1. Unramified representations and the Satake isomorphism

1.1. Unramified representations.

Definition 1.1. A smooth representations (π, V) of $G = \mathrm{GL}_n(F)$ is said to be unramified (or spherical) if the space V^{K_0} of invariants under the maximal compact subgroup $K_0 = \mathrm{GL}_n(\mathcal{O}_F)$ is non-zero.

It follows from something we proved in the section on Hecke algebras that irreducible unramified representations of G correspond to simple $\mathcal{H}(G, K_0)$-modules.

The 'unramified' terminology is natural from the point of view of Galois (or Weil group) representations. Recall that local class field theory gives an isomorphism $F^\times \cong W_F^{ab}$ between the multiplicative group of F and the abelianisation Weil group W_F (the pre-image of $\mathrm{Frob}^{\mathbb{Z}}$ in $\mathrm{Gal}(\overline{F}/F)$), so unramified chacters of W_F (characters with trivial restriction to the inertia subgroup) correspond to unramified (one-dimensional) representations of $F^\times$.

Unramified representations are very important in global contexts, since all but finitely local factors of an automorphic reprensentation are unramified.

Example. We can similarly think about unramified representations of the diagonal torus $T \subset G$, i.e. those with non-zero fixed vectors under $T_0 = T \cap K_0$. An irreducible such representation is necessarily one-dimensional, and corresponds to a homomorphism

$$x : \mathcal{H}(T, T_0) \to \mathbb{C}.$$

Since T is commutative, we can identify $\mathcal{H}(T, T_0)$ with the (commutative!) group algebra $\mathbb{C}[T/T_0]$ of the quotient group. We have $T/T_0 \cong \mathbb{Z}^n$, and we can identify it with the group of cocharacters $X_\bullet(T)$ by sending $\lambda \in X_\bullet(T)$ to $\lambda(\varpi) \in T$.

So unramified representations of T correspond to homomorphisms $x \in \mathrm{Hom}_{\mathbb{Z}}(X_\bullet(T), \mathbb{C}^\times) =: \widehat{T}(\mathbb{C}) \cong \mathbb{C}^n$, which are point of the dual torus $\widehat{T} = \mathrm{Spec}(\mathbb{C}[X_\bullet(T)])$.

Now we go back to representations of $G = \mathrm{GL}_n(F)$. We can write down many example of unramified representations, as follows: let $\chi : T \to \mathbb{C}^\times$ be an unramified character of T, and consider the induced representation $I_\chi = n\text{-}\mathrm{Ind}_B^G \chi$. It follows from the Iwasawa decomposition ($G = BK_0$) and the fact that χ is trivial on $B \cap K_0$, that

$$(I_\chi)^{K_0} = \mathbb{C} \cdot v_\chi$$

is one-dimensional, with basic vector v_χ the *spherical vector* (i.e. fixed under K_0) given by $f(tnk_0) = \delta_B^{1/2}\chi(t)$.

For most choices of χ, I_χ is irreducible (we have discussed the case $n = 2$ a few times). In general, since taking K_0-invariants is an exact functor, we can say that there is a unique irreducible sub-quotient of I_χ with a non-zero space of K_0-invariants. We denote this irreducible unramified representation of G by π_χ.

Example. When $n = 2$ and $\chi = \chi_1 \otimes \chi_2$, with $\chi_i : F^\times \to \mathbb{C}^\times$ unramified characters with $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$, $\pi_\chi = I_\chi$ is an irreducible unramified representation. When $\chi_1/\chi_2 = |\cdot|$, π_χ is one-dimensional with character $\delta_B^{-1/2}\chi$.

Since I_χ and I_{χ^w} have isomorphic irreducible constituents for $w \in W$ (this was stated last week), we know that $\pi_\chi \cong \pi_{\chi^w}$ for all $w \in W$.

1.2. **Satake transform.**

Definition 1.2. The Satake transform $S : \mathcal{H}(G, K_0) \to \mathcal{H}(T, T_0)$ is given by

$$(Sf)(t) = \delta_B(t)^{1/2} \int_N f(tn) d\mu_N(n),$$

where μ_N is the left Haar measure on N with $\mu_N(N \cap K_0) = 1$.

(It's not hard to check that this does indeed take values in the Hecke algebra $\mathcal{H}(T, T_0)$.)

To motivate this definition, let's think about how the spherical Hecke algebra acts on the spherical vector $v_\chi \in (I_\chi)^{K_0}$.

Since $(I_\chi)^{K_0}$ is one-dimensional, $f \in \mathcal{H}(G, K_0)$ acts on v_χ as multiplication by a scalar, which we denote by $\pi_\chi(f)$. To work out this scalar, we use the fact that we have $\mu_G = \mu_T \times \mu_N \times \mu_{K_0}$, where the respective (left) Haar measures are all normalized to give the intersection of K_0 with the relevant group volume 1. In other words,

$$\int_G f(g) d\mu_G = \int_T \int_N \int_{K_0} f(tnk) d\mu_{K_0}(k) d\mu_N(n) d\mu_T(t).$$

Now we can compute our scalar as

$$\begin{aligned}\int_G f(tnk) v_\chi(tnk) dkdndt = \int_T \int_N \int_{K_0} f(tn) \delta_B^{1/2} \chi(t) dkdndt &= \int_T \int_N f(tn) \delta_B^{1/2} \chi(t) dndt \\ &= \int_T (Sf)(t) \chi(t) dt.\end{aligned}$$

We can write this as $\pi_\chi(f) = Sf(\chi)$ where on the right hand side evaluating a function in $\mathcal{H}(T, T_0) = \mathbb{C}[T/T_0]$ at the point $\chi \in \widehat{T}(\mathbb{C})$.

So the Satake transform captures the action of $\mathcal{H}(G, K_0)$ on unramified principal series representations.

Theorem 1.3. *The Satake theorem is a $\mathbb{C}$-algebra homomorphism which maps $\mathcal{H}(G, K_0)$ isomorphically to the subalgebra $\mathcal{H}(T, T_0)^W \subset \mathcal{H}(T, T_0)$ of Weyl group invariants.*

Remark 1.4. Note that $\mathcal{H}(T, T_0)^W = \mathbb{C}[X_\bullet(T)]^W = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^W$, so the invariants are given by the algebra $\mathbb{C}[e_1, e_2, \ldots, e_n, e_n^{-1}]$, where the e_i are the standard symmetric polynomials in the t_i.

Proof. First we check that the map is a homomorphism and takes values in the W-invariants. This can be done using the formula $\pi_\chi(f) = Sf(\chi)$. Note that if we know $(w \cdot Sf)(\chi) = Sf(\chi)$ for all w and all χ (or even a Zariski dense set of χ), then we have $Sf \in \mathcal{H}(T, T_0)^W$. Since $\pi_\chi \cong \pi_{\chi^w}$ we do indeed have this equality.

For the homomorphism property, it is clear from the definition that S is additive. It follows from the definition that $S(e_{K_0})$ is supported on T_0, and for $t \in T_0$ we have

$$S(e_{K_0})(t) = \int_N e_{K_0}(tn) d\mu_N(n) = \int_N e_{K_0 \cap N}(n) d\mu_N(n) = 1,$$

so $S(e_{K_0}) = e_{T_0}$. If we know $S(f_1 * f_2)(\chi) = ((Sf_1) * (Sf_2))(\chi)$ for all χ then we get $S(f_1 * f_2) = S(f_1) * S(f_2)$. This follows from $\pi_\chi(f_1 * f_2) = \pi_\chi(f_1)\pi_\chi(f_2)$ (the eigenvalue of $f_1 * f_2$ is the product of the eigenvalues of the f_i).

This leaves showing that the map is an isomorphism. This is the hard part! We now have a map

$$\mathcal{H}(G, K_0) \to \mathbb{C}[X_\bullet(T)]^W$$

and we can identify bases on both sides indexed by $\lambda \in X_\bullet(T)^W = X_\bullet(T)^+$, where a cocharacter is in $X_\bullet(T)^+$ if it is given by a map $z \mapsto \mathrm{diag}(z^{a_1}, z^{a_2}, \ldots, z^{a_n})$ with $a_1 \geq a_2 \geq \cdots \geq a_n$.

For $\mathcal{H}(G, K_0)$ we use the Cartan decomposition, so $e_\lambda := \mathbf{1}_{K_0\lambda(\varpi)K_0}, \lambda \in X_\bullet(T)^+$ give a basis. For $\mathbb{C}[X_\bullet(T)]^W$, basis vectors are given by $c_\lambda = \sum_{w\in W}[w\lambda]$.

We write $S(e_\lambda) = \sum_\mu a_{\lambda,\mu}c_\mu$, for $a_{\lambda,\mu} \in \mathbb{C}$. To prove the theorem, it suffices to show that the matrix $a_{\lambda,\mu}$ is upper triangular, with respect to some ordering on $X_\bullet(T)^+$, with non-zero diagonal entries. To compute the matrix coefficients, we have

$$a_{\lambda,\mu} = S(e_\lambda)(\mu(\varpi)) = \delta(\mu(\varpi))^{1/2}\int_N e_\lambda(\mu(\varpi)n)dn = \delta(\mu(\varpi))^{1/2}\mu_N(N\cap\mu(\varpi)^{-1}K_0\lambda(\varpi)K_0).$$

For the diagonal entries, we easily see that $a_{\lambda,\lambda} \neq 0$. A fact (which has a beautiful geometric interpretation in terms of the affine Grassmannian) is that $a_{\lambda,\mu}$ is zero *unless* $\lambda \geq \mu$, with respect to the partial ordering given by asking that $\lambda - \mu$ is a non-negative linear combination of positive coroots. More explicity, if λ, μ correspond to decreasing n-tuples of integers $(a_i), (b_i)$, this condition says that $a_1 + a_2 + \cdots a_r \geq b_1 + b_2 + \cdots b_r$ for all $1 \leq r \leq n-1$ and $a_1 + a_2 + \cdots a_n = b_1 + b_2 + \cdots b_n$. This partial ordering can be extended into a total ordering (for example, the lexicographic ordering), and so we win. Satake originally just proves that $a_{\lambda,\mu}$ is zero unless λ is above μ in the lexicographic ordering (this is a weaker statement than the fact mentioned above), this can be proved using elementary divisors in a relatively elementary way. As an example, let's do $n = 2$. We have a matrix $\mu(\varpi)\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\varpi^{b_1} & \varpi^{b_1}n\\ 0 & \varpi^{b_2}\end{pmatrix}$ which is in the Cartan cell indexed by λ. Comparing determinants, we see that $b_1 + b_2 = a_1 + a_2$. Looking at 1-minors, we see that $\inf(b_1, b_2, b_1 + \mathrm{ord}_\varpi(n)) = a_2$. We deduce that $a_2 \leq b_2$, and hence (from the preceding equality) $a_1 \geq b_1$. □

Corollary 1.5. *The irreducible principal series representations π_χ give all the isomorphism classes of irreducible unramified representations of G.*

Proof. We have compute the $\mathcal{H}(G, K_0) = \mathcal{H}(T, T_0)^W$-modules $(\pi_\chi)^{K_0}$ and we can now see that they cover all isomorphism classes of simple $\mathcal{H}(G, K_0)$-modules. □

Corollary 1.6 (Unramified local Langlands)**.** *Fix an embedding $\widehat{T}(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$. There is a bijection between isomorphism classes irreducible unramified representations of G and semsimple unramified representations $\phi : W_F \to \mathrm{GL}_n(\mathbb{C})$.*

Proof. The point is that an isomorphism class of semisimple unramified representations is determined by the conjugacy class of the semisimple element $\phi(\mathrm{Frob}) \in \mathrm{GL}_n(\mathbb{C})$. Standard theory of maximal tori implies that the conjugacy class intersected with $\widehat{T}(\mathbb{C})$ is a W-conjugacy class of elements in $\widehat{T}(\mathbb{C})$. These biject with homomorphisms $\mathbb{C}[X_\bullet(T)]^W \to \mathbb{C}$, which are identified with isomorphism classes of simple $\mathcal{H}(G, K_0)$-modules by the Satake transform. □

2. Local Langlands for $\mathrm{GL}_n(F)$

To finish the course let's finish with a quick discussion of the local Langlands correspondence for $G = \mathrm{GL}_n(F)$. We have discussed the objects on one side of the correspondence: irreducible smooth representations of G.

On the other side are Weil–Deligne representations: these are pairs (r, N), where $r : W_F \to \mathrm{GL}(V)$ is a representation of the Weil group on a $\mathbb{C}$-vector space V, which is trivial on a finite index subgroup of intertia I_F, and $N : V \to V$ is a (nilpotent) linear map which satisfies $r(\sigma)Nr(\sigma)^{-1} = q^{n(\sigma)}N$, where σ maps to $\mathrm{Frob}^{-n(\sigma)}$ in $\mathrm{Gal}(\overline{k}/k)$ (Frob is geometric Frobenius).

A Weil–Deligne representation (r, N) is called F-semisimple if r is semisimple. There is a natural duality on Weil–Deligne representations (take the transpose of N to get an endomorphism of the dual space).

Theorem 2.1 (Harris–Taylor,Henniart)**.** *For each $n \geq 1$ there is a bijection rec_F between isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(F)$ and n-dimensional F-semisimple Weil–Deligne representations, uniquely determined by a number of conditions:*

- *When $n = 1$ the bijection is given by local class field theory.*
- *For a locally constant character $\chi : F^\times \to \mathbb{C}^\times$,*

$$\mathrm{rec}_K(\pi \otimes (\chi \circ \det)) = \mathrm{rec}_K(\pi) \otimes \mathrm{rec}_K(\chi)$$

- $\det \mathrm{rec}_K(\pi) = \mathrm{rec}_K(\omega_\pi)$
- $\mathrm{rec}_K(\pi^\vee) = \mathrm{rec}_K(\pi)^\vee$
- *Two more conditions involving local L-factors and ϵ-factors for pairs of representations.*

Classification results of Bernstein and Zelevinsky which we have touched on (describing how all irreducibles contribute to parabolic inductions of supercuspidals) show that it suffices to construct bijections (satisfying the above properties) between isomorphism classes of cuspidal representations of $\mathrm{GL}_n(F)$ and irreducible n-dimensional representations of W_F.

If $\gcd(n, q) = 1$, there is a simple description of these irreducible Weil group representations (see Proposition 10.1 in Prasad–Raghuram). When $n = 2$ (and q is odd) there is an corresponding explicit description of the irreducible cuspidal representations of $\mathrm{GL}_2(F)$ using a version of the Weil oscillator representation.

We already described the case of unramified principal series representations in the previous secton. More generally if $I_\chi = n\text{-}\mathrm{Ind}_B^G \chi$ is an irreducible principal series, $\mathrm{rec}_K(I_\chi)$ is the Weil–Deligne representation $(\chi_1 \oplus \chi_2 \oplus \cdots \chi_n, 0)$ where we interpret $\chi_i : F^\times \to \mathbb{C}^\times$ as a character of W_F using the Artin reciprocity map.

For the Steinberg representation, $\mathrm{rec}_K(St) = (|\cdot|^{1/2} \oplus |\cdot|^{-1/2}, N)$. We define N: if e_1 is the basis vector on which W_F acts by $|\cdot|^{1/2}$ and e_2 is the basis vector with action by $|\cdot|^{-1/2}$, then $Ne_1 = 0$ and $Ne_2 = e_1$.