeta_p)}}$ is irreducible, then this result is also proven in [GK] (assuming the projective image of $\rho|_{G_{F^+(\zeta_p)}}$ is not conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$ if $p = 5$)

this profusion of Shimura curves is more or less essential to the detailed comparison between quaternionic Shimura curves and PEL unitary Shimura curves.

In Section 3 we explain how the theory of [Del79, 2.7] allows us to relate the cohomology of Shimura varieties with the cohomology of their neutral components (which are connected Shimura varieties). We could not find this material in the literature, but the situation is quite simple: Shimura varieties are obtained from connected Shimura varieties by an 'induction' construction [Del79, 2.7.3], so the cohomology of a Shimura variety is given by a representation theoretic induction of the cohomology of its neutral component. In Section 4 we apply this theory to our Shimura curves, and define the mod p and p-adic local systems which will appear as the coefficients in our cohomology groups.

In Section 5 we define the notion of being modular of a Serre weight with central character, and discuss how to transfer weights between the different Shimura curves. The most subtle issue is transferring between two Shimura curves with rather different Shimura data, which is done in Corollary 5.4.3. This can probably be done using the set up of Section 3, but we decided the proof was more transparent when carried out via a comparison of the action of Hecke operators on the two curves.

In Section 6 we follow [BLGG], applying modularity lifting results to show modularity of predicted weights on a PEL unitary Shimura curve. Finally, in Section 7 we transfer these results to quaternionic Shimura curves.

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2. Shimura curves

2.1. **The basic set-up.** Let F^+ be a totally real number field of degree $d > 1$, with real places $\tau_1, ..., \tau_d$. Denote by Σ_p the places over p of F^+, where p is a fixed prime. Let D denote a quaternion algebra with centre F^+, such that D is split at the place τ_1 and non-split at the other real places. We also assume that D is split at all places in Σ_p. We denote by G the reductive group over $\mathbb{Q}$ defined by

$$G(R) = (D \otimes_{\mathbb{Q}} R)^{\times}$$

for $\mathbb{Q}$-algebras R. We denote by T the torus $\mathrm{Res}_{F^+/\mathbb{Q}}(\mathbb{G}_m)$, by Z the centre of G (which is isomorphic to T), by G_1 the derived subgroup of G, and by $\nu : G \to T$ the reduced norm map. Note that G_1 is the kernel of ν.

Fix an imaginary quadratic field E in which p splits. Denote by F the compositum EF^+, and denote by $c \in \mathrm{Gal}(F/F^+)$ the non-trivial element of the Galois group. We assume that if x is a place of F which is not split over F^+ then $D \otimes_{F^+} F_x$ is split. We will now define two more reductive groups over $\mathbb{Q}$, which depend on the choice of E and have derived subgroup G_1. We fix an embedding τ_E of E into $\mathbb{C}$ — the embeddings τ_i then have unique extensions to complex embeddings $\tau_i : F \hookrightarrow \mathbb{C}$ which restrict to τ_E on E. For any $\mathbb{Q}$-algebra R, we have an $F^+ \otimes_{\mathbb{Q}} R$-algebra endomorphism $z \mapsto z^c$ of $F \otimes_{\mathbb{Q}} R$, given by $c \otimes 1$. Denote by T_F the torus $\mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$, and let U_F denote the subgroup of T_F defined by

$$U_F(R) = \{z \in T_F(R) : zz^c = 1\}$$

for $\mathbb{Q}$-algebras R. We also denote by T' the subtorus $\mathbb{G}_m \times U_F$ of $T \times U_F$. Denote by G'' the reductive group obtained by taking the amalgamated product $G \underset{Z}{\times} T_F$. Recall that this is the quotient of $G \times T_F$ by the central subgroup $\{(g, g^{-1}) : g \in Z\}$. Denote the centre of G'' by Z'' (it is isomorphic to T_F).

Define a morphism $\nu' : G'' \to T \times U_F$ by $\nu'(g, z) = (\nu(g)z\overline{z}, z/\overline{z})$. We will write T'' for the torus $T \times U_F$. Define G' to be the inverse image under ν' of T'. We also denote the centre of G' by Z'. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & G_1 \arrow[r] \arrow[d, equal] & G' \arrow[r, "\nu'"] \arrow[d] & T' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G_1 \arrow[r] & G'' \arrow[r, "\nu'"] & T'' \arrow[r] & 1 \\
1 \arrow[r] & G_1 \arrow[r] \arrow[u, equal] & G \arrow[r, "\nu"] \arrow[u, "a"] & T \arrow[r] \arrow[u, "b"] & 1.
\end{tikzcd}$$

In this diagram, the map a is defined by $a(g) = (g, 1)$, and b is defined by $b(z) = (z, 1)$. The other vertical arrows are the natural inclusions.

We fix $\iota_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$, and denote by u_E the place of E induced by the embedding $\iota_p^{-1} \circ \tau_E : E \hookrightarrow \overline{\mathbb{Q}}_p$. For each $v \in \Sigma_p$ we denote by $\tilde{v}$ the unique place of F satisfying $\tilde{v}|_{F^+} = v$ and $\tilde{v}|_E = u_p$. We denote the other place of F above v by $\tilde{v}^c$. We will also fix an identification of $\overline{\mathbb{F}}_p$ with the residue field of $\overline{\mathbb{Q}}_p$.

2.2. **Some Shimura data.** Now we describe Shimura data for each of the groups G, G' and G''. The group $G(\mathbb{R})$ is isomorphic to $\mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H}^\times)^{d-1}$, where $\mathbb{H}$ denotes the quaternions, so we have a homomorphism

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_\mathbb{R}$$

which on real points sends $x + iy$ to $\left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1}, 1, ..., 1\right)$. The $G(\mathbb{R})$-conjugacy class of h, which we denote by X, is isomorphic to $\mathbb{C} - \mathbb{R}$. The pair (G, h) is a Shimura datum, in the sense of [Del79], with reflex field F^+, so we have an inverse system of curves $S_K(G, h)/F^+$, indexed by compact open subgroups K of $G(\mathbb{A}_f)$.

We note here that we are using the conventions of [Del79] and [Car86a], but we correct the sign error in the definition of canonical models, as described in the note at the end of [Mil05, §12]. We also adopt the conventions that the Artin reciprocity map sends uniformisers to geometric Frobenius elements and group actions on schemes are right actions.

The complex embeddings $\tau_1, ..., \tau_d$ of F give an isomorphism $T_F(\mathbb{R}) \cong (\mathbb{C}^\times)^d$, so we have a homomorphism

$$h_F : \mathbb{S} \to (T_F)_\mathbb{R}$$

which on real points sends z to $(1, z^{-1}, ..., z^{-1})$ (note that this depends on the choice of τ_E). The map $h \times h_F$ now induces a homomorphism

$$h \times h_F : \mathbb{S} \to G''_\mathbb{R}.$$

We can also consider the composite

$$h : \mathbb{S} \to G_\mathbb{R} \hookrightarrow G''_\mathbb{R}.$$

The map $\nu' \circ (h \times h_F)$ has image contained in T', so $h \times h_F$ has image contained in $G'_\mathbb{R}$. It is clear that the $G''(\mathbb{R})$-conjugacy class of $h \times h_F$ is naturally isomorphic to X, as is the $G''(\mathbb{R})$-conjugacy class of h. On the other hand, the $G'(\mathbb{R})$-conjugacy class of

$h\times h_F$, denoted X', is isomorphic to a single copy of the complex upper half plane (since the norm of a quaternion is positive). The pairs $(G', h\times h_F)$, $(G'', h\times h_F)$ and (G'', h) are all Shimura data, the first two with reflex field F and the last with reflex field F^+. We obtain inverse systems of curves $S_{K'}(G', h\times h_F)$, $S_{K''}(G'', h\times h_F)$ and $S_{K''}(G'', h)$ over their respective reflex fields, indexed by compact open subgroups of $G'(\mathbb{A}_f)$ and $G''(\mathbb{A}_f)$ as appropriate.

The natural embeddings $G\hookrightarrow G''$ and $G'\hookrightarrow G''$ induce morphisms of Shimura data $(G,h)\to(G'',h)$ and $(G', h\times h_F)\to(G'', h\times h_F)$, and hence induce morphisms of the respective Shimura curves. In fact, they induce open and closed immersions, by [Del71, Theorem 1.15].

2.3. **Similitude groups.** We can also describe G' and G'' as similitude groups, following [Car86a, 2.2,2.6.1]. We denote by B the quaternion algebra $D\otimes_{F^+}F$ over F, and define V to be the $\mathbb{Q}$-vector space underlying D.

Following Carayol, we may choose the following

- an involution $\beta\mapsto\beta^*$ of the second kind on B
- an alternating non-degenerate bilinear form ψ on V

such that

(1) for all $\beta\in B$
$$\psi(\beta v, w)=\psi(v,\beta^* w)$$
(2) for all $\mathbb{Q}$-algebras R we may identify $G''(R)$ with the set of $\beta\in(B\otimes_{\mathbb{Q}}R)^\times$ such that there exists $\mu\in(F^+\otimes_{\mathbb{Q}}R)^\times$ with $\psi(v\beta, w\beta)=\mu\psi(v,w)$
(3) the above induces an identification of $G'(R)\subset G''(R)$ with the subset consisting of those β satisfying the above equation with $\mu\in R^\times$

Now we can give a rather explicit description of the groups $G'(\mathbb{Q}_p)$ and $G''(\mathbb{Q}_p)$. Recall that we have fixed embeddings $\tau_E: E\hookrightarrow\mathbb{C}$ and $\tau_1: F^+\hookrightarrow\mathbb{R}$. We also denote by τ_1 the unique embedding $\tau_1: F\hookrightarrow\mathbb{C}$ which restricts to τ_E on E and τ_1 on F^+.

Note that $G''(\mathbb{Q}_p)$ consists of $\beta\in(B\otimes_{\mathbb{Q}}\mathbb{Q}_p)^\times$ such that there exists $\mu\in(F^+\otimes_{\mathbb{Q}}\mathbb{Q}_p)^\times$ with $\psi(v\beta,w\beta)=\mu\psi(v,w)$.

For each $v\in\Sigma_p$ we choose isomorphisms $j_v: B\otimes_{F^+}F_v^+=B_{\tilde v}\oplus B_{\tilde v^c}\cong M_2(F_{\tilde v})\oplus M_2(F_{\tilde v^c})$ such that $j_v(\beta^*)={}^tj_v(\beta)^c$. This determines an isomorphism
$$j_p: G''(\mathbb{Q}_p)\cong(F_p^+)^\times\times\prod_{v\in\Sigma_p}\mathrm{GL}_2(F_v^+)$$
given by $j_p(\beta)=(\mu,(j_v(\beta_v)_{\tilde v^c})_{v\in\Sigma_p})$. Here $j_v(\beta_v)_{\tilde v^c}$ denotes the projection of $j_v(\beta_v)$ to its component in $\mathrm{GL}_2(F_{\tilde v^c})$. We identify this group with $\mathrm{GL}_2(F_v^+)$ via the canonical isomorphism $\mathrm{GL}_2(F_{\tilde v^c})\cong\mathrm{GL}_2(F_v^+)$. The isomorphism j_p restricts to an isomorphism
$$j_p: G'(\mathbb{Q}_p)\cong\mathbb{Q}_p^\times\times\prod_{v\in\Sigma_p}\mathrm{GL}_2(F_v^+).$$

Definition 2.3.1. *We denote by K_0'' the compact open subgroup of $G''(\mathbb{Q}_p)$ given by*
$$K_0'':=j_p^{-1}(\mathscr{O}_{F^+,p}^\times\times\prod_{v\in\Sigma_p}\mathrm{GL}_2(\mathscr{O}_v)).$$
Denote by K_0' the intersection of K_0'' with $G'(\mathbb{Q}_p)$ and denote by K_0 the intersection of K_0'' with $G(\mathbb{Q}_p)$.

We denote by K_1'' the compact open subgroup of $G''(\mathbb{Q}_p)$ given by
$$K_1'':=j_p^{-1}(\mathscr{O}_{F^+,p}^\times\times\prod_{v\in\Sigma_p}(1+\varpi_v M_2(\mathscr{O}_v))).$$

Denote by K_1' the intersection of K_1'' with $G'(\mathbb{Q}_p)$ and denote by K_1 the intersection of K_1'' with $G(\mathbb{Q}_p)$.

Note that
$$K_0' = j_p^{-1}(\mathbb{Z}_p^\times \times \prod_{v\in\Sigma_p} \mathrm{GL}_2(\mathscr{O}_v))$$
and
$$K_0 = j_p^{-1}\{(\mu, (g_v)_{v\in\Sigma_p}) \in \mathscr{O}_{F^+,p}^\times \times \prod_{v\in\Sigma_p} \mathrm{GL}_2(\mathscr{O}_v) | \mu_v = \det(g_v)\}.$$
We have similar descriptions of K_1' and K_1. The following, slightly smaller, compact open subgroups will be useful in Section 6.

Definition 2.3.2. *Denote by $K_{1,1}'$ the compact open subgroup of $G'(\mathbb{Q}_p)$ given by*
$$K_{1,1}' = j_p^{-1}((1+p\mathbb{Z}_p) \times \prod_{v\in\Sigma_p} (1+\varpi_v M_2(\mathscr{O}_v)))$$

Definition 2.3.3. *Suppose K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$. We will say that K^p is* sufficiently small *if $S_{K^pK_0}(G,h)$ has no elliptic points and*
$$S_{K^pK_1}(G,h) \to S_{K^pK_0}(G,h)$$
is a finite Galois cover of schemes with Galois group K_0/K_1. We make the analogous definitions for compact open subgroups of $G'(\mathbb{A}_f^p)$ and $G''(\mathbb{A}_f^p)$.

Note that it doesn't matter whether we use the Shimura datum (G'', h) or $(G'', h \times h_F)$ for this definition in the case of subgroups of $G''(\mathbb{A}_f^p)$. Comparing with the definition of 'sufficiently small' in [BDJ10] (before Definition 2.1), we are additionally imposing the condition that $S_{K^pK_0}(G,h)$ has no elliptic points. Nevertheless, it is still the case that any compact open subgroup of $G(\mathbb{A}_f^p)$ will contain a sufficiently small subgroup. The same goes for the groups G' and G''.

3. Connected components and weakly canonical models: generalities

In this section we discuss some general results which will allow us to relate the étale cohomology groups of the various Shimura curves defined in Section 2.2. For these purposes we use the formalism of [Del79, §2.7]. This is recalled in sections 3.1 and 3.2. In section 3.3 we explain the cohomological consequences of this formalism. The (presumably well-known) moral is that the 'induction' construction which Deligne applies to obtain (canonical models of) Shimura varieties from (weakly canonical models of) connected Shimura varieties, translates into genuine induction (in the sense of representation theory) when we pass to cohomology.

3.1. **Preliminary definitions.**

Definition 3.1.1. *For a locally compact, totally disconnected group Γ, a scheme (over some base scheme T) with* continous Γ-action *is an inverse limit $S = \varprojlim_K S_K$ of quasi-projective schemes (over some fixed base), indexed by compact open subgroups K of Γ and equipped with a right Γ-action arising from isomorphisms $\gamma : S_K \to S_{\gamma^{-1}K\gamma}$. We moreover require that these isomorphisms are the identity when $\gamma \in K$, and that for $K \subset K' \subset \Gamma$ with K an open normal subgroup of the compact open K', the map $S_K \to S_{K'}$ from the inverse system induces an isomorphism $S_K/(K'/K) \cong S_{K'}$. This implies that the maps in the inverse limit are finite, hence S is a scheme.*

Note that taking the inverse limit over K of the isomorphisms $S_K/(K'/K) \cong S_{K'}$ induces an isomorphism $S/K' \cong S_{K'}$.

Suppose π is a profinite set, with a continuous transitive right action of Γ. We suppose that the orbits of a compact open subgroup are open — equivalently, for $e \in \pi$, with stabiliser Δ, the bijection

$$\Delta\backslash\Gamma \cong \pi$$

is a homeomorphism. Moreover, we assume that the action of Γ on π makes π a torsor for an Abelian quotient of Γ. In particular, the subgroup Δ is normal in Γ with Abelian quotient, and it is independent of the choice of e.

Definition 3.1.2. *Suppose T is a scheme with continuous Δ-action. Then Δ acts on the scheme $\Gamma \times T$ by $(\gamma, s)\delta = (\delta^{-1}\gamma, s\delta)$. We define $I_\Delta^\Gamma(T)$ to be the quotient $(\Gamma \times T)/\Delta$.*

Lemma 3.1.3. [Del79, pp.285-6] *Suppose we are in the setting of Definition 3.1.2. Then $S := I_\Delta^\Gamma(T)$ is a scheme with continuous Γ-action, where the Γ action is induced by $(\gamma_0, t)\gamma = (\gamma_0\gamma, t)$. Moreover, for $K \subset \Gamma$ a compact open subgroup, we have*

$$S_K = \coprod_{[\gamma]\in\Delta\backslash\Gamma/K} T/(\Delta \cap \gamma K\gamma^{-1}).$$

Proof. It is clear that we have a canonical isomorphism between S and the inverse limit of the $S_K := [(\Gamma \times T)/\Delta]/K$. So it suffices to check that

$$\coprod_{[\gamma]\in\Delta\backslash\Gamma/K} T/(\Delta \cap \gamma K\gamma^{-1}) = [(\Gamma \times T)/\Delta]/K.$$

This canonical isomorphism is induced by the maps taking the class of t in the γ component of the left hand side to the class of (γ, t) in the right hand side. Note that the group $\Delta \cap \gamma K\gamma^{-1}$ is independent of the choice of coset representative γ for $[\gamma] \in \Delta\backslash\Gamma/K$. □

We now put ourselves in the situation where the scheme S has a continuous Γ-action and a Γ-equivariant continuous map $S \to \pi$. By saying this map is continuous we just mean that it is induced from an inverse limit of maps $S_K \to \pi_K$. Now for $e \in \pi$ with stabiliser $\Delta \subset \Gamma$ the fibre S_e has a continuous Δ-action, and Lemme 2.7.3 of [Del79] says the following:

Lemma 3.1.4. *The functor $S \to S_e$ is an equivalence of categories between schemes S with continuous Γ-action (over some base) and a Γ-equivariant continuous map $S \to \pi$, and schemes T with continuous Δ-action.*

Proof. The inverse to the functor $S \to S_e$ is given by sending T to $I_\Delta^\Gamma(T)$. □

3.2. **Galois actions.** We now refine these constructions to take account of a Galois action. Let $L \subset \overline{\mathbb{Q}}$ be a number field, and suppose the profinite set π is equipped with a continuous right action of $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$, commuting with the Γ-action — i.e. we have a (continuous) right action of $\Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/L)$. Suppose S is a scheme over L, endowed with a continuous Γ-action and a continuous Γ and $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-equivariant map from $S_{\overline{\mathbb{Q}}}$ to π. We denote the category of schemes over L endowed with these extra structures by $\mathbf{A}_L$. Fix $e \in \pi$ with stabiliser in Γ denoted Γ_e, and stabiliser in $\Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ denoted $\mathscr{E}$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_e \arrow[r] \arrow[d] & \mathscr{E} \arrow[r] \arrow[d] & \mathrm{Gal}(\overline{\mathbb{Q}}/L) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Gamma \arrow[r] & \Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/L) \arrow[r] & \mathrm{Gal}(\overline{\mathbb{Q}}/L) \arrow[r] & 1.
\end{tikzcd}$$

The fibre S_e is a scheme over $\overline{\mathbb{Q}}$, with continuous $\mathscr{E}$-action compatible with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ on $\overline{\mathbb{Q}}$. We denote the category of schemes over $\overline{\mathbb{Q}}$ endowed with these extra structures by $\mathbf{B}_L$. Combining the above lemma with Galois descent, one obtains [Del79, Lemme 2.7.9]:

Lemma 3.2.1. *The functor $S \to S_e$ is an equivalence of categories between $\mathbf{A}_L$ and $\mathbf{B}_L$.*

Proof. We define an inverse functor from $\mathbf{B}_L$ to $\mathbf{A}_L$. We start with a scheme T over $\overline{\mathbb{Q}}$, with continuous $\mathscr{E}$-action, compatible with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ on $\overline{\mathbb{Q}}$. We set $\overline{S} = I_{\mathscr{E}}^{\Gamma\times\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(T)$. By Galois descent, $\overline{S}$ descends to a scheme S over L with the desired extra structures. $\square$

We now depart a little from the setting of [Del79] and consider certain equivariant sheaves on our schemes with continuous groups actions. In section 3.3 we then consider the cohomology of these sheaves.

Definition 3.2.2. *A* smooth Γ-sheaf *on S is defined to be a torsion étale sheaf $\mathscr{F}$ of Abelian groups on S, arising from a system of finite locally constant sheaves $\mathscr{F}_K$ on S_K for some cofinal system of compact open subgroups $K \subset \Gamma$, such that*

- *For each inclusion $K' \subset K$ in the cofinal system, $\mathscr{F}_{K'}$ is the pullback to $S_{K'}$ of $\mathscr{F}_K$*
- *$\mathscr{F}$ comes equipped with isomorphisms*
$$\rho_\gamma : \mathscr{F} \cong \gamma^*\mathscr{F}$$
for each $\gamma \in \Gamma$ satisfying
  1. *The isomorphisms ρ_γ are continuous — i.e. they arise from compatible systems of isomorphisms $\mathscr{F}_K \cong \gamma^*\mathscr{F}_{\gamma^{-1}K\gamma}$.*
  2. *For γ, δ in Γ, we have*
$$\delta^*(\rho_\gamma) \circ \rho_\delta = \rho_{\gamma\delta}.$$

In practice we will make use of the following extension of the previous definition:

Definition 3.2.3. *Suppose we have a sheaf $\mathscr{F}$ of $\overline{\mathbb{F}}_p$-vector spaces on S which is the extension of scalars to $\overline{\mathbb{F}}_p$ of a smooth Γ-sheaf $\mathscr{F}'$ (of vector spaces over a finite field of characteristic p). Then we again refer to $\mathscr{F}$ as a smooth Γ-sheaf. We denote by $\mathscr{F}_K$ the local system $\mathscr{F}'_K \otimes \overline{\mathbb{F}}_p$ of $\overline{\mathbb{F}}_p$-vector spaces on S_K (for small enough K).*

Definition 3.2.4. *Suppose $\mathscr{F}$ is a smooth Γ-sheaf on S. Then we denote by $\mathscr{F}_e$ the sheaf on S_e obtained by pulling back $\mathscr{F}$. This is endowed with the structure of a smooth $\mathscr{E}$-sheaf.*

3.3. **Cohomology.** In this section we suppose we have S, π, Γ, L as in section 3.2, and suppose $\mathscr{F}$ is a smooth Γ-sheaf on S.

Definition 3.3.1. *We define*
$$H^\bullet(S_{\overline{\mathbb{Q}}}, \mathscr{F}) = \varinjlim_K H^\bullet_{et}((S_K)_{\overline{\mathbb{Q}}}, \mathscr{F}_K)$$
and
$$H^\bullet(S_e, \mathscr{F}_e) = \varinjlim_K H^\bullet_{et}((S_e)_{\Delta\cap K}, (\mathscr{F}_e)_{\Delta\cap K}).$$
Here the limits are taken over the compact open $K \subset \Gamma$ such that $\mathscr{F}_K$ is defined.

Definition 3.3.2. *Suppose a topological group H acts on an Abelian group M. We say that $m \in M$ is a* smooth *element if m is fixed by an open subgroup of H. If every element of M is smooth, we say that M is a smooth H-representation.*

We similarly extend this definition to actions of H on $\overline{\mathbb{F}}_p$-vector spaces, as in Definition 3.2.3.

Lemma 3.3.3. *The natural $\Gamma\times\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-action on $H^\bullet(S_{\overline{\mathbb{Q}}},\mathscr{F})$ makes $H^\bullet(S_{\overline{\mathbb{Q}}},\mathscr{F})$ a smooth $\Gamma\times\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-representation.*

Proof. The image of $H^\bullet_{et}((S_K)_{\overline{\mathbb{Q}}},\mathscr{F}_K)$ in $H^\bullet(S_{\overline{\mathbb{Q}}},\mathscr{F})$ is fixed by K. Moreover, it is a finitely generated Abelian group (or the extension of scalars to $\overline{\mathbb{F}}_p$ of such), so the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ factors through a finite quotient. Since each element of $H^\bullet(S_{\overline{\mathbb{Q}}},\mathscr{F})$ is in the image of $H^\bullet_{et}((S_K)_{\overline{\mathbb{Q}}},\mathscr{F}_K)$ for some K, the action of $\Gamma\times\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ is smooth. □

In exactly the same way, we see that $H^\bullet(S_e,\mathscr{F}_e)$ is a smooth $\mathscr{E}$-representation.

Definition 3.3.4. *Suppose we have a topological group H_1 and a closed subgroup H_2. Suppose the Abelian group M is a smooth representation of H_2. Then the* smooth induction $\mathrm{Ind}_{H_2}^{H_1}(M)$ *is a smooth representation of H_1, defined to be the Abelian group*

$$\{f: H_1\to M \textit{ uniformly locally constant}|f(hx)=h\cdot f(x) \textit{ for all } h\in H_2\}$$

with H_1 acting by right translation. Here 'f is uniformly locally constant' means that there exists a compact open subgroup K of H_1 such that $f(xk)=f(x)$ for all $x\in M$ and $k\in K$. This construction defines a functor $\mathrm{Ind}_{H_2}^{H_1}$.

The following Lemma is standard:

Lemma 3.3.5 (Frobenius reciprocity)**.** *The functor $\mathrm{Ind}_{H_2}^{H_1}$ is right adjoint to the functor restricting smooth H_1-representations to smooth H_2-representations.*

The canonical $\mathscr{E}$-equivariant map $S_e\to S_{\overline{\mathbb{Q}}}$ gives an $\mathscr{E}$-equivariant map

$$H^\bullet(S_{\overline{\mathbb{Q}}},\mathscr{F})\to H^\bullet(S_e,\mathscr{F}_e).$$

By Frobenius reciprocity this induces a canonical $\Gamma\times\mathrm{Gal}(\overline{\mathbb{Q}}/L)$-equivariant map

$$\alpha: H^\bullet(S_{\overline{\mathbb{Q}}},\mathscr{F})\to \mathrm{Ind}_{\mathscr{E}}^{\Gamma\times\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(H^\bullet(S_e,\mathscr{F}_e)).$$

Proposition 3.3.6. *The map*

$$\alpha: H^\bullet(S_{\overline{\mathbb{Q}}},\mathscr{F})\to \mathrm{Ind}_{\mathscr{E}}^{\Gamma\times\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(H^\bullet(S_e,\mathscr{F}_e))$$

is an isomorphism.

Proof. We can make the map α completely explicit: recall that by Lemma 3.1.3 we have

$$(S_K)_{\overline{\mathbb{Q}}}=\coprod_{[\gamma]\in\Delta\backslash\Gamma/K} S_e/(\Delta\cap\gamma K\gamma^{-1}).$$

Choose a set of coset representatives R for the finite set $\Delta\backslash\Gamma/K$. Then we have isomorphisms

$$\begin{aligned}H^\bullet((S_K)_{\overline{\mathbb{Q}}},\mathscr{F}_K)&\cong\bigoplus_{\gamma\in R}H^\bullet((S_e)_{\Delta\cap\gamma K\gamma^{-1}},(\gamma^*\mathscr{F}_K)_e)\\&\cong\bigoplus_{\gamma\in R}H^\bullet((S_e)_{\Delta\cap\gamma K\gamma^{-1}},(\mathscr{F}_{\gamma K\gamma^{-1}})_e).\end{aligned}$$

The first isomorphism comes from applying γ^* to the cohomology of the fibre over $e\gamma K\in\pi/K=\Delta\backslash\Gamma/K$ of $(S_K)_{\overline{\mathbb{Q}}}$. The second isomorphism comes from applying ρ_γ^{-1}. On the other hand, the choice of the transversal R also determines an isomorphism

$$\left(\mathrm{Ind}_{\mathscr{E}}^{\Gamma\times\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(H^\bullet(S_e,\mathscr{F}_e))\right)^K \cong \bigoplus_{\gamma\in R} H^\bullet(S_e,\mathscr{F}_e)^{\Delta\cap\gamma K\gamma^{-1}}.$$

It is straightforward to check that the map α is then given by taking the direct limit of the direct sums of the natural maps

$$\alpha_{\gamma,K} : H^\bullet((S_e)_{\Delta\cap\gamma K\gamma^{-1}}, (\mathscr{F}_{\gamma K\gamma^{-1}})_e) \to H^\bullet(S_e,\mathscr{F}_e)^{\Delta\cap\gamma K\gamma^{-1}}.$$

We can now show that α is an isomorphism. Suppose we have f in the kernel of $\alpha_{\gamma,K}$. Recall that f is an element of $H^\bullet((S_e)_{\Delta\cap\gamma K\gamma^{-1}}, (\mathscr{F}_{\gamma K\gamma^{-1}})_e)$, which may also be thought of as an element of $H^\bullet((S_K)_{\overline{\mathbb{Q}}},\mathscr{F}_K)$. By the definition of $H^\bullet(S_e,\mathscr{F}_e)$, the image of f in $H^\bullet((S_e)_{\Delta\cap\gamma K'\gamma^{-1}}, (\mathscr{F}_{\gamma K'\gamma^{-1}})_e)$ will be zero for some $K'\subset K$. It is then easy to check that for any $\gamma'\in\Gamma$ satisfying $\Delta\gamma' K = \Delta\gamma K$ the image of f in $H^\bullet((S_e)_{\Delta\cap\gamma' K'(\gamma')^{-1}}, (\mathscr{F}_{\gamma' K'(\gamma')^{-1}})_e)$ will also be zero. Hence the image of f in $H^\bullet((S_{K'})_{\overline{\mathbb{Q}}},\mathscr{F}_{K'})$ is zero. From this, we may conclude the α is injective.

The surjectivity of α is clear from the fact that for fixed γ, K, each element of $H^\bullet(S_e,\mathscr{F}_e)^{\Delta\cap\gamma K\gamma^{-1}}$ is in the image of $H^\bullet((S_e)_{\Delta\cap\gamma K'\gamma^{-1}}, (\mathscr{F}_{\gamma K'\gamma^{-1}})_e)$ for some $K'\subset K$. □

4. Shimura curves: connected components and systems of coefficients

4.1. **Connected components of Shimura curves.** We now discuss how some of the formalism of the previous section applies to the Shimura curves of interest in this paper. We will be using the notation and definitions of Section 2.

Definition 4.1.1. *Denote by $\mathfrak{K}$ the directed system whose objects are the compact open subgroups $K_0K^p \subset G(\mathbb{A}_f)$, where K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$. The morphisms in $\mathfrak{K}$ are the inclusions between subgroups. Denote by $\mathfrak{K}'$ and $\mathfrak{K}''$ the analogously defined directed systems for the groups G' and G'', where the factors of the compact open subgroups at places over p are K_0' and K_0'', respectively.*

Definition 4.1.2. *We define a scheme over F^+, $S_0(G,h)$, to be the inverse limit $\lim_{\mathfrak{K}}(S_K(G,h))$. Similarly define schemes over F,*

$$S_0(G', h\times h_F) = \lim_{\mathfrak{K}'}(S_{K'}(G', h\times h_F))$$

and

$$S_0(G'', h\times h_F) = \lim_{\mathfrak{K}''}(S_{K''}(G'', h\times h_F)),$$

and a scheme over F^+,

$$S_0(G'', h) = \lim_{\mathfrak{K}''}(S_{K''}(G'', h)).$$

We also define profinite sets π, π' and π'' to be the sets of geometric connected components of $S_0(G,h)$, $S_0(G', h\times h_F)$ and $S_0(G'', h\times h_F)$ respectively. Note that π'' is also the set of geometric connected components of $S_0(G'',h)$. We use the same letter e to denote the elements of π, π' and π'' arising from the connected components containing the images of

$$\{1\}\times X^+ \subset G(\mathbb{A}_f)\times X, \{1\}\times X' \subset G'(\mathbb{A}_f)\times X' \text{ and } \{1\}\times X^+ \subset G''(\mathbb{A}_f)\times X$$

respectively.

Definition 4.1.3. *We set $\Gamma = G(\mathbb{A}_f^p)$, $\Gamma' = G'(\mathbb{A}_f^p)$ and $\Gamma'' = G''(\mathbb{A}_f^p)$.*

Lemma 4.1.4. *The natural actions of the groups Γ, Γ' and Γ'' on $S_0(G,h)$, $S_0(G', h\times h_F)$ and $S_0(G'', h\times h_F)$ and $S_0(G'', h)$ respectively are continuous (in the sense of Definition 3.1.1).*

Proof. This follows immediately from the definition of a canonical model — for example, see [Del71, Définition 3.1, Remarque 3.2]. $\square$

For an algebraic group $H/\mathbb{Q}$ we denote by $H(\mathbb{Q})^+$ the intersection of $H(\mathbb{Q})$ with the neutral component of $H(\mathbb{R})$. We furthermore denote by $\overline{H(\mathbb{Q})^+}$ the closure of $H(\mathbb{Q})^+$ in $H(\mathbb{A}_f)$.

Proposition 4.1.5.
- *The morphism $\nu : G \to T$ induces an isomorphism*
$$\pi \xrightarrow{\sim} \overline{T(\mathbb{Q})^+}\backslash T(\mathbb{A}_f)/T(\mathbb{Z}_p).$$
- *The morphism $\nu' : G' \to T'$ induces an isomorphism*
$$\pi' \xrightarrow{\sim} T'(\mathbb{Q})^+\backslash T'(\mathbb{A}_f)/T'(\mathbb{Z}_p).$$
- *The morphism $\nu' : G'' \to T''$ induces an isomorphism*
$$\pi'' \xrightarrow{\sim} \overline{T''(\mathbb{Q})^+}\backslash T''(\mathbb{A}_f)/T''(\mathbb{Z}_p).$$

The induced actions of Γ, Γ' and Γ'' on the right hand sides of these isomorphisms are given by applying the maps ν, ν' and ν' respectively, then multiplying.

Proof. As in [Car86a, 1.2, 3.2.1], this follows from [Del71, Variante 2.5]. Note that $T'(\mathbb{Q})^+$ is already closed in $T'(\mathbb{A}_f)$, so we do not need to take a closure in this case. $\square$

Now we apply the construction of Section 3.2. We obtain extensions
$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \mathscr{E} & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/F^+) & \longrightarrow & 1 \\
1 & \longrightarrow & \Delta' & \longrightarrow & \mathscr{E}' & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/F) & \longrightarrow & 1 \\
1 & \longrightarrow & \Delta'' & \longrightarrow & \mathscr{E}''(h \times h_F) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/F) & \longrightarrow & 1 \\
1 & \longrightarrow & \Delta'' & \longrightarrow & \mathscr{E}''(h) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/F^+) & \longrightarrow & 1.
\end{array}$$
Here Δ, Δ', Δ'' denote the stabilisers of e in the relevant adelic group (Γ, Γ' or Γ''), and $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{E}''(h \times h_F)$, $\mathscr{E}''(h)$ denote the stabilisers of e in the product of the adelic group and the absolute Galois group of the reflex field. For the definition of $\mathscr{E}''(h \times h_F)$ we regard e as a geometric connected component of $S_0(G'', h \times h_F)$, whilst for the definition of $\mathscr{E}''(h)$ we regard e as a geometric component of $S_0(G'', h)$. The reciprocity laws for the various curves give explicit descriptions of these groups, as in [Car86a, §4.1.3].

We will also find it useful to consider $\mathscr{E}_F$ and $\mathscr{E}''(h)_F$, the pullbacks of $\mathscr{E}$ and $\mathscr{E}''$ via the inclusion $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F^+)$.

Lemma 4.1.6. *The natural injections*
$$\Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/F^+) \hookrightarrow \Gamma'' \times \mathrm{Gal}(\overline{\mathbb{Q}}/F^+)$$
and
$$\Gamma' \times \mathrm{Gal}(\overline{\mathbb{Q}}/F) \hookrightarrow \Gamma'' \times \mathrm{Gal}(\overline{\mathbb{Q}}/F)$$
induce injections
$$\mathscr{E} \hookrightarrow \mathscr{E}''(h)$$
and
$$\mathscr{E}' \hookrightarrow \mathscr{E}''(h \times h_F).$$

Proof. This follows from compatibilities between, on the one hand, the reciprocity laws for $S(G, h)$ and $S(G'', h)$, and on the other hand, the reciprocity laws for $S(G', h \times h_F)$ and $S(G'', h \times h_F)$. $\square$

We will, via these injections, regard $\mathscr{E}$ as a subgroup of $\mathscr{E}''(h)$ and $\mathscr{E}'$ as a subgroup of $\mathscr{E}''(h \times h_F)$.

Lemma 4.1.7. *We have open and closed immersions (of schemes over F^+ and F respectively)*

$$S_0(G,h) \hookrightarrow S_0(G'',h)$$

and

$$S_0(G',h\times h_F) \hookrightarrow S_0(G'',h\times h_F)$$

which are equivariant with respect to the action of Γ and Γ' respectively.

Proof. We give the proof for G and G''. The map induced by the obvious morphism of Shimura data $(G,h) \to (G'',h)$ gives an open and closed immersion $S(G,h) \hookrightarrow S(G'',h)$. We need to check that this induces an open and closed immersion $S_0(G,h) \hookrightarrow S_0(G'',h)$. Since $K_0 \subset K_0''$ we certainly have an induced map $S_0(G,h) \to S_0(G'',h)$. The lemma follows from applying the proof of [Del71, Theorem 1.15] with fixed levels at p. $\square$

Lemma 4.1.8. *There are isomorphisms of schemes over $\overline{\mathbb{Q}}$*

$$S_0(G,h)_e \cong S_0(G'',h)_e,$$

and

$$S_0(G',h\times h_F)_e \cong S_0(G'',h\times h_F)_e,$$

which are equivariant for the $\mathscr{E}$ and $\mathscr{E}'$ actions respectively.

Proof. This follows immediately from the previous lemma. $\square$

We now want to compare the cohomology of our various Shimura curves. First, we introduce a couple more general notions.

Definition 4.1.9. *Suppose H_1 and H_2 are topological groups, and V is a smooth representation of H_1 on a finite-dimensional $\overline{\mathbb{F}}_p$-vector space. Suppose that M is a smooth $H_1\times H_2$-representation on an $\overline{\mathbb{F}}_p$-vector space. Define $M(V)$ to be the Abelian group $\mathrm{Hom}_{H_1}(V,M)$. We endow $M(V)$ with a left group action of H_2 as follows: for $h\in H_2$ and $\alpha\in M(V)$, let $(h\cdot\alpha)(v) = h\cdot\alpha(v)$ for any $v\in V$.*

Suppose N is a smooth H_2-representation on an $\overline{\mathbb{F}}_p$-vector space. Then we define the tensor product $V\otimes N$ to be $V\otimes_{\overline{\mathbb{F}}_p} N$ with a left group action of $H_1\times H_2$ as follows: for $h_1\in H_1$, $h_2\in H_2$, $v\in V$ and $n\in N$, let $(h_1,h_2)\cdot(v\otimes n) = (h_1v)\otimes(h_2n)$.

Lemma 4.1.10. *The constructions of the above Definition give a smooth H_2-representation $M(V)$ and a smooth $H_1\times H_2$-representation $V\otimes N$. We have $\mathrm{Hom}_{H_1\times H_2}(V\otimes N, M) = \mathrm{Hom}_{H_2}(N, M(V))$.*

Proof. We first prove the smoothness of $M(V)$. We have

$$\mathrm{Hom}(V,M) = \varinjlim_{K_2} \mathrm{Hom}(V, M^{K_2}) = \varinjlim_{K_2} M(V)^{K_2},$$

where the limit is over open subgroups of H_2. Indeed, any homomorphism from the finite-dimensional V lands inside a finite-dimensional subspace of M, which is invariant under some open subgroup of $H_1\times H_2$ (by smoothness of M), and hence invariant under some K_2. Similar considerations show smoothness of the tensor product $V\otimes N$.

We now prove the second part of the Lemma. We have

$$\mathrm{Hom}_{H_1\times H_2}(V\otimes N, M) = \mathrm{Hom}_{\overline{\mathbb{F}}_p}(V\otimes N, M)^{H_1\times H_2},$$

where the action of $g\in H_1\times H_2$ on $\lambda\in\mathrm{Hom}_{\overline{\mathbb{F}}_p}(V\otimes N,M)$ is given by

$$(g\lambda)(x) = g(\lambda(g^{-1}x)).$$

By the tensor–hom adjunction we have

$$\begin{aligned}\mathrm{Hom}_{\overline{\mathbb{F}}_p}(V\otimes N, M)^{H_1\times H_2} &= \mathrm{Hom}_{\overline{\mathbb{F}}_p}(N, \mathrm{Hom}_{\overline{\mathbb{F}}_p}(V, M))^{H_1\times H_2}\\ &= \mathrm{Hom}_{\overline{\mathbb{F}}_p}(N, \mathrm{Hom}_{\overline{\mathbb{F}}_p}(V, M)^{H_1})^{H_2}\\ &= \mathrm{Hom}_{\overline{\mathbb{F}}_p}(N, M(V))^{H_2} = \mathrm{Hom}_{H_2}(N, M(V)).\end{aligned}$$

□

Suppose we have a smooth Γ''-sheaf of $\overline{\mathbb{F}}_p$-vector spaces, $\mathscr{F}$, on $S_0(G'', h)$. This pulls back, via the immersion of Corollary 4.1.7, to a smooth Γ-sheaf (which we again denote by $\mathscr{F}$) on $S_0(G, h)$. We denote the pullback of $\mathscr{F}$ to $S_0(G'', h)_e$ by $\mathscr{F}_e$. Note that $\mathscr{F}_e$ is a smooth $\mathscr{E}''$-sheaf on S_e. By restriction, we can also view it as a smooth $\mathscr{E}$-sheaf.

Corollary 4.1.11. *Suppose that V is a finite-dimensional continuous representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ *over $\overline{\mathbb{F}}_p$ such that $H^1(S_0(G'', h)_{\overline{\mathbb{Q}}}, \mathscr{F})(V) \neq 0$. Then*

$$H^1(S_0(G, h)_{\overline{\mathbb{Q}}}, \mathscr{F})(V) \neq 0.$$

Proof. For brevity we set $M := H^1(S_0(G, h)_{\overline{\mathbb{Q}}}, \mathscr{F})$ and $M'' := H^1(S_0(G'', h)_{\overline{\mathbb{Q}}}, \mathscr{F})$. The Γ-equivariant inclusion $S_0(G, h) \hookrightarrow S_0(G'', h)$ induces a $\Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/F^+)$-equivariant inclusion $M \hookrightarrow M''$ (since it identifies $S_0(G, h)$ as an open and closed subscheme of $S_0(G'', h)$). Now suppose $M''(V) \neq 0$. By Lemma 4.1.10, we have

$$\mathrm{Hom}_{\Gamma''\times \mathrm{Gal}(\overline{\mathbb{Q}}/F)}(V\otimes M''(V), M'') = \mathrm{End}(M''(V)) \neq 0.$$

On the other hand, Frobenius reciprocity and Proposition 3.3.6 imply that

$$\mathrm{Hom}_{\Gamma''\times \mathrm{Gal}(\overline{\mathbb{Q}}/F)}(V\otimes M''(V), M'') = \mathrm{Hom}_{\mathscr{E}''(h)_F}(V\otimes M''(V), H^1(S_0(G'', h)_e, \mathscr{F}_e)).$$

From here, since we have an $\mathscr{E}_F$-equivariant isomorphism

$$S_0(G, h)_e \cong S_0(G'', h)_e$$

we conclude that we have $\mathrm{Hom}_{\mathscr{E}_F}(V\otimes M''(V), H^1(S_0(G, h)_e, \mathscr{F}_e)) \neq 0$. Applying Frobenius reciprocity and Proposition 3.3.6 again we have

$$\mathrm{Hom}_{\Gamma\times \mathrm{Gal}(\overline{\mathbb{Q}}/F)}(V\otimes M''(V), M) \neq 0.$$

Now Lemma 4.1.10 implies that $\mathrm{Hom}_\Gamma(M''(V), M(V)) \neq 0$, and hence

$$M(V) \neq 0.$$

□

Remark 4.1.12. The above corollary is stated for representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ rather than $\mathrm{Gal}(\overline{\mathbb{Q}}/F^+)$ simply because it is the former case which we will need in applications.

In exactly the same way, if $\mathscr{F}$ is a smooth Γ''-sheaf on $S_0(G'', h\times h_F)$, and if we also denote by $\mathscr{F}$ the pullback to $S_0(G', h\times h_F)$, then we obtain

Corollary 4.1.13. *Suppose that V is a finite-dimensional continuous representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ *over $\overline{\mathbb{F}}_p$ such that $H^1(S_0(G'', h\times h_F)_{\overline{\mathbb{Q}}}, \mathscr{F})(V) \neq 0$. Then*

$$H^1(S_0(G', h\times h_F)_{\overline{\mathbb{Q}}}, \mathscr{F})(V) \neq 0.$$

4.2. **Serre weights, local systems.** We follow [BDJ10] in defining Serre weights. First we denote by $\mathbf{G}$ the finite group

$$\prod_{v\in\Sigma_p} \mathrm{GL}_2(k_v).$$

Note that the map j_p defined in Section 2.3 induces an isomorphism $K_0''/K_1'' \cong \mathbf{G}$, which restricts to give isomorphisms $K_0'/K_1' \cong \mathbf{G}$ and $K_0/K_1 \cong \mathbf{G}$. Using these isomorphisms, we from now on identify all these groups with $\mathbf{G}$.

Suppose that we have a finite Galois cover of schemes

$$S_1 \to S_0$$

with Galois group $\mathbf{G}$. Suppose S_0 is connected. Then for x any geometric point of S_0, we obtain an identification of $\mathbf{G}$ with a quotient of $\pi_1(S_0, x)$.

Definition 4.2.1. *Suppose that W is a finite dimensional $\overline{\mathbb{F}}_p$-representation of $\mathbf{G}$. If W is irreducible, we say that it is a Serre weight. Suppose that $S_1 \to S_0$ is a finite Galois cover of schemes with Galois group $\mathbf{G}$. Moreover, suppose that S_0 is connected. Pick x a geometric point of S_0. Then we define a locally constant étale sheaf of finite dimensional $\overline{\mathbb{F}}_p$-vector spaces $\mathscr{F}_W(S_0, S_1)$ on S_0 to be the local system attached to the action of $\pi_1(S_0, x)$ on W via its quotient $\mathbf{G}$. More generally, if S_0 is not necessarily connected, we may still define a locally constant étale sheaf $\mathscr{F}_W(S_0, S_1)$ on S_0 by applying the above procedure to each connected component of S_0.*

This construction will be applied to the Galois covers appearing in Definition 2.3.3. It is straightforward to list all possible Serre weights. We have the following standard description

Definition 4.2.2. *Suppose*

$$\mathbf{a} = (a^v_{\tau,1}, a^v_{\tau,2}) \in \prod_{v\in\Sigma_p} \prod_{\tau: k_v \hookrightarrow \overline{\mathbb{F}}_p} \mathbb{Z}^2$$

satisfies

$$0 \leq a^v_{\tau,1} - a^v_{\tau,2} \leq p-1$$

for each v and τ. Then we refer to $\mathbf{a}$ as a weight vector, and define a Serre weight $W_{\mathbf{a}}$ by

$$W_{\mathbf{a}} = \bigotimes_{v\in\Sigma_p} \bigotimes_{\tau: k_v\hookrightarrow\overline{\mathbb{F}}_p} \det{}^{a^v_{\tau,2}} \otimes \mathrm{Sym}^{a^v_{\tau,1}-a^v_{\tau,2}} k_v^2 \otimes_{k_v,\tau} \overline{\mathbb{F}}_p,$$

equipped with its natural action of $\mathbf{G}$.

Lemma 4.2.3. *Suppose W is a Serre weight. Then $W \cong W_{\mathbf{a}}$ for some weight vector $\mathbf{a}$.*

Proof. This is standard. One way of proving it is to find the number of isomorphism classes of Serre weights using Brauer theory, then verify that the representations $W_{\mathbf{a}}$ define precisely this many non-isomorphic Serre weights. For example, see [Bon11, Theorem 10.1.8] for the determination of the irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$. $\square$

Note that, since $\det^{q_v-1}$ is a trivial character of $\mathrm{GL}_2(k_v)$, different weight vectors may give rise to the same Serre weight. If $\mathbf{a}$ is a weight vector, and $S_1 \to S_0$ is as in Definition 4.2.1, then we abbreviate $\mathscr{F}_{W_{\mathbf{a}}}(S_0, S_1)$ by $\mathscr{F}_{\mathbf{a}}(S_0, S_1)$.

We will need a p-adic variant of the construction of the mod p local systems $\mathscr{F}_{\mathbf{a}}(S_0, S_1)$ in a particular situation.

Definition 4.2.4. *Suppose*
$$\lambda = (\lambda^v_{\sigma,1}, \lambda^v_{\sigma,2}) \in \prod_{v\in\Sigma_p} \prod_{\sigma:F_v^+\hookrightarrow\overline{\mathbb{Q}}_p} \mathbb{Z}^2$$
satisfies
$$\lambda^v_{\sigma,1} - \lambda^v_{\sigma,2} \geq 0$$
for each v and σ. Then we refer to λ as a weight.

Now we recall a special case of a standard construction of p-adic local systems on Shimura varieties (see [Car86b, 2.1],[HT01, III.2] and [BDJ10, §2]). Suppose λ is a weight. Then we have an irreducible algebraic representation of G' over $\overline{\mathbb{Q}}_p$ defined by
$$\xi_\lambda = \bigotimes_{v\in\Sigma_p} \bigotimes_{\sigma:F_v^+\hookrightarrow\overline{\mathbb{Q}}_p} \det{}^{\lambda^v_{\sigma,2}} \otimes \mathrm{Sym}^{\lambda^v{}_{\sigma,1}-\lambda^v{}_{\sigma,2}}(F_v^+)^2 \otimes_{F_v^+,\sigma} \overline{\mathbb{Q}}_p,$$
where G' acts via the isomorphism
$$G' \times \overline{\mathbb{Q}}_p \cong \mathbb{G}_m \times \prod_{v,\sigma} \mathrm{GL}_2$$
induced by the map j_p. Of course, ξ_λ is defined over some finite subextension $L/\mathbb{Q}_p$. For small enough compact open subgroups $K \subset G'(\mathbb{A}_f)$ (for example, with $K_p \subset K'_0$ and $S_K(G', h \times h_F)$ having no elliptic points), we may define, following Carayol [Car86b, 2.1.4], an $\mathscr{O}_L$-sheaf $\mathscr{F}^0_{\lambda,K}$ on $S_K(G', h \times h_F)$ associated to the representation ξ_λ (more precisely we fix the $\mathscr{O}_L$-lattice ξ^0_λ in ξ_λ arising from tensor products of symmetric powers of $\mathscr{O}_L^2$ and construct our $\mathscr{O}_L$-sheaves using this lattice). More generally, if ξ is any irreducible algebraic representation of G' over $\overline{\mathbb{Q}}_p$ we denote by $\mathscr{F}^0_{\xi,K}$ the $\mathscr{O}_L$-sheaf given by this construction. Since L is a subfield of $\overline{\mathbb{Q}}_p$ and we have identified the residue field of $\overline{\mathbb{Q}}_p$ with $\overline{\mathbb{F}}_p$, we obtain an induced map $\mathscr{O}_L/\mathfrak{m}_L \to \overline{\mathbb{F}}_p$ and hence an $\overline{\mathbb{F}}_p$-sheaf $\mathscr{F}^0_{\lambda,K} \otimes_{\mathscr{O}_L} \overline{\mathbb{F}}_p$ on $S_K(G', h\times h_F)$.

Definition 4.2.5. *Suppose* $\mathbf{a}$ *is a weight vector and λ is a weight. We say that λ is a* lift *of* $\mathbf{a}$ *if for each $v \in \Sigma_p$ and $\tau : k_v \hookrightarrow \overline{\mathbb{F}}_p$ there is one embedding $\sigma : F_v^+ \hookrightarrow \overline{\mathbb{Q}}_p$ lifting τ such that $\lambda^v_\sigma = a^v_\tau$, and for the other σ' lifting τ we have $\lambda^v_{\sigma'} = 0$.*

Lemma 4.2.6. *Suppose λ is a lift of* $\mathbf{a}$*. Let $K^p \subset G'(A^p_f)$ be a sufficiently small compact open subgroup. Then*
$$\mathscr{F}^0_{\lambda,K^pK'_0} \otimes_{\mathscr{O}_L} \overline{\mathbb{F}}_p \cong \mathscr{F}_{\mathbf{a}}(S_{K^pK'_0}(G', h\times h_F), S_{K^pK'_1}(G', h\times h_F)).$$

Proof. This follows immediately from the fact that if λ is a lift of $\mathbf{a}$ then $\xi^0_\lambda \otimes_{\mathscr{O}_L} \overline{\mathbb{F}}_p$ is isomorphic to $W_{\mathbf{a}}$ as a $K'_0/K'_1 \cong \mathbf{G}$ representation. □

We now discuss how our Serre weights fit into the framework of Section 4.1.

Definition 4.2.7. *Suppose W is a Serre weight. Then the construction of Definition 4.2.1 gives rise to*

- *a smooth Γ-sheaf of $\overline{\mathbb{F}}_p$-vector spaces on $S_0(G,h)$, which we denote by $\mathscr{F}_W$*
- *a smooth Γ'-sheaf of $\overline{\mathbb{F}}_p$-vector spaces on $S_0(G', h\times h_F)$, which we denote by $\mathscr{F}'_W$*
- *a smooth Γ''-sheaf of $\overline{\mathbb{F}}_p$-vector spaces on $S_0(G'', h\times h_F)$, which we denote by $\mathscr{F}''_W(h\times h_F)$*
- *a smooth Γ''-sheaf of $\overline{\mathbb{F}}_p$-vector spaces on $S_0(G'', h)$, which we denote by $\mathscr{F}''_W(h)$.*

If $\mathbf{a}$ *is a weight vector, then we denote $\mathscr{F}_{W_{\mathbf{a}}}$ by $\mathscr{F}_{\mathbf{a}}$, and similarly for the other sheaves defined above.*

We need a mild variant of the above definition in Section 6.

Definition 4.2.8. *Suppose W is a Serre weight and $n \in \mathbb{Z}$. Denote by $W(n)$ the representation of $K_0'/K_{1,1}' \cong \mathbb{F}_p^\times \times \mathbf{G}$ whose underlying $\mathbf{G}$ representation is W and on which $\mathbb{F}_p^\times$ acts by multiplication by nth powers. Then we define $\mathscr{F}_W'(n)$ to be the smooth Γ'-sheaf of $\overline{\mathbb{F}}_p$-vector spaces on $S_0(G', h \times h_F)$ obtained by replacing K_1' with $K_{1,1}'$ (see Definition 2.3.2), and using the representation $W(n)$ of $K_0'/K_{1,1}'$ in the construction of Definition 4.2.1. Similarly, for a weight λ we define algebraic representations $\xi_\lambda(n)$ and characteristic zero local systems $\mathscr{F}^0_{\lambda, K^pK_0'}(n)$.*

Note that for a Serre weight W, we can identify $\mathscr{F}_W'(0)$ and $\mathscr{F}_W'$.

Lemma 4.2.9. *Suppose W is a Serre weight. The pullback of $\mathscr{F}_W''(h)$ to $S_0(G, h)$, via the inclusion of Corollary 4.1.7, is isomorphic to $\mathscr{F}_W$. Similarly, the pullback of $\mathscr{F}_W''(h \times h_F)$ to $S_0(G', h \times h_F)$ is isomorphic to $\mathscr{F}_W'$.*

Proof. This is immediate from Corollary 4.1.7. □

5. Transfer of Serre weights

Corollary 4.1.11 will allow us to compare the Galois representations appearing in the cohomology of $S_0(G, h)$ and of $S_0(G'', h)$. Similarly, Corollary 4.1.13 will allow us to compare the representations appearing in the cohomology of $S_0(G', h \times h_F)$ and of $S_0(G'', h \times h_F)$. In the main result of this section, Corollary 5.4.3, we relate the cohomology of $S_0(G'', h)$ with that of $S_0(G'', h \times h_F)$. This is essentially an application of the fact that the neutral components of these Shimura curves are isomorphic, by the uniqueness of weakly canonical models of connected Shimura varieties. However, we will need to keep track of central characters, to allow us to take account of the 'twisted' relationship between the groups $\mathscr{E}''(h \times h_F)$, $\mathscr{E}''(h)$ (see [Car86a, 4.2.1]).

5.1. Modularity.

We begin with the definition of being modular of a certain Serre weight, with a certain central character. This is essentially a refinement of [BDJ10, Definition 2.1] — we need to keep track of a central character when we move between $S_0(G'', h)$ and $S_0(G'', h \times h_F)$.

In this section we fix a Shimura datum (G_0, h_0), with reflex field F_0. We denote by Z_0 the centre of G_0. We will assume that (G_0, h_0) is one of (G, h), $(G', h \times h_F)$, $(G'', h \times h_F)$ or (G'', h). We fix compact open subgroups K_0 and K_1 of $G_0(\mathbb{Q}_p)$ (which will be those described in Definition 2.3.1), together with an isomorphism $K_0/K_1 \cong \mathbf{G}$. Via this isomorphism, we can view representations of $\mathbf{G}$ as representations of K_0/K_1, and by inflation as representations of K_0.

We denote by $H(G_0, h_0; \overline{\mathbb{F}}_p)$ the $\overline{\mathbb{F}}_p[G_{F_0} \times G_0(\mathbb{A}_f)]$-representation

$$\varinjlim_{\substack{K \subset G_0(\mathbb{A}_f) \\ \text{cpt. open}}} H^1_{et}(S_K(G_0, h_0)_{\overline{\mathbb{Q}}}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p.$$

Suppose

$$\psi_0 : \overline{Z_0(\mathbb{Q})} \backslash Z_0(\mathbb{A}_f) \to \overline{\mathbb{F}}_p^\times$$

is a continuous character. Then we denote by $H(G_0, h_0; \overline{\mathbb{F}}_p)[\psi_0]$ the subrepresentation of $H(G_0, h_0; \overline{\mathbb{F}}_p)$ where $Z_0(\mathbb{A}_f)$ acts via the character ψ_0. Note that the action of $Z_0(\mathbb{A}_f)$ on $H(G_0, h_0; \overline{\mathbb{F}}_p)$ factors through the quotient $\overline{Z_0(\mathbb{Q})} \backslash Z_0(\mathbb{A}_f)$, since $\overline{Z_0(\mathbb{Q})}$ acts trivially on $S(G_0, h_0)$.

Definition 5.1.1. *Suppose F_1 is a finite extension of F_0 (in practice, F_1 will be either F^+ or F) and suppose*

$$\rho : G_{F_1} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is a continuous irreducible representation. Let W be a Serre weight and let $\psi : \overline{Z_0(\mathbb{Q})}\backslash Z_0(\mathbb{A}_f) \to \overline{\mathbb{F}}_p^\times$ be a continuous character. We say that ρ is (G_0, h_0)-modular of weight W and central character ψ if

$$\mathrm{Hom}_{G_{F_1}\times K_0}(\rho \otimes W^\vee, H(G_0, h_0; \overline{\mathbb{F}}_p)[\psi]) \neq 0.$$

We say that ρ is (G_0, h_0)-modular of weight W if there exists ψ as above such that ρ is (G_0, h_0)-modular of weight W and central character ψ. Similarly, we simply say that ρ is (G_0, h_0)-modular if there exists a Serre weight W such that ρ is (G_0, h_0)-modular of weight W.

Remark 5.1.2. Note that if ρ is (G_0, h_0)-modular of weight W and central character ψ, then we are forced to have $\psi|_{K_0 \cap Z_0(\mathbb{A}_f)}$ equal to the central character of the representation $W^\vee$. Suppose we have ψ, W such that this is satisfied. Then we can alternatively (and equivalently) define ρ to be (G_0, h_0)-modular of weight W and central character ψ if

$$\mathrm{Hom}_{G_{F_1}\times K_0 Z_0(\mathbb{A}_f)}(\rho \otimes W^{\vee,\psi}, H(G_0, h_0; \overline{\mathbb{F}}_p)) \neq 0,$$

where $W^{\vee,\psi}$ denotes the representation of $K_0 Z_0(\mathbb{A}_f) \subset G_0(\mathbb{A}_f)$ with underlying K_0 representation $W^\vee$ and $z \in Z_0(\mathbb{A}_f)$ acting via ψ. The representation $W^{\vee,\psi}$ is well-defined because of the assumed compatibility between ψ and W.

Remark 5.1.3. Our conventions differ from those of [BDJ10] by a twist, so ρ is (G, h)-modular of weight W implies that the Tate twist $\rho(1)$ is modular of weight W in the sense of [BDJ10, Definition 2.1]. Conversely, if $\rho(1)$ is modular of weight W in the sense of [BDJ10] then there exists a quaternion algebra D as in 2.1 such that ρ is (G, h)-modular of weight W.

We can now record a consequence of Corollaries 4.1.11 and 4.1.13 as a lemma:

Lemma 5.1.4. *Suppose $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous irreducible representation. Let W be a Serre weight.*

- *If ρ is (G'', h)-modular of weight W, it is (G, h)-modular of weight W.*
- *If ρ is $(G'', h \times h_F)$-modular of weight W, it is $(G', h \times h_F)$-modular of weight W.*

If $\mathbf{a}$ is a weight vector, then we say that ρ is modular of weight $\mathbf{a}$ if it is modular of weight $W_{\mathbf{a}}$ (and similarly with a prescribed central character).

Definition 5.1.5. *Let $\mathbf{a}$ be a weight vector. We define a character*

$$\psi_{\mathbf{a}} : \mathscr{O}_{F_p}^\times = \prod_{w|p} \mathscr{O}_{F_w}^\times \to \overline{\mathbb{F}}_p^\times$$

by

$$\psi_{\mathbf{a}}(z_{\tilde{v}}) = 1$$
$$\psi_{\mathbf{a}}(z_{\tilde{v}^c}) = \prod_{\tau : k_v \to \overline{\mathbb{F}}_p} \tau([z_{\tilde{v}^c}])^{a_{\tau,1}+a_{\tau,2}}.$$

Here we denote by $[z_{\tilde{v}^c}]$ the class of $z_{\tilde{v}^c} \in \mathscr{O}_{F_{\tilde{v}^c}} = \mathscr{O}_{F_v^+}$ in the residue field k_v.

Lemma 5.1.6. *Suppose $\mathbf{a}$ is a weight vector. Consider the representation $W_{\mathbf{a}}$ of K_0''. Let $z \in \mathscr{O}_{F_p}^\times \hookrightarrow K_0''$. Then*

$$z.v = \psi_{\mathbf{a}}(z) v$$

for all v in $W_{\mathbf{a}}$.

Proof. This is a direct calculation. ∎

Lemma 5.1.7. *Suppose*
$$\rho_0 : G_{F_1} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$
is a continuous irreducible representation. Let W be a Serre weight and
$$\psi : \overline{Z_0(\mathbb{Q})}\backslash Z_0(\mathbb{A}_f) \to \overline{\mathbb{F}}_p^\times$$
a continuous character. Then ρ is (G_0, h_0)-modular of weight W and central character ψ if and only if
$$\mathrm{Hom}_{G_{F_1}}(\rho, H^1_{et}(S_{K^pK_0}(G_0,h_0)_{\overline{\mathbb{Q}}}, \mathscr{F}_W)[\psi^p]) \neq 0,$$
for some sufficiently small compact open subgroup $K^p \subset G_0(\mathbb{A}_f^p)$.

Proof. This follows from combining Hochschild–Serre, as in [BDJ10, Lemma 2.4 (a)], with the fact that ψ is determined by ψ^p and $\psi|_{K_0 \cap Z_0(\mathbb{A}_f)}$ (W determines the latter, by remark 5.1.2). □

Lemma 5.1.8. *Suppose $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is $(G', h \times h_F)$-modular of weight W and central character ψ'. Then ρ is $(G'', h \times h_F)$-modular of weight W and central character ψ'', with ψ'' some character of $Z''(\mathbb{A}_f)/\overline{Z''(\mathbb{Q})}$ satisfying $\psi''|_{Z'(\mathbb{A}_f)} = \psi'$.*

Proof. By Lemma 5.1.7, we have
$$\mathrm{Hom}_{G_F}(\rho, H^1_{et}(S_{K^pK_0'}(G', h \times h_F)_{\overline{F}}, \mathscr{F}_W)[\psi'^p]) \neq 0,$$
for some compact open subgroup $K^p \subset G'(\mathbb{A}_f^p)$. By Corollary 4.1.7 we have an open and closed immersion (of schemes over F) $S_{K^pK_0'}(G', h \times h_F) \hookrightarrow S_{K''^pK_0''}(G'', h \times h_F)$ for some sufficiently small compact open subgroup $K''^p \subset G''(\mathbb{A}_f^p)$. This embedding is equivariant with respect to the action of $Z'(\mathbb{A}_f^p)$, and by Lemma 4.2.9 the sheaf $\mathscr{F}''_{\mathbf{a},K''^p}$ pulls back to $\mathscr{F}'_{\mathbf{a},K^p}$. Taken together, this gives a $G_F \times Z'(\mathbb{A}_f^p)$-equivariant embedding
$$H^1_{et}(S_{K^pK_0'}(G', h \times h_F)_{\overline{F}}, \mathscr{F}_W) \hookrightarrow H^1_{et}(S_{K''^pK_0''}(G'', h \times h_F)_{\overline{F}}, \mathscr{F}_W).$$
Applying Lemma 5.1.7 again, we see that there is a character ψ'' which restricts to ψ' on $Z'(\mathbb{A}_f)$ such that ρ is $(G'', h \times h_F)$-modular of weight W and central character ψ''. □

Similarly we obtain

Lemma 5.1.9. *Suppose $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is (G, h)-modular of weight W and central character ψ. Then ρ is (G'', h)-modular of weight W and central character ψ'', with ψ'' some character of $Z''(\mathbb{A}_f)/\overline{Z''(\mathbb{Q})}$ satisfying $\psi''|_{Z(\mathbb{A}_f)} = \psi$.*

5.2. **Hecke operators.** Our main result in this section is a comparison between the notions of (G'', h)-modularity and $(G'', h \times h_F)$-modularity. We initially hoped to make use of the material in Section 3 to do this, but it is in fact significantly easier to make this comparison using the action of certain Hecke operators. The simplification arises because it is enough to work with Hecke operators at places which split in F/F^+. Fix a sufficiently small compact open subgroup K^p of $G''(\mathbb{A}_f^p)$. By the Betti comparison isomorphism, since $S_{K^pK_0''}(G'', h)_{\mathbb{C}}$ and $S_{K^pK_0''}(G'', h \times h_F)_{\mathbb{C}}$ are isomorphic as topological spaces, we have an isomorphism
$$H^1_{et}(S_{K^pK_0''}(G'', h)_{\overline{F}}, \mathscr{F}_{\mathbf{a}}) \xrightarrow{\sim} H^1_{et}(S_{K^pK_0''}(G'', h \times h_F)_{\overline{F}}, \mathscr{F}_{\mathbf{a}}),$$
which is equivariant with respect to the Hecke operator $[K^pgK^p]$, for any $g \in G''(\mathbb{A}_f^p)$. Of course, this isomorphism is not equivariant for the Galois action. We want to understand the relationship between the Galois actions on the Hecke eigenspaces of these two cohomology groups.

Recall that we have fixed embeddings $\tau_E : E \hookrightarrow \mathbb{C}$ and $\tau_1 : F^+ \hookrightarrow \mathbb{R}$. We also denote by τ_1 the unique embedding $\tau_1 : F \hookrightarrow \mathbb{C}$ which restricts to τ_E on E and τ_1

on F^+. In this section we will be interested in places v of F^+ which split in F. We fix a rational prime l which splits in E and let v be a prime of F^+ lying over l. Therefore, v splits in F. We also assume that D_v is split. Let w be one of the places of F which divides v, so $v = ww^c$. Denote by u the place $w|_E$ of E. We fix a uniformiser ϖ_v of F_v^+ and denote by ϖ_w the uniformiser of F_w obtained from ϖ_v under the canonical isomorphism $F_v^+ \cong F_w$.

We recall that $G''(\mathbb{Q}_l)$ consists of $\beta \in (B \otimes_{\mathbb{Q}} \mathbb{Q}_l)^\times$ such that there exists $\mu \in (F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_l)^\times$ with $\psi(v\beta, w\beta) = \mu\psi(v, w)$.

We choose an isomorphism $j_v : B \otimes_{F^+} F_v^+ = B_w \oplus B_{w^c} \cong M_2(F_w) \oplus M_2(F_{w^c})$ such that $j_v(\beta^*) = {}^t j_v(x)^c$. This determines an isomorphism

$$j_w : G''(\mathbb{Q}_l) \cong (F_l^+)^\times \times \mathrm{GL}_2(F_v^+) \times \prod_{v'|l, v' \neq v} D_{v'}^\times,$$

just as in Section 2.3 (see also [Car86a, 2.6.3]).

We now suppose that we have chosen K^p, w and j_v such that $j_w((K^p)_l) = (\mathscr{O}_{F_v^+})^\times \times \mathrm{GL}_2(\mathscr{O}_{F_v^+}) \times \cdots$. We denote by T_w the Hecke operator

$$[K^p j_w^{-1}(1, \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix}, 1, ..., 1) K^p]$$

acting on $H^1_{et}(S_{K^pK_0''}(G'', h)_{\overline{F}}, \mathscr{F}_{\mathbf{a}})$ and $H^1_{et}(S_{K^pK_0''}(G'', h \times h_F)_{\overline{F}}, \mathscr{F}_{\mathbf{a}})$.

5.3. Eichler-Shimura relations.

Eichler-Shimura for (G'', h).

Lemma 5.3.1. *Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible continuous representation. Fix a finite set of places S of F such that ρ is unramified outside S, and S contains all the places over p, and all the places over places where D is non-split. Then*

$$\mathrm{Hom}_{G_{F^+}}(\rho, H^1_{et}(S_{K^pK_0''}(G'', h)_{\overline{F}}, \mathscr{F}_{\mathbf{a}})[\psi^p]) \neq 0$$

if and only if there exists a non-zero $f_\rho \in H^1_{et}(S_{K^pK_0''}(G'', h)_{\overline{F}}, \mathscr{F}_{\mathbf{a}})[\psi^p]$ satisfying

- *for w any place of F, not in S, and split over F^+ we have*

$$\psi(\iota_w^{-1}(\varpi_v, 1, 1, ..., 1)) T_w(f_\rho) = Tr(\rho(Frob_w)) f_\rho$$

Here $Frob_w$ denotes a geometric Frobenius element at w.

Proof. This follows from [Car86a, Proposition 10.3]. □

Eichler-Shimura for $(G'', h \times h_F)$.

Lemma 5.3.2. *Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible continuous representation. Fix a finite set of places S of F such that ρ is unramified outside S, and S contains all the places over p, and all the places over places where D is non-split. Then*

$$\mathrm{Hom}_{G_{F^+}}(\rho, H^1_{et}(S_{K^pK_0''}(G'', h)_{\overline{F}}, \mathscr{F}_{\mathbf{a}})[\psi^p]) \neq 0$$

if and only if there exists a non-zero $f_\rho \in H^1_{et}(S_{K^pK_0''}(G'', h)_{\overline{F}}, \mathscr{F}_{\mathbf{a}})[\psi^p]$ satisfying

- *for w any place of F, not in S, and split over F^+ we have*

$$\psi(\iota_w^{-1}(N_{F_v^+/\mathbb{Q}_l}(\varpi_v), 1, 1, ..., 1)) T_w(f_\rho) = Tr(\rho(Frob_w)) f_\rho$$

Proof. This follows from [Car86a, 10.4]. □

Remark 5.3.3. We have

$$\iota_w^{-1}(\varpi_v, 1, 1, ..., 1) = \varpi_w \in F_w^\times \hookrightarrow F_l^\times \hookrightarrow G''(\mathbb{Q}_l)$$

and

$$\iota_w^{-1}(N_{F_v^+/\mathbb{Q}_l}(\varpi_v), 1, 1, ..., 1) = N_{F_w/E_u}(\varpi_w) \in E_u^\times \hookrightarrow F_w^\times \hookrightarrow F_l^\times \hookrightarrow G''(\mathbb{Q}_l).$$

5.4. **Transferring weights.**

Definition 5.4.1. *Suppose* $\psi : \overline{Z''(\mathbb{Q})}\backslash Z''(\mathbb{A}_f) \to \overline{\mathbb{F}}_p^{\times}$ *is a continuous character. Denote by* S_ψ *the (finite) set of finite places* q *of* $\mathbb{Q}$ *where* $\psi|_{Z''(\mathbb{Z}_q)}$ *is non-trivial. We denote by*

$$\varphi_\psi : G_F \to \overline{\mathbb{F}}_p^{\times}$$

the unique character satisfying

$$\varphi_\psi(Frob_w) = \psi(N_{F_w/E_u}(\varpi_w)/\varpi_w)$$

for all geometric Frobenius elements $Frob_w$ *at places* w *which are split over* F^+ *and do not divide any of the places* $q \in S_\psi$*.*

Remark 5.4.2. If we denote by ϕ the map of algebraic groups $\phi : Z'' \to Z''$ given by $z \mapsto N_{F/E}(z)/z$ then φ_ψ is the Galois character attached by class field theory to the character

$$\psi \circ \phi : \overline{Z''(\mathbb{Q})}\backslash Z''(\mathbb{A}_f) \to \overline{\mathbb{F}}_p^{\times}.$$

Corollary 5.4.3. *Suppose* $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *is a continuous irreducible representation. Let* W *be a Serre weight and* $\psi : \overline{Z''(\mathbb{Q})}\backslash Z''(\mathbb{A}_f) \to \overline{\mathbb{F}}_p^{\times}$ *a continuous character. Let* $\varphi_\psi : G_F \to \overline{\mathbb{F}}_p^{\times}$ *be the character defined above. Then* ρ *is* (G'', h)*-modular of weight* W *and central character* ψ *if and only if* $\rho \otimes \varphi_\psi$ *is* $(G'', h \times h_F)$*-modular of weight* W *and central character* ψ*.*

Proof. This follows from comparing the statements of Lemmas 5.3.1 and 5.3.2. $\square$

Remark 5.4.4. Suppose ρ is (G'', h)-modular of weight $\mathbf{a}$ and central character ψ. Then we have $\psi|_{K_0'' Z''(\mathbb{A}_f)} = \psi_{\mathbf{a}}$. Hence, by Lemma 5.1.6, φ_ψ is unramified at the places $\tilde{v}$ (but will often be ramified at the places $\tilde{v}^c$).

6. Serre weights for $(G', h \times h_F)$

In this section, we concentrate exclusively on Serre weight conjectures for the unitary similitude group G'. Our approach to investigating the weights of modular Galois representations for the group G' is identical to that of [BLGG] in the case of definite unitary groups. The crucial fact which facilitates this approach in the case of unitary groups (but not in the quaternionic setting) is Lemma 4.2.6.

6.1. **Lifting modular Galois representations.**

Proposition 6.1.1. *Suppose* $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *is a continuous irreducible representation. Let* $\mathbf{a}$ *be a weight vector, and let* λ *be a lift of* $\mathbf{a}$*. Finally, let*

$$\psi : Z'(\mathbb{Q})\backslash Z'(\mathbb{A}_f) \to \overline{\mathbb{F}}_p^{\times}$$

be a continuous character. Then ρ *is* $(G', h \times h_F)$*-modular of weight* W *and central character* ψ *if and only if there exists* K^p *a sufficiently small compact open subgroup of* $G'(\mathbb{A}_f^p)$*,* $L/\mathbb{Q}_p$ *a finite subextension of* $\overline{\mathbb{Q}}_p$ *over which we can (and do) define* ξ_λ*, a continuous character*

$$\tilde{\psi}^p : Z'(\mathbb{A}_f^p) \to \mathscr{O}_L^{\times}$$

and a continuous representation

$$\tilde{\rho} : G_F \to \mathrm{GL}_2(\mathscr{O}_L)$$

such that

- $\tilde{\psi}^p \otimes_{\mathscr{O}_L} \overline{\mathbb{F}}_p = \psi^p$*,*
- $\tilde{\rho} \otimes_{\mathscr{O}_L} \overline{\mathbb{F}}_p \cong \rho$ *and*
- $\mathrm{Hom}_{G_F}(\tilde{\rho}, H^1_{et}((S_{K_0'K^p}(G', h \times h_F))_{\overline{\mathbb{Q}}}, \mathscr{F}^0_{\lambda, K_0'K^p})[\tilde{\psi}^p]) \neq 0.$

Proof. This follows from Lemma 4.2.6 and the proof of [BDJ10, Proposition 2.10] (where we adjust the latter by keeping track of the central character). □

Remark 6.1.2. By Matsushima's formula, and the Eichler-Shimura relation for the Shimura curves $S_{K_0'K^p}(G', h\times h_F)$ the equivalent conditions of the above proposition are also equivalent to the existence of a suitable automorphic representation of $G'(\mathbb{A})$.

6.2. **Sets of weights determined by lifting properties of local Galois representations.** In this section we describe some sets of Serre weights given by the existence of lifts of local Galois representations. We need a few preliminary definitions.

Definition 6.2.1. *Let K be a finite extension of $\mathbb{Q}_p$ and suppose we have a de Rham (or merely Hodge-Tate) representation $\rho_K : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$. Let $\sigma : K \hookrightarrow \overline{\mathbb{Q}}_p$ be a continuous embedding. Then the multiset $\mathrm{HT}_\sigma(\rho_K)$ of Hodge-Tate weights of ρ_K with respect to σ is defined by the property that it contains an integer i with multiplicity $\dim_{\overline{\mathbb{Q}}_p}(\rho_K \otimes_{\sigma,K} \hat{\overline{K}}(i))^{G_K}$. With this definition the p-adic cyclotomic character has Hodge-Tate weight -1.*

Definition 6.2.2. *Suppose we have, for some $v \in \Sigma_p$,*

$$\lambda^v = (\lambda^v_{\sigma,1}, \lambda^v_{\sigma,2}) \in \prod_{\sigma : F_v^+ \hookrightarrow \overline{\mathbb{Q}}_p} \mathbb{Z}^2$$

satisfying

$$\lambda^v_{\sigma,1} - \lambda^v_{\sigma,2} \geq 0$$

for each σ. Then we say that a de Rham representation

$$\rho_v : G_{F_v^+} = G_{F_{\tilde{v}}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

has weight λ^v if for each $\sigma : F_v^+ \hookrightarrow \overline{\mathbb{Q}}_p$ we have

$$\mathrm{HT}_\sigma(\rho_v) = \{\lambda^v_{\sigma,1} + 1, \lambda^v_{\sigma,2}\}.$$

Definition 6.2.3. *Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation. We denote by $W_{cr}(\rho)$ the set of Serre weights comprising those $W_{\mathbf{a}}$ such that there is a weight λ lifting $\mathbf{a}$ and for each $v \in \Sigma_p$ there is a crystalline lift of $\rho|_{G_{F_{\tilde{v}}}}$ with weight λ^v.*

Similarly, we denote by $W_{pd}(\rho)$ the set of Serre weights comprising those $W_{\mathbf{a}}$ such that there is a weight λ lifting $\mathbf{a}$ and for each $v \in \Sigma_p$ there is a crystalline and potentially diagonalisable (in the sense of [BLGGT10, 1.4]*) lift of $\rho|_{G_{F_{\tilde{v}}}}$ with weight λ^v.*

Corollary 6.2.4. *Suppose ρ is $(G', h \times h_F)$-modular of weight W. Then $W \in W^{cr}(\rho)$.*

Proof. Pick a weight λ lifting $\mathbf{a}$. It follows from Proposition 6.1.1 that we have a lift $\tilde{\rho}$ of ρ (necessarily absolutely irreducible), with

$$\mathrm{Hom}_{G_F}(\tilde{\rho}, H^1_{et}((S_{K_0'K^p})_{\overline{\mathbb{Q}}}, \mathscr{F}^0_{\lambda,K_0'K^p})) \neq 0.$$

The representation $\tilde{\rho} \otimes_{\mathscr{O}_L} L$ is crystalline at places dividing p (as in [HT01, Lemma III.4.2]), and we can compute its labelled Hodge-Tate weights using [HT01, Proposition III.2.1] to obtain the corollary (since the representations $\tilde{\rho}|_{G_{F_{\tilde{v}}}}$ have weight λ^v). Note that the assumptions of the book [HT01] exclude the situation (which we want to allow here) where D is split at all finite places, but everything we need goes through in this case. □

We have the following conjecture in the other direction, originally stated by Gee in the case of Hilbert modular forms:

Conjecture. *Suppose ρ is $(G', h \times h_F)$-modular, and $W \in W_{cr}(\rho)$. Then ρ is $(G', h \times h_F)$-modular of weight W.*

6.3. **Serre weight conjectures and modularity lifting theorems.** In this section we apply modularity lifting theorems, as in [BLGG], to show that, under certain technical hypotheses, if ρ is $(G', h \times h_F)$-modular, then it is modular of weight W for every $W \in W_{pd}(\rho)$. We first need to relate automorphic representations of $G'(\mathbb{A})$ and automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. These are applications of base change and Jacquet-Langlands functorialities. We will need some notation, predominantly borrowed from [HT01]

Definition 6.3.1.

- *We call a regular, algebraic, conjugate self-dual, cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_F)$ an RACSDC automorphic representation.*
- *For an algebraic Hecke character ψ of $\mathbb{A}_E^\times$ we denote by $\mathrm{rec}(\psi)$ the unique continuous character $G_E \to \overline{\mathbb{Q}}_p^\times$ satisfying*
$$\iota_p \circ \mathrm{rec}(\psi) \circ \mathrm{Art}_{E_x} = \psi|_{E_x^\times}$$
for every finite place $x \nmid p$ of E (Art_{E_x} denotes the local Artin map). We denote by $\overline{\mathrm{rec}}(\psi)$ the character $G_E \to \overline{\mathbb{F}}_p^\times$ obtained by reducing $\mathrm{rec}(\psi)$ mod p.
- *For an RACSDC automorphic representation Π, we denote by $r(\Pi)$ the two-dimensional representation $G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ attached to Π, ι_p, as in* [BLGHT11, Theorem 1.2]. *We denote by $\overline{r}(\Pi)$ the semisimplified reduction of this representation mod p.*
- *For ξ an irreducible algebraic representation of G' over $\overline{\mathbb{Q}}_p$, we write $\xi' = \iota_p(\xi)$ for the corresponding irreducible algebraic representation of G' over $\mathbb{C}$ and ξ'_E for the representation of $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ obtained from ξ' by the procedure described at the start of* [HT01, VI.2]. *We also adopt from loc. cit. the definition of an irreducible admissible representation of $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ being cohomological for ξ'_E.*

Proposition 6.3.2. *Suppose we have K^p a sufficiently small compact open subgroup of $G'(\mathbb{A}_f^p)$, a weight λ, $L/\mathbb{Q}_p$ a finite subextension of $\overline{\mathbb{Q}}_p$ over which we can (and do) define ξ_λ, a continuous character*
$$\tilde{\psi}^p : Z'(\mathbb{A}_f^p) \to \mathscr{O}_L^\times$$
and a continuous representation
$$\tilde{\rho} : G_F \to \mathrm{GL}_2(\mathscr{O}_L)$$
such that
$$\mathrm{Hom}_{G_F}(\tilde{\rho}, H^1_{et}((S_{K'_0K^p})_{\overline{\mathbb{Q}}}, \mathscr{F}^0_{\lambda,K'_0K^p})[\tilde{\psi}^p]) \neq 0.$$
Then there exists an RACSDC automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_F)$, with central character ψ_Π, and an algebraic Hecke character $\tilde{\psi}_E$ of $\mathbb{A}_E^\times$ such that $\tilde{\psi}_E|_{\mathbb{A}^p_{E,f}} = \tilde{\psi}^p|_{\mathbb{A}^p_{E,f}}$ and
$$r(\Pi) \cong \tilde{\rho} \otimes \mathrm{rec}(\tilde{\psi}_E)|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}.$$

The representation Π and character $\tilde{\psi}_E$ are unramified at all places dividing p, and we have
$$\tilde{\psi}^p = \iota_p^{-1} \circ (\tilde{\psi}_E \times \psi_\Pi)|_{Z'(\mathbb{A}_f^p)}.$$

Proof. When D is non-split at some finite place, this follows from Remark 6.1.2, [HT01, Theorem VI.2.1] (base change) and [HT01, Theorem VI.1.1] (Jacquet-Langlands). When D is split at every finite place, the necessary ingredients are provided by [Rog90]. □

Proposition 6.3.3. *Suppose we have an RACSDC automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_F)$, with central character ψ_Π, an algebraic Hecke character $\tilde{\psi}_E$ of $\mathbb{A}_E^\times$ such that $\psi_\Pi|_{\mathbb{A}_E^\times} = \tilde{\psi}_E^c/\tilde{\psi}_E$, and an algebraic representation ξ of G' over $\overline{\mathbb{Q}}_p$ such that $\Pi_\infty = \Pi|_{\mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{R})}$ is cohomological for ξ'_E, and $\xi'|^{-1}_{E_\infty^\times} = \tilde{\psi}_E|^c_{E_\infty^\times}$. Moreover, suppose that Π_w is square-integrable at each finite place w of F where B is ramified, and that Π and $\tilde{\psi}_E$ are unramified at all places dividing p.*

Then there exists K^p a sufficiently small compact open subgroup of $G'(\mathbb{A}_f^p)$, $L/\mathbb{Q}_p$ a finite subextension of $\overline{\mathbb{Q}}_p$ over which we can (and do) define ξ, a continuous character

$$\tilde{\psi}^p : Z'(\mathbb{A}_f^p) \to \mathscr{O}_L^\times$$

such that

$$\tilde{\psi}^p = \iota_p^{-1} \circ (\tilde{\psi}_E \times \psi_\Pi)|_{Z'(\mathbb{A}_f^p)}$$

and a continuous representation

$$\tilde{\rho} : G_F \to \mathrm{GL}_2(\mathscr{O}_L)$$

such that

$$r(\Pi) \cong \tilde{\rho} \otimes \mathrm{rec}(\tilde{\psi}_E)|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}$$

and

$$\mathrm{Hom}_{G_F}(\tilde{\rho}, H^1_{et}((S_{K'_0K^p})_{\overline{\mathbb{Q}}}, \mathscr{F}^0_{\xi,K'_0K^p})[\tilde{\psi}^p]) \neq 0.$$

Proof. When D is non-split at some finite place, the result follows from [HT01, Theorem VI.2.9] (characterising the image of base change), [HT01, Theorem VI.1.1] and Remark 6.1.2. When D is split at every finite place, the necessary ingredients are again provided by [Rog90]. □

Definition 6.3.4. *If R is a commutative ring and $r : G_F \to \mathrm{GL}_2(R)$ is a representation, we say that r has* split ramification *if $r|_{G_{F_w}}$ is unramified for any finite place w of F which is not split over F^+.*

Before we give the main result of this section, we need to prove a lemma regarding twisting by certain characters and $(G', h \times h_f)$-modularity. This is an analogue of [BDJ10, Lemma 2.3]. First we need a small generalisation of Definition 5.1.1.

Definition 6.3.5. *Suppose $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous irreducible representation. Let W be a Serre weight and $n \in \mathbb{Z}$. We say that ρ is $(G', h \times h_F)$-modular of weight $W(n)$ if*

$$\mathrm{Hom}_{G_F \times K'_0}(\rho \otimes W^\vee(-n), H(G', h \times h_F; \overline{\mathbb{F}}_p)) \neq 0.$$

Equivalently, ρ is $(G', h \times h_F)$-modular of weight $W(n)$ if

$$\mathrm{Hom}_{G_F}(\rho, H^1_{et}(S_{K^pK'_0}(G', h \times h_F)_{\overline{\mathbb{Q}}}, \mathscr{F}_W(n))) \neq 0,$$

for some sufficiently small compact open subgroup $K^p \subset G'(\mathbb{A}_f^p)$.

Now we can state our twisting lemma:

Lemma 6.3.6. *Suppose $\chi : G_{\mathbb{Q}} \to \overline{\mathbb{F}}_p^\times$ is a continuous character, with $\chi|_{I_p} = \omega^n$ for $n \in \mathbb{Z}$, where ω denotes the mod p cyclotomic character. Suppose that $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous irreducible representation. Let W be a Serre weight. Then ρ is $(G', h \times h_F)$-modular of weight W if and only if $\rho \otimes (\chi|_{G_F})$ is modular of weight $W(-n)$.*

Proof. This is proved just as [BDJ10, Lemma 2.3], using the reciprocity law for the Shimura datum $(G', h \times h_F)$. This reciprocity law is essentially described in [Car86a, 3.2.2] — however, it is in fact the inverse of the map described there (because of the sign error in [Del79]). Note that, similarly, the reciprocity map for the Shimura datum (G, h) is the identity, rather than the inverse of the identity as stated in [Car86a, 1.2]. We give some details of the proof. Fix a sufficiently small compact open subgroup K^p of $G'(\mathbb{A}_f^p)$ and set $U = K^p K_0'$, $U' = K^p K_{1,1}'$. We denote the curve $S_U(G', h \times h_F)$ by S and denote the curve $S_{U'}(G', h \times h_F)$ by S'.

Class field theory gives an isomorphism

$$\mathrm{Gal}(F^{ab}/F) \cong \overline{T_F(\mathbb{Q})^+} \backslash T_F(\mathbb{A}_f).$$

We obtain a map

$$r : \mathrm{Gal}(F^{ab}/F) \to T'(\mathbb{Q})^+ \backslash T'(\mathbb{A}_f)$$

by composing the isomorphism of class field theory with the map induced by a certain morphism of algebraic groups $\mathscr{R} : T_F \to T'$. Denote by ϕ the map of algebraic groups $\phi : T_F \to T_F$ given by $z \mapsto N_{F/E}(z)/z$. Then we define $\mathscr{R}$ to be the map

$$z \mapsto (N_{F/\mathbb{Q}}(z), \phi(z)/\phi(z)^c).$$

The map r is the reciprocity map for the Shimura datum $(G', h \times h_F)$. In particular, if we denote by π_U the finite set $T'(\mathbb{Q})^+ \backslash T'(\mathbb{A}_f)/\nu'(U)$ and denote by $\pi_{U'}$ the finite set $T'(\mathbb{Q})^+ \backslash T'(\mathbb{A}_f)/\nu'(U')$, then the map r induces a right action of $\mathrm{Gal}(F^{ab}/F)$ on these sets (by right multiplication) and hence we canonically obtain associated 0-dimensional schemes over F, which we again denote by π_U, $\pi_{U'}$. Then the map ν' induces a diagram of schemes over F

$$\begin{tikzcd} S' \arrow[r, "\nu'"] \arrow[d] & \pi_{U'} \arrow[d] \\ S \arrow[r, "\nu'"] & \pi_U. \end{tikzcd}$$

We denote by $\mathscr{F}_n$ the sheaf on π_U induced by the $\overline{\mathbb{F}}_p^\times$-valued character $(a, b) \mapsto a^{-n}$ mod p of $\nu'(U)$. The sheaf $\mathscr{F}_{\mathbf{1}}(-n)$ on S is isomorphic to the pullback of $\mathscr{F}_n$ via the map $S \to \pi_U$. This map induces a bijection of geometrically connected components, so it induces a G_F-equivariant isomorphism

$$H^0(S_{\overline{\mathbb{Q}}}, \mathscr{F}_{\mathbf{1}}(-n)) \cong H^0(\pi_{U,\overline{\mathbb{Q}}}, \mathscr{F}_n).$$

Let ψ be the $\overline{\mathbb{F}}_p^\times$-valued character of $T'(\mathbb{Q})^+ \backslash T'(\mathbb{A}_f)$ given by mapping to the first component of the torus $T' = \mathbb{G}_m \times U_F$ and then applying the character associated to χ by class field theory. We have $\psi \circ r = \chi|_{G_F}$. Shrinking U^p, we may suppose that ψ is trivial on $\nu'(U')$. Since $\chi|_{I_p} = \omega^n$, we have $\psi|_{\nu'(U)}(a, b) = a^{-n}$. Hence, the character ψ gives rise to an element of $H^0(\pi_{U,\overline{\mathbb{Q}}}, \mathscr{F}_n)$, non-trivial on each geometrically connected component of $\pi_{U,\overline{\mathbb{Q}}}$, and on which G_F acts via the restriction of the character χ. If we denote by α the image of this element in $H^0(S_{\overline{\mathbb{Q}}}, \mathscr{F}_{\mathbf{1}}(-n))$ then cupping with α gives a G_F-equivariant isomorphism

$$H^1(S_{\overline{\mathbb{Q}}}, \mathscr{F}_W) \otimes \overline{\mathbb{F}}_p^\times \cdot \alpha \cong H^1(S_{\overline{\mathbb{Q}}}, \mathscr{F}_W(-n)).$$

The result now follows from unwinding definitions. □

The following is the main result of this section

Proposition 6.3.7. *Suppose $p > 2$, $\zeta_p \notin F$ and that $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is $(G', h \times h_F)$-modular. We moreover suppose that*

- *ρ has split ramification,*

- *$\rho(G_{F(\zeta_p)})$ is adequate, in the sense of* [BLGG, Definition A.1.1].

Then for each $W \in W_{pd}(\rho)$, ρ is $(G', h \times h_F)$-modular of weight W.

Moreover, if ρ is $(G', h \times h_F)$-modular of weight W_0 and central character ψ, for some Serre weight W_0, then for each $W \in W_{pd}(\rho)$, ρ is $(G', h \times h_F)$-modular of weight W and central character ψ_W satisfying $\psi_W|_{U_F(\mathbb{A}_f)} = \psi|_{U_F(\mathbb{A}_f)}$.

Proof. It suffices to prove the stronger statement giving information about central characters. Starting from ρ, W_0 and ψ as in the statement of the theorem, we apply Propositions 6.1.1 and 6.3.2 to obtain an RACSDC automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_F)$, unramified at all places over p, with central character ψ_Π, and an algebraic Hecke character $\tilde{\psi}_E$ of $\mathbb{A}_E^\times$ such that $\tilde{\psi}_E|_{\mathbb{A}^p_{E,f}} = \tilde{\psi}^p|_{\mathbb{A}^p_{E,f}}$ and

$$r(\Pi) \cong \tilde{\rho} \otimes \mathrm{rec}(\tilde{\psi}_E)|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}.$$

Here, $\tilde{\psi}^p$ is as in the statement of Proposition 6.1.1. We have

$$\tilde{\psi}^p = (\tilde{\psi}_E \times \psi_\Pi)|_{Z'(\mathbb{A}^p_f)}.$$

Now we suppose $W \in W_{pd}(\rho)$. By the definition of $W_{pd}(\rho)$ we have $W = W_{\mathbf{a}}$, λ a lift of $\mathbf{a}$ and for each $v \in \Sigma_p$ we have $\tilde{\rho}_{\tilde{v}}$ a potentially diagonalisable lift of $\rho|_{G_{F_{\tilde{v}}}}$ with weight λ^v.

We now apply [BLGG, Theorem 3.1.2]. This tells us that we can find a RACSDC automorphic representation Π_W of $\mathrm{GL}_2(\mathbb{A}_F)$, unramified at all places over p, such that

- $\rho \otimes \overline{\mathrm{rec}}(\tilde{\psi}_E)|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)} \cong \overline{r}(\Pi_W)$
- Π_W is square integrable at every finite place of F where B is ramified
- for $v \in \Sigma_p$ and $\sigma : F_v^+ \hookrightarrow \overline{\mathbb{Q}}_p$ we have

$$\mathrm{HT}_\sigma(r(\Pi_W)|_{G_{F_{\tilde{v}}}}) = \mathrm{HT}_\sigma(\tilde{\rho}_{\tilde{v}} \otimes \mathrm{rec}(\tilde{\psi}_E)|_{G_{F_{\tilde{v}}}}).$$

Now we must return to the group G'. For this, we combine [HT01, Lemma VI.2.10] and Proposition 6.3.3. Lemma VI.2.10 gives us a character $\tilde{\psi}'_E$ of $\mathbb{A}_E^\times/E^\times$ satisfying

$$\psi_{\Pi_W}|_{\mathbb{A}_E^\times} = \tilde{\psi}'^c_E/\tilde{\psi}'_E.$$

Comparing this equation with the equation satisfied by $\tilde{\psi}_E$, and the fact that $\overline{r}(\Pi_W) \cong \overline{r}(\Pi)$, we see that $\chi := \overline{\mathrm{rec}}(\tilde{\psi}_E/\tilde{\psi}'_E)$ extends to a character of $G_{\mathbb{Q}}$. We also obtain an algebraic representation ξ' of G' over $\mathbb{C}$, which will have the form $(\xi_\lambda(n))'$ for some integer n, such that $\Pi_{W,\infty}$ is cohomological for ξ'_E and $\xi'|^{-1}_{E_\infty^\times} = \tilde{\psi}_E|^c_{E_\infty^\times}$. We apply Proposition 6.3.3 to Π_W, the character $\tilde{\psi}'_E$ and the algebraic representation $\xi_\lambda(n)$. We conclude that $\rho \otimes \chi|_{G_F} = \overline{r}(\Pi_W) \otimes \overline{\mathrm{rec}}(\tilde{\psi}'_E)|^{-1}_{G_F}$ is $(G', h \times h_F)$-modular of weight $W(n)$ and some central character, ψ'_W, where ψ'^p_W is equal to the reduction mod p of $\iota_p^{-1} \circ (\tilde{\psi}'_E \times \psi_{\Pi_W})|_{Z'(\mathbb{A}^p_f)}$.

Now it follows from Lemma 6.3.6 that ρ is $(G', h \times h_F)$-modular of weight W and some central character ψ_W. It is easy to see that the twist given by Lemma 6.3.6 does not change the restriction of the central character to $U_F(\mathbb{A}^p_f)$, so $\psi_W|_{U_F(\mathbb{A}^p_f)}$ coincides with ψ'_W. Recall that ψ^p is equal to the reduction mod p of $(\tilde{\psi}_E \times \psi_\Pi)|_{Z'(\mathbb{A}^p_f)}$. Now we are done, since χ extending to $G_{\mathbb{Q}}$ implies that the contribution of $\tilde{\psi}_E$ and $\tilde{\psi}'_E$ to the central character (mod p and restricted to U_F) are the same, and ψ_{Π_W}, ψ_Π are the central characters of two RACSDC automorphic representations whose attached Galois representations have isomorphic reduction mod p, so these again have the same contribution. $\square$

7. Serre weights for (G, h), the Buzzard-Diamond-Jarvis conjectures

We now use the results of the the previous two sections to deduce some results about Serre weights for the group G.

7.1. **Serre weight conjectures for totally real fields.**

As in the previous section, we are going to relate questions about being modular of some weight to the existence of crystalline or potentially diagonalisable lifts of certain Galois representations. We make the following definition, parallel to Definition 6.2.3:

Definition 7.1.1. *Let $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation. We denote by $W_{cr}(\rho)$ the set of Serre weights comprising those $W_{\mathbf{a}}$ such that there is a weight λ lifting $\mathbf{a}$ and for each $v \in \Sigma_p$ there is a crystalline lift of $\rho|_{G_{F_v^+}}$ with weight λ^v.*

Similarly, we denote by $W_{pd}(\rho)$ the set of Serre weights comprising those $W_{\mathbf{a}}$ such that there is a weight λ lifting $\mathbf{a}$ and for each $v \in \Sigma_p$ there is a crystalline and potentially diagonalisable (in the sense of [BLGGT10, 1.4]*) lift of $\rho|_{G_{F_v^+}}$ with weight λ^v.*

Theorem 7.1.2. *Let $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation. Suppose ρ is (G, h)-modular of weight W. Then $W \in W_{cr}(\rho)$.*

Proof. We begin by choosing an imaginary quadratic field E as in section 2.1, and setting $F = EF^+$. We choose E such that $\rho|_{G_F}$ remains irreducible. The idea of the proof is to show that $\rho|_{G_F}$ is (up to a twist by a character with controlled ramification) $(G', h \times h_F)$-modular of weight W (i.e. to transfer to the PEL unitary Shimura curve) and then apply Corollary 6.2.4.

Lemma 5.1.9 shows that ρ is (G'', h) modular of weight W and some central character ψ. Corollary 5.4.3 implies that $\rho|_{G_F} \otimes \varphi_\psi$ is $(G'', h \times h_F)$-modular of weight W. Applying Lemma 5.1.4, we see that $\rho|_{G_F} \otimes \varphi_\psi$ is $(G', h \times h_F)$-modular of weight W. Finally, Corollary 6.2.4 tells us that $W \in W_{cr}(\rho|_{G_F} \otimes \varphi_\psi)$, but since φ_ψ is unramified at the places $\tilde{v}$ (see Remark 5.4.4) we have $W_{cr}(\rho|_{G_F} \otimes \varphi_\psi) = W_{cr}(\rho)$. □

Theorem 7.1.3. *Suppose $p > 2$, that the continuous irreducible representation $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is (G, h)-modular, and that $\rho(G_{F^+(\zeta_p)})$ is adequate. Then for any $W \in W_{pd}(\rho)$, ρ is (G, h)-modular of weight W.*

Proof. Again we begin by choosing an imaginary quadratic field E and setting $F = EF^+$. We can and do choose E so that

- $\zeta_p \notin F$
- $\rho|_{G_F}$ is irreducible and has split ramification,
- $\rho(G_{F(\zeta_p)})$ is adequate.

The idea of the proof is that, up to a twist by a character, $\rho|_{G_F}$ is $(G', h \times h_F)$-modular, and we can apply Proposition 6.3.7 to show modularity with the appropriate weights in the PEL unitary Shimura curve setting. Then we can transfer back to the quaternionic Shimura curve, up to some ambiguity about a character twist, which we resolve by ensuring that the character (a priori of G_F) extends to G_{F^+}, and is unramified at places over p, hence does not affect the weights for which ρ is modular.

First we transfer ρ to the unitary setting. Lemma 5.1.9 shows that ρ is (G'', h)-modular of some weight W_0 and some central character ψ. Therefore, we deduce from Corollary 5.4.3 that $\rho|_{G_F} \otimes \varphi_\psi$ is $(G'', h \times h_F)$-modular of weight W_0 and some central character ψ. Then Lemma 5.1.4 tells us that $\rho|_{G_F} \otimes \varphi_\psi$ is $(G', h \times h_F)$-modular of weight W_0 and some central character ψ'. We will need some information on the relationship between ψ and ψ'. Lemma 5.1.8 tells us that $\rho|_{G_F} \otimes \varphi_\psi$ is

$(G'', h \times h_F)$-modular of weight W_0 and central character ψ'' with $\psi''|_{Z'(\mathbb{A}_f)} = \psi'$. Another application of Corollary 5.4.3 tells us that $\rho|_{G_F} \otimes \varphi_\psi \varphi_{\psi''}^{-1}$ is $(G'', h \times h_F)$-modular of weight W_0 and central character ψ''. We conclude that the character $\varphi_\psi \varphi_{\psi''}^{-1}$ extends to G_{F^+}.

Let $W \in W_{pd}(\rho) = W_{pd}(\rho|_{G_F} \otimes \varphi_\psi)$. Then Proposition 6.3.7 and Lemma 5.1.8 tell us that $\rho|_{G_F} \otimes \varphi_\psi$ is $(G'', h \times h_F)$-modular of weight W and some central character ψ_W with $\psi_W|_{U_F(\mathbb{A}_f)} = \psi''|_{U_F(\mathbb{A}_f)}$. Corollary 5.4.3 tells us that $\rho|_{G_F} \otimes \varphi_\psi \varphi_{\psi_W}^{-1}$ is (G, h)-modular of weight W. Since $\psi_W|_{U_F(\mathbb{A}_f)} = \psi''|_{U_F(\mathbb{A}_f)}$ and $\varphi_\psi \varphi_{\psi''}^{-1}$ extends to G_{F^+}, we see that $\varphi_\psi \varphi_{\psi_W}^{-1}$ extends to G_{F^+}. By Remark 5.4.4, the character $\varphi_\psi \varphi_{\psi_W}^{-1}$ is unramified at all places $\tilde{v}$, so the extension to G_{F^+} is unramified at all places $v \in \Sigma_p$. It follows from [BDJ10, Corollary 2.11 (2)] that $\rho|_{G_F}$ is modular of weight W, so ρ or $\rho \otimes \chi(F/F^+)$ is (G, h) modular of weight W, where $\chi(F/F^+)$ is the quadratic character corresponding to F/F^+. Since the places over p split in F/F^+, this character is also unramified at the places $v \in \Sigma_p$ and another application of [BDJ10, Corollary 2.11 (2)] tells us that ρ is (G, h)-modular of weight ρ. □

7.2. Buzzard-Diamond-Jarvis conjectures, p unramified in F^+.

Corollary 7.2.1. *Suppose $p > 2$ and that p is unramified in F^+. Let $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous, irreducible, (G, h)-modular representation, and suppose that $\rho(G_{F^+(\zeta_p)})$ is adequate. Then ρ is (G, h)-modular of weight W if and only if $W \in W_{cr}(\rho)$.*

Proof. This follows from combining Theorems 7.1.2, 7.1.3 and [GLSa, Theorem 2.12]. This last reference shows that, if p is unramified in F^+, $W_{cr}(\rho) = W_{pd}(\rho)$ (and also provides a more explicit description of these sets, see the following remark). □

Remark 7.2.2. See Remark 5.1.3 for a comparison of the notion of (G, h)-modularity with the definition of modularity used in [BDJ10]. We refer to section 4 of [BLGG] for a discussion of the various forms of the 'weight part of Serre's conjecture' in the literature for GL_2. When p is unramified in F^+ (and $p > 2$), the paper [GLSa] shows that the set $W_{cr}(\rho)$ is the same as the more explicit set denoted $W^{BDJ}(\rho)$ in [BLGG, GLSa]. The set $W^{BDJ}(\rho)$ is in most cases the same as the set of conjectured weights in [BDJ10], and is conjecturally always the same - we again refer to section 4 of [BLGG] for a discussion of the situation.

Remark 7.2.3. As mentioned in the introduction, a very recent preprint of Gee, Liu and Savitt [GLSb] proves a generalisation of [GLSa, Theorem 2.12], removing the assumption that p is unramified in F^+. This enables us to likewise remove the 'p is unramified' hypothesis from Corollary 7.2.1.

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