Martingale optimal transport

At the crossroads of mathematical finance, optimal transport and probability

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[1]With thanks to Nizar Touzi for sharing many slides!

On Some Transport problems

For some space E, consider $\Omega := E \times E$ with the canonical process

$$X(\omega) = x, \quad Y(\omega) = y \quad \text{for all } \omega = (x, y) \in \Omega.$$

Transport plans:

$$\Pi(\mu, \nu) \quad := \quad \{\mathbb{P} \in \mathrm{Prob}(\Omega) : \ \mathbb{P} \circ X^{-1} = \mu, \ \mathbb{P} \circ Y^{-1} = \nu\}$$

In our applications additional restrictions are natural:

- further measurability, e.g. Y adapted to a given filtration
- dynamics of (X, Y), e.g. is a $\mathbb{P}$–martingale, or nearly so
- more marginals: $\Omega = E^N$ or $\Omega = E^{[0,T]}$
- but maybe with less information: $\mathbb{P} \circ Y^{-1} \in \Lambda \subseteq \mathrm{Prob}(E)$
- pathspace restrictions: $(X, Y) \in \mathfrak{P} \subseteq \Omega$ $\mathbb{P}$–a.s.

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Outline

MOT and its duality

Two applications

Skorokhod Embedding Problem

Robust Hedging of Financial Derivatives

On some novel features in the MOT

Martingale Optimal Transport on the line

Let $\Omega := \mathbb{R} \times \mathbb{R}$ and introduce the canonical process

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Martingale Transport plans: μ, ν have finite first moment,

$$\mathcal{M}(\mu, \nu) \quad := \quad \left\{ \mathbb{P} \in \Pi(\mu, \nu) : \; \mathbb{E}^{\mathbb{P}}[Y|X] = X \right\}$$

i.e. $\mathbb{P}(d\omega) = \mu(dx)\mathbb{P}_x(dy)$, whose desintegration $\mathbb{P}_x$ has barycentre x

Martingale Optimal Transport problem

$$\inf_{\mathbb{P} \in \mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\left[c(X, Y)\right]$$

Martingale restriction

- $\mathbb{E}^{\mathbb{P}}[Y|X] = X$ iff $\mathbb{E}^{\mathbb{P}}[h(X)(Y - X)] = 0$ for all $h \in C_b^0$

$\Longrightarrow$ h will act as Lagrange multipliers... Denote

$$h^{\otimes}(x, y) := h(x)(y - x), \quad x, y \in \mathbb{R}$$

[complementing the standard notations $\varphi \oplus \psi$]

- Strassen '65: $\mathcal{M}(\mu, \nu) \neq \emptyset$ iff $\mu \preceq \nu$ in convex order:

$$\mu[f] \leq \nu[f] \quad \text{for all} \quad f : \mathbb{R} \longrightarrow \mathbb{R} \text{ convex}$$

- $\mathcal{M}(\mu, \nu)$ closed convex subset of $\Pi(\mu, \nu)$...

Kantorovitch dual formulation

Martingale Optimal Transport: $c : \Omega \longrightarrow \mathbb{R}$ measurable

$$\mathbf{P}(\mu,\nu) := \inf_{\mathbb{P}\in\mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c], \quad \mathcal{M}(\mu,\nu) := \left\{\mathbb{P}\in\Pi(\mu,\nu) : \ \mathbb{E}^{\mathbb{P}}[Y|X] = X\right\}$$

Pointwise Dual Problem:

$$\mathbf{D}(\mu,\nu) := \sup_{(\varphi,\psi,h)\in\mathcal{D}(c)} \mu[\varphi] + \nu[\psi]$$

where

$$\mathcal{D}(c) \ := \ \left\{(\varphi,\psi,h) : \ \varphi\oplus\psi + h^{\otimes} \le c \text{ on } \Omega\right\}$$

Duality for LSC claim

Theorem (Beiglböck, Henry-Labordère, Penkner '13)

Assume $c \in$ LSC and bounded from below. Then $\mathbf{P} = \mathbf{D}$, and existence holds for $\mathbf{P}(\mu,\nu)$ for all $\mu \preceq \nu$

Theorem (Beiglböck, Lim, O. '17)

Assume further that there exists u such that $\mu(dx)$–a.e.

$$y \to c(x,y) + u(y) \quad \textit{is convex, of linear growth.}$$

Then existence holds for extended $\mathbf{D}(\mu,\nu)$. Existence for $\mathbf{D}(\mu,\nu)$ holds when c is Lipschitz and ν has compact support.

- There are easy examples where existence for the dual fails, even for bounded c, bounded support... (Beiglböck, Henry-Labordère & Penkner, Beiglböck, Nutz & Touzi)

- The condition $c \in$ LSC is not innocent, e.g. duality may fail for the USC function $c(x,y) := -\mathbb{1}_{\{x \neq y\}}$ on $[0,1] \times [0,1]$

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Continuous-time Transport Plans

Let $\Omega := C^0([0,T],\mathbb{R})$ or $\Omega := \mathrm{RCLL}([0,T],\mathbb{R})$, with canonical process and filtration

$$X_t(\omega) = \omega(t), \quad \mathcal{F}_t := \sigma(X_s, s \le t) \quad \text{for all} \;\; 0 \le t \le T$$

Transport plans:

$$\Pi(\mu,\nu) \;\; := \;\; \left\{\mathbb{P} \in \mathrm{Prob}(\Omega) : \; \mathbb{P}\circ X_0^{-1} = \mu, \; \mathbb{P}\circ X_T^{-1} = \nu\right\}$$

A first difficulty: $\Pi(\mu,\nu)$ is not weakly compact

Continuous-time Martingale Transport

Martingale Transport plans: μ, ν have finite first moment,

$$\mathcal{M}(\mu,\nu) \quad := \quad \{\mathbb{P} \in \Pi(\mu,\nu) : \ X \text{ is } \mathbb{P} - \text{martingale}\}$$

i.e. $\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_s] = X_s$ for all $0 \le s \le t \le T$, or "equivalently":

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T h_t dX_t\Big] = 0 \quad \text{for} \quad \mathbb{F} - \text{meas. bdd } h : [0,T] \times \Omega \longrightarrow \mathbb{R}$$

Martingale Optimal Transport: $c : (\Omega, \mathcal{F}_T) \longrightarrow \mathbb{R}$ measurable

$$\mathbf{P}(\mu,\nu) \quad := \quad \inf_{\mathbb{P} \in \mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X_t : t \le T)]$$

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Dual Problem:

$$\mathbf{D}(\mu,\nu) := \sup_{(\varphi,\psi,h)\in\mathcal{D}(c)} \mu[\varphi]+\nu[\psi]$$

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$$\mathcal{D}(c) := \Big\{(\varphi,\psi,h) : \ \varphi(X_0)+\psi(X_T)+\underbrace{\int_0^T h_t dX_t}_{h \text{ s.t. } \dots \ !!!} \le c \text{ on } \Omega\Big\}$$

Theorem (Dolinsky & Soner '14; Hou & O. '16)

Let $\mu \preceq \nu$. Then $\mathbf{P} = \mathbf{V}$ for a unif. continuous and bounded c.

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Extensions

- Martingale optimal transport in $\mathbb{R}^d$
- Multiple marginals (easy in DT, hard in CT)
- All marginals specified,

  e.g. fake Brownian motion: $\mu_t = \mathcal{N}(0, t)$ for all $t \geq 0$

- Partial specification of marginal distributions
- Pathspace restrictions

Some more references...

Pioneered by Pierre Henry-Labordère,

Discrete-time: Beiglböck, Burzoni, Campi, Davis, De March, Frittelli, Ghoussoub, Griessler, Henry-Labordère, Hobson, Hou, Kim, Klimmek, Lim, Martini, Maggis, Neuberger, Nutz, O., Penkner, Juillet, Schachermayer, Touzi

Continuous-time: Beiglböck, Bayraktar, Claisse, Cox, Davis, Dolinsky, Galichon, Guo, Hou, Henry-Labordère, Hobson, Huesmann, Perkowski, Proemel, Kallblad, Klimmek, O., Siorpaes, Soner, Spoida, Stebegg, Tan, Touzi, Zaev

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Two applications

- Skorokhod Embedding Problem
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On some novel features in the MOT

Formulation of the SEP

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ filtered probability space, B Brownian motion

SEP(μ, ν): Find a stopping time τ such that

$$\mathbb{P} \circ (B_0)^{-1} = \mu, \quad \mathbb{P} \circ (B_\tau)^{-1} = \nu \quad \text{and} \quad B_{\cdot \wedge \tau} \quad \text{UI}$$

- on $\mathbb{R}$, infinity of solutions for any $\mu \preceq \nu$
- on $\mathbb{R}^d$ a stronger relation is required (Rost)
- UI requirement needed for a meaningful solution
- originally, and in many applications, $\mu = \delta_{x_0}$.
- also considered in a weak formulation
- goes back to Skorokhod in 1961, see my (outdated!) survey paper

(Original) Motivation of the SEP

SEP originally used to show Invariance Principles, such as the Central Limit Theorem or the Law of Iterated Logarithm, etc.

E.g.: Weak law of large numbers $\Longrightarrow$ Central Limit Theorem

$X_i \sim \mu$ iid, where μ is centred and $\int x^2 \mu(dx) < \infty$.

$X_i = B^i_{\tau_i}$, with $\tau_i \sim$ iid, and $B^i_t := B_{\tau_{i-1}+t} - B_{\tau_{i-1}}$ iid BM. Then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i = \frac{1}{\sqrt{n}} B_{nT_n}, \quad \text{where} \quad T_n := \frac{1}{n} \sum_{i=1}^{n} \tau_i \xrightarrow{\mathbb{P}} \mathbb{E}[\tau] = \mathbb{E}[X_i^2]$$

and $B^n_t = \frac{1}{\sqrt{n}} B_{nt}$ converges in law to a BM independent of B.

Some solutions of the SEP

- Skorokhod, Doob, Hall, Chacon and Walsh,
- Root
- Azéma-Yor
- Vallois

Perkins, Jacka, Bertoin and Le Jan, and many many more

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- Azéma-Yor $\Longrightarrow \max_\tau \mathbb{E}[\phi(\sup_{t\le\tau} B_t)]$, $\phi' > 0$
- Vallois $\Longrightarrow \max_\tau \mathbb{E}[\phi(L_\tau)]$, $\phi' > 0$

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Connection with Martingale Transport

The process $\left(X_t = B_{\frac{t}{T-t}\wedge\tau} : t \in [0, T]\right)$ is a martingale transport: $X_0 = B_0 \sim \mu$ and $X_T = B_\tau \sim \nu$

Conversely, every martingale is a time-changed Brownian motion

Martingale Optimal Transport $\Longrightarrow$ find a solution τ of the SEP for a given optimality criterion...

Geometry of optimality $\Longrightarrow$ characterisation of support of $(B_{t\wedge\tau} : t \geq 0)$ analogous to c-cyclical monotonicity
Monotonicity Principle of Beiglböck, Cox & Huesmann (IM, 2016) recovers all known optimality properties!

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A simple financial setup with traded options

- Consider a risky asset $S = (S_0, \dots, S_N)$. Trading at no cost:

$$\sum_{t=0}^{N-1} h_t(S_0, \dots, S_t)(S_{t+1} - S_t)$$

- Suppose call options with maturity N are traded at prices $C(K)$.
- If $\mathbb{P}$ is a model and pricing via expectation then

$$\mathbb{E}^{\mathbb{P}}[(S_N - K)^+] = C(K), \quad \text{i.e.} \quad \int_K^\infty (s - K)\mathbb{P}(S_N \in ds) = C(K).$$

Differentiating twice: $S_N \sim \nu_N$ under $\mathbb{P}$, where $\nu_N = C''$.

- Arbitrage considerations $\Longrightarrow \nu_N$ a probability measure and if call options for maturities t_1, t_2 available then $\nu_{t_1} \preceq \nu_{t_2}$.

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Hedging (trading) instruments

Consider a two-time snapshot: $S = (S_1, S_2)$.

- Prices C_1, C_2 of calls with maturities $1, 2$ available for all strikes
- A generic Vanilla payoff $\varphi \in C^2$ may be synthesised:

$$\begin{aligned}\varphi(S_i) = {} & \varphi(x_0) + (S_i - x_0)\varphi'(x_0) \\ & + \int_{x_0}^{\infty}(S_i - K)^+\varphi''(K)dK + \int_{-\infty}^{x_0}(K - S_i)^+\varphi''(K)dK\end{aligned}$$

- By linearity of pricing rules, with $\nu_i = C_i''$,

$$\mathrm{Price}(\varphi(S_i)) = \mathbb{E}^{\mathbb{P}}[\varphi(S_i)] = \int \varphi(s)\mathbb{P}(S_i \in ds) = \nu_i[\varphi]$$

- In addition, dynamic trading for zero cost

$$h_1(S_1)(S_2 - S_1) = h_1^{\otimes}(S_1, S_2)$$

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Robust / Model-Free Subhedging Problem

Exotic option defined by the payoff $c(S_1, S_2)$ at time 2:

$$c : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

Robust sub-hedging problem naturally formulated as:

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i.e. as the MOT Kantorovitch dual, where

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The dual "pricing problem" is: $\mathbf{P}(\mu, \nu) = \inf_{\mathbb{P}\in\mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c]$

All the quantities of direct financial relevance: value of $\mathbf{P} = \mathbf{D}$, optimal hedging in $\mathbf{D}$, structure of optimal $\mathbb{P}$ for $\mathbf{P}$.

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One natural extension: American options

• Consider N times, $(S_0, S_1, \ldots, S_N)$, $\mu = \delta_{S_0}$ and $c = (c_t)$ the payoff of an American option $\rightsquigarrow$ a game situation

• dual natural: inequality required at all times $t \leq N$

• first attempt at primal: $\sup_{\mathbb{P} \in \mathcal{M}(\mu,\nu)} \sup_{\tau \leq N} \mathbb{E}^{\mathbb{P}}[c_\tau]$ gives a duality gap! (Hobson & Neuberger, Bayraktar & Zhou)

• this is because we lost the Bellman principle
$\rightsquigarrow$ need to transfer the terminal condition into a starting one

• consider transport for ∞ of assets with given initial prices alternatively consider Measures Valued Martingales: $X_t = \mathcal{L}(S_N|\mathcal{F}_t)$, see Aksamit, Deng, O. & Tan '17.

• Also useful in continuous time: MOT $\rightsquigarrow$ ∞–dim stoch. opt. control, see Eldan '16, Cox & Kallblad '17.

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where $\mathcal{D}(c) := \left\{(\varphi,\psi,h) : \ \varphi \oplus \psi + h^{\otimes} \leq c \text{ on } \Omega\right\}$.

For $c \in$ LSC, $\mathbf{P} = \mathbf{D}$ and existence holds for $\mathbf{P}(\mu,\nu)$ for all $\mu \preceq \nu$.
Duality for $\mathbf{D}$ requires convexity* of c.

Quasi-sure dual formulation

Definition

$\mathcal{M}(\mu,\nu)-$q.s. (quasi surely) means $\mathbb{P}-$a.s. for all $\mathbb{P}\in\mathcal{M}(\mu,\nu)$

- The quasi-sure robust sub-hedging cost

$$\mathbf{D}^{qs} := \sup_{(\varphi,\psi,h)\in\mathcal{D}^{qs}} \{\mu[\varphi]+\nu[\psi]\}$$

$$\mathcal{D}^{qs} := \{(\varphi,\psi,h)\in\hat{L}(\mu,\nu)\times\mathbb{L}^0 : \varphi\oplus\psi+h^{\otimes}\le c,\ \mathcal{M}(\mu,\nu)-\text{q.s.}\}$$

is also natural... $(\hat{L}(\mu,\nu)\supset\mathbb{L}^1(\mu)\times\mathbb{L}^1(\nu))$

- Then, $\mathbf{D}(\mu,\nu)\le\mathbf{D}^{qs}(\mu,\nu)\ \le\ \mathbf{P}(\mu,\nu)$

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Structure of polar sets in (standard) optimal transport

$\mathcal{N}_\mu := \{\mu -$ null sets$\}$, $\mathcal{N}_\nu$...

Theorem (Kellerer)

For $N \subset \mathbb{R} \times \mathbb{R}$, TFAE:

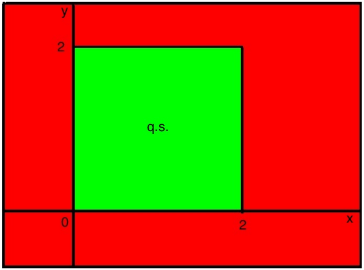
- $\mathbb{P}[N] = 0$ *for all* $\mathbb{P} \in \Pi(\mu, \nu)$
- $N \subset (N_\mu \times \mathbb{R}) \cup (\mathbb{R} \times N_\nu)$ *for some* $N_\mu \in \mathcal{N}_\mu$, $N_\nu \in \mathcal{N}_\nu$

$\Longrightarrow$ no difference between the pointwise and the quasi-sure formulations in standard optimal transport

Pointwise versus Quasi-sure superhedging I

Suppose Supp$(\mu) = [0,2]$ =Supp$(\nu) = [0,2]$, then

- $\mathcal{M}(\mu,\nu)-$q.s. only involves the values $(x,y) \in [0,2]^2$
- Pointwise superhedging involves all values $(x,y) \in \mathbb{R}^2$



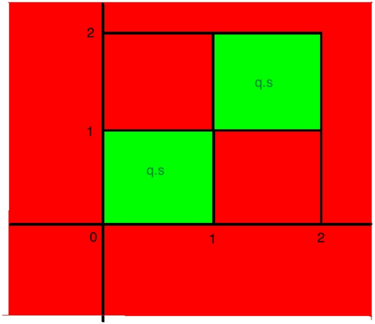
Pointwise versus Quasi-sure superhedging II

Suppose Supp(μ) =Supp(ν) $= [0,2]$, and $C_\mu(1) = C_\nu(1)$

$$\mathbb{E}\big[(X-1)^+\big] \;=\; \mathbb{E}\big[(Y-1)^+\big] \;\geq\; \mathbb{E}\big[(X-1)^+\big]$$

by Jensen's inequality, and then $\{X \geq 1\} = \{Y \geq 1\}$

$\Longrightarrow$ many more MOT polar set than OT ones!



Duality and existence under quasi-sure formulation in $\mathbb{R}$

Theorem (Beiglböck, Nutz & Touzi '15)

Let $\mu \preceq \nu$ and $c \geq 0$ measurable. Then

$$\mathbf{P}(\mu, \nu) \quad = \quad \mathbf{D}^{qs}(\mu, \nu)$$

and existence holds for $\mathbf{D}^{qs}$, whenever finite

Many examples where $\mathbf{D}(\mu, \nu) < \mathbf{D}^{qs}(\mu, \nu) = \mathbf{P}(\mu, \nu)$.

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Description of MOT polar sets (O. & Siorpaes)

Convex functions allow to study MOT polar sets in $\mathbb{R}^d$:

$$\varphi'' \geq 0 \text{ and } (\nu - \mu)[\varphi] = 0 \quad \Longrightarrow \quad \varphi \text{ "is affine" } \mathcal{M}(\mu,\nu)-q.s. \quad (*)$$

Let $A_x(\varphi)$ be the largest relatively open set containing x on which φ is affine. Then, for any convex Lip φ with $(\nu-\mu)[\varphi]=0$,

$$\kappa(x, \overline{A_x(\varphi)}) = 1 \; \mu(dx)\text{–a.e.} \quad \forall\, \mathbb{P} = \mu \otimes \kappa \in \mathcal{M}(\mu,\nu)$$

Extend the notion to sequences of functions $(\nu-\mu)[\varphi_n] \to 0$ and take μ–essential infimum of r.v. $x \to \overline{A_x(\varphi_n)}$:

$$E_x(\mu,\nu) := \mu - \text{ess} \bigcap_{\varphi_n : (\nu-\mu)[\varphi_n] \to 0} \overline{A_x(\varphi_n)}$$

Finally, the convex component is the r.i. of the face F_x:

$$C_x(\mu,\nu) := \text{ri}(F_x(E_x(\mu,\nu))) \quad \text{form a partition of } \mathbb{R}^d \text{ \& satisfy } (*)$$

In general uncountably many components. In $\mathbb{R}$ all explicit: at most countably many intervals C_i + points (B-N-T '15).

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Description of MOT polar sets

Full description given in De March & Touzi '19 via duality.

For any $\pi \in \mathcal{M}(\mu, \nu)$, $\pi = \mu \otimes \kappa$, we have

$$\kappa(x, \overline{C_x(\mu, \nu)}) = 1 \quad \mu - a.e.$$

Recently, Schachermayer & Tschiderer '24 show that the stretched BM attains the paving

$$\overline{C_x(\mu, \nu)} = \text{closed convex hull of support of } \kappa(x, \cdot) \quad \mu - a.e.$$

Extensions – discrete time

- Geometry of MOT on the line, Brenier–type thm
- Geometry of Super/Sub-Martingale Optimal Transport
- Many papers on duality under relaxed conditions
  - only finitely many constraints on the marginals
  - CPS (ϵ-martingale transports)
- Extension to $\mathbb{R}^n$:
  - Lim '16: 1-dim marginals constraints $(\mu_i, \nu_i)_{1 \leq i \leq n}$
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Extensions – continuous time

- Continuous–time transport and Skorokhod embedding
  - Beiglböck, Cox & Huesmann ('16,'17) on geometry of solutions to the optimal SEP
  - Ghoussoub, Kim & Lim on optimal SEP for radially symmetric distributions in $\mathbb{R}^d$
  - O. & Spoida '15, Cox, O. & Touzi '16 on iterated SEP
  - Duality in different setups in several papers. Also in $\mathbb{R}^d$ and with multiple maturities. Require stronger continuity of c. "Complete" duality still open!
  - Optimal Local Martingale Transport in Cox, Hou & O. '16

THANK YOU!

(and I am happy to discuss any of the above if you are interested)