L2: Existence & some properties of optimal transport plans. ©Jan Obłój

Weierstrass criterion for existence of minimizers (direct method in calc. of variations)

Prokhorov's thm & similar preliminaries

Def A function $f : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semi-continuous (lsc) if $\forall_{x_n \to x}$ $f(x) \le \liminf_{n\to\infty} f(x_n)$.

Thm (W.) If $f : \mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ is lsc & $\mathcal{X}$ is compact then $\inf_{x\in\mathcal{X}} f(x)$ is attained by some $\bar{x}\in\mathcal{X}$.

Proof If $f \equiv +\infty$ we are done. Otherwise let $\ell = \inf_{x\in\mathcal{X}} f(x)$ $(\in \mathbb{R}\cup\{-\infty\})$. Let x_n be a minimizing sequence. Pick a converging subsequence $x_{n_k} \to \bar{x}$. Then $\ell \le f(\bar{x}) \le \liminf f(x_{n_k}) = \ell$ so we have $=$. (& $\ell \in \mathbb{R}$). ∎

Def We say that a sequence of measures $\mu_n \in \mathcal{P}(\mathcal{X})$ converges weakly (or narrowly) to $\mu \in \mathcal{P}(\mathcal{X})$ if
$$\int \varphi(x)\,\mu_n(dx) \longrightarrow \int \varphi(x)\,\mu(dx) \quad \forall_{\varphi\in C_b(\mathcal{X})}.$$
We write $\mu_n \Rightarrow \mu$ or $\mu_n \longrightarrow \mu$.

Rk $\mathcal{X} = \mathbb{R}$ this is equivalent to $F_{\mu_n}(x) \longrightarrow F_\mu(x)$ $\forall_x$ point of continuity of F_μ.

Def A family of measures $\{\mu_i : i\in\mathcal{I}\}$ in $\mathcal{P}(\mathcal{X})$ is said to be tight if $\forall_{\varepsilon>0}$ $\exists$ $K_\varepsilon \subseteq \mathcal{X}$ compact s.t. $\mu_i(\mathcal{X}\setminus K_\varepsilon) < \varepsilon$ $\forall_{i\in\mathcal{I}}$.

Thm (Prokhorov) Let $\mathcal{X}$ be Polish & $(\mu_n)_{n\ge1} \subseteq \mathcal{P}(\mathcal{X})$. (μ_n) is relatively compact (ie, $\forall_{(\mu_{n_k})\subseteq(\mu_n)}$, $\exists (\mu_{n_{k_l}})$ $\mu_{n_{k_l}} \to \mu$ for some $\mu\in\mathcal{P}(\mathcal{X})$) iff $(\mu_n)_{n\ge1}$ is tight.

Rk $\mu(\mathcal{X}\setminus K_\varepsilon) \le \liminf \mu_n(\mathcal{X}\setminus K_\varepsilon) < \varepsilon$ by portmanteau thm so $\overline{(\mu_n)}$ is also tight.
($\mathcal{X}\setminus K_\varepsilon$ open)

Ideas of Proof "$\Leftarrow$"

For a compact $K \subseteq \mathcal{X}$ we have $C_0(K) = C_b(K) = C(K)$ so the dual is the space of measures & (μ_n) is a bounded sequence so (by Banach–Alaoglu since $C_0(K)$ is separable) has a weakly converging subsequence: $\mu_{m_n}|_K \longrightarrow \mu_K$. Take $K_{1/i} \subseteq K_{1/(i+1)}$ s.t. through a diagonal argument build one subsequence μ_{m_n} which converges weakly to some ν_i on $K_{1/i}$.

Let $\mu(A) := \sup_i \nu_i(A\cap K_{1/i})$. For $\varphi\in C_b(\mathcal{X})$, $\int_{\mathcal{X}} \varphi\, d(\mu_{m_n} - \mu) \le 2\|\varphi\|_\infty \cdot \frac{1}{i} + \int_{K_i} \varphi\, d(\ldots)$
& $\mu(\mathcal{X}) = 1$.
$\longrightarrow 2\|\varphi\|_\infty/i$ K_i
so $\longrightarrow 0$.

"$\Rightarrow$" $\forall_{r>0}$, we can cover $\mathcal{X}$ by open balls $B_1, B_2, \ldots$ of radius r.

Let $G_k = B_1 \cup \ldots \cup B_k$. Then $\lim_{k\to\infty} \inf_n \mu_n(G_k) = 1$. $(*)$

Indeed, otherwise $\sup_k \mu_{n_k}(G_{n_k}) = c < 1$. Taking subsequence, $\mu_{n_{k_\ell}} \to \mu$ &

$\mu(G_m) \le \liminf \mu_{n_{k_\ell}}(G_m) \le \liminf \mu_{n_{k_\ell}}(G_{n_{k_\ell}}) = c < 1$. & $m \to \infty$ gives $1 = \mu(X) < 1$. contr.

Take $r = \frac{1}{m}$ & write G^m_k. $\forall_{\varepsilon>0}$, by $(*)$, $\exists k_1, k_2, \dots$ $\inf_n \mu_n(G^m_{k_m}) \ge 1 - \varepsilon 2^{-m}$

Let $A := \bigcap_m G^m_{k_m}$ then $\inf_n \mu_n(\bar{A}) \ge 1 - \varepsilon$ & $\bar{A}$ is complete & totally bounded $\Rightarrow$ compact.

uses Ax. of Ch.

$$\left(\mu_n(\bar{A}^c) \le \mu_n(A^c) = \mu_n\left(\bigcup (G_\cdot)^c\right) \le \sum_i \mu_n\left((G^m_{k_m})^c\right) \le \varepsilon \sum_{i=1}^{\infty} 2^{-m} = \varepsilon\right)$$

(can be covered by a finite union of balls of any given radius).

Existence of solutions to OT pb

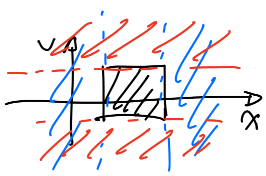
Lemma 21 Let $P \subseteq \mathcal{P}(X)$ & $Q \subseteq \mathcal{P}(Y)$ be tight. Then

$$\Pi(P, Q) = \{\pi \in \mathcal{P}(X \times Y) : \mathrm{proj}_X \# \pi \in P \ \& \ \mathrm{proj}_Y \# \pi \in Q\} \text{ is tight.}$$

Proof Fix $\varepsilon > 0$ & K_X, K_Y compacts with ...

$\forall \pi \in \Pi(P, Q)$, $\pi(X \times Y \setminus K_X \times K_Y) \le \pi(X \times Y \setminus K_X \times Y) + \pi(X \times Y \setminus X \times K_Y)$

$= \mu(X \setminus K_X) + \nu(Y \setminus K_Y) < 2\varepsilon$.



Lemma 22 $\Pi(\mu, \nu)$ is compact.

Proof It is relatively compact by Prokhorov & the above lemma so we just have to establish closedness.

Let π be a limit of π_n. Then $\forall_{\varphi, \psi \in C_b}$ $\int \varphi(x) + \psi(y)\, d\pi = \lim_n \int \varphi(x) + \psi(y)\, d\pi_n = \int \varphi\, d\mu + \int \psi\, d\nu$

so $\pi \in \Pi(\mu, \nu)$.

Rk We used that $C_b(X)$ determine element in $\mathcal{P}(X)$. In fact one can construct a countable family of functions that does that.

Lemma 2.3 Suppose $c: \mathcal{X}\times\mathcal{Y} \mapsto \mathbb{R}\cup\{+\infty\}$ is lsc and bounded from below. Then

$$\pi \longmapsto \int c\, d\pi \quad \text{is lsc on } \mathcal{P}(\mathcal{X}\times\mathcal{Y}) \text{ with topology of weak conv.}$$

Lemma 2.4. For $c: \mathcal{Z} \mapsto \mathbb{R}\cup\{+\infty\}$ bounded from below

$$c \text{ is lsc} \iff c(z) = \sup_k f_k(z) \text{ for a family } \{f_k\}_{k\geq 1} \text{ of Lipschitz functions on } \mathcal{Z}.$$

Proof (2.4).

$\Leftarrow$ ① $f_k(x) \leq \liminf_n f_k(x_n) \leq \liminf_n c(x_n)$ since $c \geq f_k$.

& taking $\sup_k$ $\quad c(x) \leq \liminf_n c(x_n)$.

(Rk) More generally a sup of lsc functions is lsc.

② (2nd proof) c lsc $\iff$ the epigraph $\{(z,u): u \geq c(z)\}$ is closed in $\mathcal{Z}\times\mathbb{R}$

but — " — of sup $= \bigcap$ epigraphs.

$\Rightarrow$ Wlog $c \geq 0$. Let $f_k(z) = \inf_{u\in\mathcal{Z}} \left(c(u) + k\, d(z,u)\right)$

- $|f_k(z_1) - f_k(z_2)| = \left| \inf_{u_1\in\mathcal{Z}} \left(c(u_1) + k\,d(z_1,u_1)\right) - \inf_{u_2\in\mathcal{Z}} \left(c(u_2) + k\,d(z_2,u_2)\right)\right|$ (wlog suppose >0)

$$\leq \inf_{u_1\in\mathcal{Z}} c(u_1) + k\,d(z_1,u_1) - c(u_1) - k\,d(z_2,u_1)$$

$$= k \inf_{u\in\mathcal{Z}} d(z_1,u_1) - d(z_2,u_1) \leq k\, d(z_1,z_2) \quad \text{is } k\text{-Lip.}$$

- $f_k \leq f_{k+1} \leq c$
- $\lim_k f_k(z) = \sup_k f_k(z) \leq c(z)$ Suppose the $=$ does not hold for some $z\in\mathcal{Z}$ $\quad \ell := \lim_k f_k(z) < c(z) \leq \infty$

$\forall_k$ pick $u_k \in \mathcal{Z}$ s.t. $\quad c(u_k) + k\,d(u_k,z) < f_k(z) + 1/k \leq \ell + 1/k$

$$d(u_k,z) \leq \frac{\ell + 1/k - c(u_k)}{k} \leq \frac{\ell + 1/k}{k} \longrightarrow 0$$

taking limits in we get $c(z) \leq \liminf c(u_k) \leq \ell$ a contradiction. ☒

Rk Taking $g_k = f_k \wedge k$ we may assume the sequence is of bdd functions

Proof (2.3)

We know we can take a sequence $c_n \nearrow c$ of Lip & bounded functions.

Then $\pi \mapsto J_n(\pi) = \int c_n\, d\pi$ is cont $\implies \int c\, d\pi \overset{MCT}{=} \lim J_n(\pi) = \sup J_n(\pi)$ is lsc

(by the Rk above). ☒

Thm 25 Let $\mathcal{X},\mathcal{Y}$ be Polish & $\mu\in\mathcal{P}(\mathcal{X})$, $\nu\in\mathcal{P}(\mathcal{Y})$ & $c:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ lsc and bounded below. Then the Kantorovich OT problem is attained

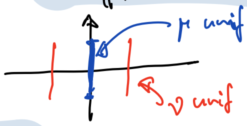
$$P(\mu,\nu)=\inf_{\pi\in\Pi(\mu,\nu)}\int c\,d\pi=\int c\,d\pi^* \quad \text{for some } \pi^*\in\Pi(\mu,\nu)$$

Proof We know that $\Pi(\mu,\nu)$ is compact, $\pi\mapsto\int c\,d\pi$ is lsc so we conclude by Weierstrass. ∎

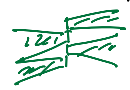
Rk We already noted that while $\Pi(\mu,\nu)$ is non-empty, the set of transports $T(\mu,\nu)=\{\pi\in\Pi(\mu,\nu):\ \exists T:\mathcal{X}\to\mathcal{Y}\ \ \pi=(Id,T)_\#\mu\}$ may well be empty (e.g. $\mu=\delta_{\{0\}}$, $\nu=N(0,1)$).

Lemma If $\mu,\nu\in\mathcal{P}(\mathbb{R}^d)$ & μ is atomless then $T(\mu,\nu)\neq\emptyset$. If μ,ν are supported on a compact then $T(\mu,\nu)$ is dense in $\Pi(\mu,\nu)$ & for c continuous: $\inf_{\pi\in T(\mu,\nu)}\int c\,d\pi=\min_{\pi\in\Pi(\mu,\nu)}\int c\,d\pi$.

Example:



$c(x,y)=|x-y|^2 \Rightarrow \pi^*$ splits mass ⟷ but can be approximated with



Extensions to MOT We consider here $\mathcal{Y}=\mathcal{X}$ & ONE step martingales.

Recall the $\mathcal{M}(\mu,\nu)=\{\pi\in\Pi(\mu,\nu):\ \mathbb{E}_\pi[Y|\sigma(X)]=X\ \ \mu\text{-a.s.}\}=\{\pi\in\Pi(\mu,\nu):\ \pi=\mu\otimes\theta\ \&\ \int y\,\theta(x,dy)=x\ \ \mu(dx)\text{-a.e.}\}$

This set may be empty. In fact:

Thm (Strassen) Let $\mu,\nu\in\mathcal{P}_1(\mathbb{R}^d)=\{\gamma\in\mathcal{P}(\mathcal{X}):\int|x|\,\gamma(dx)<\infty\}$

$$\mathcal{M}(\mu,\nu)\neq\emptyset \quad \text{iff} \quad \mu\le_{cx}\nu, \text{ i.e., } \int f\,d\mu\le\int f\,d\nu \quad \forall f:\mathbb{R}^d\to\mathbb{R} \text{ convex}$$

Lemma If $\mathcal{M}(\mu,\nu)\neq\emptyset$ then $\mathcal{M}(\mu,\nu)$ is compact.

Proof (for $\mathbb{R}$ i.e. $d=1$) Indeed, it is a subset of a compact set so we just need to prove it is closed. Recall that $\pi\in\Pi(\mu,\nu)$ belong to $\mathcal{M}(\mu,\nu)$ iff $\int_{\mathcal{X}\times\mathcal{Y}}\varphi(x)(y-x)\,d\pi=0 \quad \forall\varphi\in C_b(\mathcal{X})$

$$\| \quad \int_{\mathcal{X}}\varphi(x)\left(x-\int_{\mathcal{Y}}y\,\theta(x,dy)\right)\mu(dx)$$

Let $\pi_n\in\mathcal{M}(\mu,\nu)$ conv. weakly to $\pi\in\Pi(\mu,\nu)$. Fix $K>0$ & $f_K=\begin{cases}1 & \text{on } [-K,K]^2\\ 0 & \text{on } \mathbb{R}^2\setminus(-K-1,K+1)^2\end{cases}$ cont. bdd by 1.

$g_K=\varphi(x)(y-x)f_K(x,y)$ is C_b so $\int g_K\,d\pi_n\to\int g_K\,d\pi$

$\forall_{\varepsilon>0}$ $\exists K$ s.t. $\int |g - g_K| \, d\pi \leq \int_{\mathbb{R}^2 \setminus [-K,K]^c} b\,(|y|+|x|)\, d\pi = b\left(\int_{[-K,K]^c} |x|\, d\mu + \int_{[-K,K]^c} |y|\, d\nu\right) \leq \varepsilon$

$\|\varphi\| \leq b$

$\Rightarrow \left|\int g \, d\pi_n - d\pi\right| \leq \left|\int (g - g_K)\, d\pi_n - d\pi\right| + \left|\int g_K \, d\pi_n - d\pi\right| \leq 3\varepsilon \to 0 \Rightarrow \pi \in \mathcal{M}(\mu,\nu)$ ∎

Rk ~~The~~ The above extends to mg with finite discrete time $\mathcal{M}(\mu_1,\dots,\mu_n)$ but fails, e.g., in continuous time with $\mathcal{M}_{[0,1]}(\mu_0,\mu_1)$.

Rk Going back to OT, a restriction of an optimal tr. plan is still optimal:

Prop 2.6. In the setting of Thm 2.5, if π is a minimiser & $\pi' \leq \pi$ is a non-negative measure with $\pi'(\mathcal{X}\times\mathcal{Y})>0$ then $\hat{\pi} := \frac{\pi'}{\pi'(\mathcal{X}\times\mathcal{Y})}$ is an optimal tr. plan for marginals $\hat{\mu} = \mathrm{proj}_{\mathcal{X}\#}\hat{\pi}$ $\hat{\nu}$ — " — $\mathcal{Y}$ — " —

Proof (Ex ?)

If $\hat{\pi}$ not optimal then take minimiser $\overline{\pi}$, $\int c\, d\overline{\pi} < \int c\, d\hat{\pi}$, $\overline{\pi}, \hat{\pi} \in \Pi(\hat{\mu},\hat{\nu})$

Let $\tilde{\pi} := (\pi - \pi') + \pi'(\mathcal{X}\times\mathcal{Y}) \cdot \overline{\pi} = \pi + \pi'(\mathcal{X}\times\mathcal{Y}) \cdot (\overline{\pi} - \hat{\pi}) \in \Pi(\mu,\nu)$

& $\int c\, d\tilde{\pi} < \int c\, d\pi \Rightarrow$ contradiction. ∎

Some properties of the optimal solutions

A natural way to try to improve a given ~~transport~~ plan π is to consider if we can lower the cost via a cyclical relabelling of points.

Def (c-cyclical monotonicity) For $c : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathbb{R} \cup \{+\infty\}$ a subset $\Gamma \subseteq \mathcal{X} \times \mathcal{Y}$ is said to be c-cyclically monotone if $\forall N \in \mathbb{N}$ $\forall (x_1, y_1), \dots, (x_N, y_N) \in \Gamma$

$$\underbrace{\sum_{i=1}^{N} c(x_i, y_i)}_{\text{current cost in } \Gamma} \leq \underbrace{\sum_{i=1}^{N} c(x_i, y_{i+1})}_{\text{cost after cyclical re-routing}}, \quad \text{where } y_{N+1} = y_1 .$$

A transport plan $\pi \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is said to be c-cyclically monotone if it is concentrated on a — " — " — set.

Intuition: $\pi^* \in \Pi(\mu, \nu)$ optimal $\Longrightarrow$ π^* is c-cyclically monotone

Insight from duality: $\Longleftarrow$ also holds.