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4. MOT & SEP

Discrete time

Ex Revisit the proof of Duality to show that $\forall \mu,\nu \in \mathcal{P}_1(\mathbb{R})$, $\mu \leq_{cx} \nu$, $\forall c$ lsc & ...

$$P^{M}(\mu,\nu) = \inf_{\pi \in \mathcal{M}(\mu,\nu)} \int c\, d\pi = \sup_{\substack{\varphi,\psi,h \in C_b \\ \underbrace{\varphi(x)+\psi(y)+h(x)(y-x)}_{\varphi\oplus\psi+h^{\otimes}} \leq c(x,y)}} \int \varphi\, d\mu + \int \psi\, d\nu$$

More generally: for $\mu_1 \leq_{cx} \dots \leq \mu_n$ & $c: \mathbb{R}^n \to \mathbb{R}$

$$\inf_{\pi \in \mathcal{M}(\mu_1,\dots,\mu_n)} \int c\, d\pi = \sup\left\{ \sum_i \int \varphi_i\, d\mu_i : \sum_i \varphi_i(x_i) + \sum_{i=1}^{n-1} h_i(x_1,\dots,x_i)(x_{i+1}-x_i) \leq c(x_1,\dots,x_n) \right\}$$

Financial interpretation: Stock market prices are modelled with a stochastic process $(S_t)_{1\leq t\leq n}$.
At time t, we can use the available info $\mathcal{F}_t \leadsto h(S_1,\dots,S_t)$ to buy $h(S_1,\dots,S_t)$ shares at price $h(S_1,\dots,S_t)S_t$. We sell these at time $t+1$ for $h(S_1,\dots,S_t)S_{t+1}$. We repay the loan used to buy shares & end up with:

$$h(S_1,\dots,S_t)(S_{t+1}-S_t).$$

$\Rightarrow$ A "self-financing" trading strategy is of the form $\sum_{t=1}^{n-1} h_t(S_1,\dots,S_t)(S_{t+1}-S_t) = (h\circ S)_n$

Then if S is defined on $(\Omega,\mathcal{F},\mathbb{P})$ & we want to use $\mathbb{E}_{\mathbb{P}}$ to price we need $\mathbb{E}_{\mathbb{P}}[(h\circ S)_n]=0$ $\forall h$ above $\overset{\Leftarrow}{\Rightarrow}$ (moments ass.) S is a $\mathbb{P}$-martingale.

Suppose then that in the market we can observe the prices of call & put options, say

$$C(K,t) = \mathbb{E}_{\mathbb{P}}\left[(S_t-K)^+\right], \quad K\in\mathbb{R},\ t=1,\dots,n.$$

$$\Rightarrow \frac{\partial C}{\partial K}(K,t) = \frac{\partial}{\partial K}\int_K^\infty (x-K)\,\mathcal{L}(S_t)(dx) = -\mathcal{L}(S_t)([K,+\infty)) = -\mathbb{P}(S_t \geq K)$$

(B-L)

$$\Rightarrow \frac{\partial^2 C}{\partial K^2}(dK) \quad \frac{\partial}{\partial K} \quad = \mathcal{L}(S_t)(dK) \text{ gives the distribution of } S_t. \quad \text{(B-L)}$$

Let $\mu_t := \frac{\partial^2}{\partial K^2}C(t,dK) \Longrightarrow$ compatible $\mathbb{P}$s are $\mathcal{M}(\delta_{S_0},\mu_1,\dots,\mu_n)$.

If we now want to understand the range on admissible prices for an exotic/non-traded option ξ, this will be

$$\left[\inf_{\pi\in\mathcal{M}(\mu_0,\dots,\mu_n)} \mathbb{E}_\pi[\xi],\ \sup_{\pi\in\mathcal{M}(\mu_0,\dots,\mu_n)} \mathbb{E}_\pi[\xi]\right].$$

So the bounds are given by MOT values!

We have, by duality,

$$\inf_{\pi\in\mathcal{M}(\mu_0,\dots,\mu_n)} \int \xi \, d\pi = \sup\Big\{ \overbrace{x+\underbrace{\sum_{i=1}^{t}\sum_{j=1}^{m_t}\alpha_j^t C(K_j^t,t)}_{\text{price to setup}}}^{\sum\int \varphi_t \, d\mu_t} : x+(h\circ S)_n + \sum\sum \alpha_j^t (S_t-K_j^t)^+ \le \xi(S_1,\dots,S_n)\Big\}$$

$\mathbb{R}$

Each such element on the right can be written as:

$$x+\sum_i\sum_j \alpha_j^t C(K_j^t,t) + \underbrace{(h\circ S)_n + \sum_i\sum_j \alpha_j^t\big((S_t-K_j^t)^+ - C(K_j^t,t)\big)}_{H_n := \ \text{self financing}} \le \xi(\cdot)$$

φ_t^j $\qquad \int \varphi_t^j = \psi_t$

So the lowest admissible price = the cost of the most expensive strategy sub-replicating ξ.

At any lower price I can make riskless profit buying ξ at $p <$ above & setup hedge

$$-p+\xi - H_n \ge -p + \underset{\text{(above)}}{x+\sum\sum \alpha_j^t C(K_j^t,t)} > 0.$$

Likewise, at price higher than sup, I sell & hedge.

Continuous time

Consider now a continuous time setting $(S_t : t\le T)$. As before (take limits)

$(H\circ S)_T = \int_0^T h_t \, dS_t$ models outcomes of a self-financing strategy.

We also need some admissibility constraint to avoid ∞-credit lines (e.g. $(H\circ S)_t \ge -K,\ t\le T$)

$\mathbb{E}_{\mathbb{P}}\big[(H\circ S)_T\big]=0 \implies S$ is a $\mathbb{P}$-mg.

$C(K)=\mathbb{E}_{\mathbb{P}}\big[(S_T-K)^+\big]$ given $\implies S_T \sim_{\mathbb{P}} \nu$ given. $\quad S_0=s$ also given.

We are interested in $\inf/\sup\ \mathbb{E}\big[\xi(S_t : t\le T)\big]$ over all cont. mg $S_0=s$, $S_T\sim\nu$.

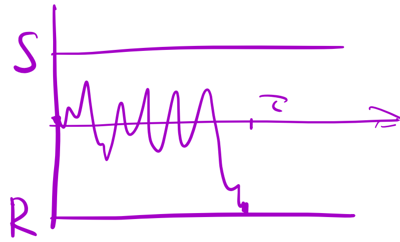
Suppose $\xi(\cdot)$ is invariant under time changes (cont.), e.g., $\xi((S_t:t\le T))=\mathbb{1}_{\max_{t\le T} S_t \ge b}$

Then using D-D-Sch. $\inf/\sup\ \mathbb{E}\big[\xi(B_t : t\le\tau)\big]$ over all st. times $B_\tau\sim\nu$ & $(B_{t\wedge\tau})$ is a UI mg.

Such a st. time is called an embedding, or a solution to: $\mathbb{E}[\tau]<\infty$ (if $\int x^2 \nu<\infty$)

(SEP) Given a centred $\nu\in\mathcal{P}_1(\mathbb{R})$, find a st. time τ s.t. $B_\tau\sim\nu$ & $(B_{\tau\wedge t})$ UI.

A simple solution using randomised stopping times (Hall '68)



Let $\rho_\nu(dr, ds) = \dfrac{s-r}{\int_0^\infty x\,\nu(dx)}\ \mathbb{1}_{r\le 0\le s}\ \nu(dr)\,\nu(ds)$

Let $(R,S)\sim \rho_\nu$ indep of (B_t) & $\tau := \inf\{t\ge 0:\ B_t \notin (R,S)\}$

indeed $\mathbb{E}[f(B_\tau)] = \mathbb{E}\left[f(S)\dfrac{-R}{S-R} + f(R)\dfrac{S}{S-R}\right]$

$$= \int_{-\infty}^{0}\int_0^\infty -r f(s)\,\frac{\nu(dr)\,\nu(ds)}{\int_0^\infty x\,\nu(dx)} + \int_{-\infty}^{0}\int_0^\infty s f(r)\,\frac{\nu(dr)\,\nu(ds)}{\int_0^\infty x\,\nu(dx)}$$

$\int_{-\infty}^{\infty} x\,\nu(dx) = 0$

$$= \int_{-\infty}^{\infty} f(x)\,\nu(dx).$$

Many solutions exist in the natural filtration $\mathcal{F}^B = (\mathcal{F}^B_t)_{t\ge 0}$ ~~are important to us~~:

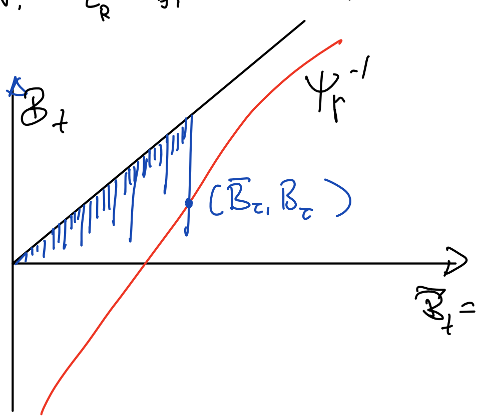
- AY : $\exists\ \psi_\nu$ s.t. $\tau_{AY} = \inf\{t\ge 0:\ \psi_\nu(B_t) \le \bar{B}_t\}$ solves (SEP)

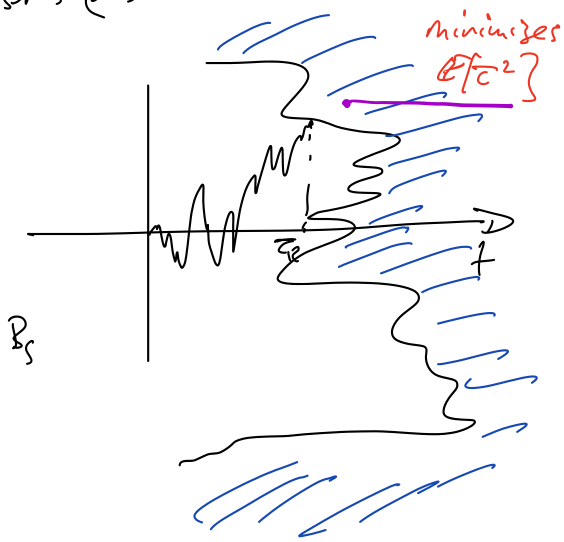
  $= \inf\{t\ge 0:\ B_t \le \psi_\nu^{-1}(\bar{B}_t)\}$ & $\bar{B}_t := \sup_{s\le t} B_s$.

- Root: $\exists$ barrier $R \subseteq \mathbb{R}_+\times\mathbb{R}$, $(x,t)\in R \Rightarrow (x,s)\in R\ \forall s\ge t$.

  s.t. $\tau_R = \inf\{t\ge 0:\ (t, B_t)\in R\}$ solves (SEP)

maximises $\mathbb{E}[f(\bar{B}_\tau)]$ for increasing $f\ge 0$.





OT & SEP (Beiglböck, Cox & Huesmann '17)

$u(y,t) := U\,\mathcal{L}(B_{t\wedge\tau})(y) = -\mathbb{E}\,|B_{t\wedge\tau} - y| \ge U_\nu(y)$

Randomised stopping times

A measure $\xi \in \mathcal{P}(C(\mathbb{R}_+)\times\mathbb{R}_+)$ s.t. $\xi(d\omega, dt) = \mathbb{W}(d\omega)\otimes\xi_\omega(dt)$ is a randomised stopping time if

$A^\xi_u(t) := \xi_\omega([0,t])$ is an optional process (here adapted w.r.t. completed nat. filtr.)

(eq) on $C(\mathbb{R}_+)\times[0,1]$ with $\mathbb{W}\otimes Leb$ $\ \rho(\omega,u) = \inf\{t\ge 0:\ \xi_\omega([0,t])\ge u\}$ is an $\bar{\mathcal{F}}_t = \sigma(\mathcal{F}^X_t \vee \mathcal{B}([0,1]))^a$ - stopping time

For an optional Y, $\mathcal{L}(Y_\xi) := Y_\#\,\xi$, where $Y: C(\mathbb{R}_+)\times\mathbb{R} \to \mathbb{R}$, $(\omega, t)\mapsto Y_t(\omega)$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a p-ty space with a BM (B_t) & $\mathcal{F}_0$ supports an indep $U[0,1]$ r.v..

<u>Rk</u> RST $\equiv$ $(\mathcal{F}_t)$ st. times.

Let $\mu \in \mathcal{P}_2(\mathbb{R})$, $\int x \, d\mu = 0$ & $\gamma : S \to \mathbb{R}$
$(\omega, t) \to \gamma(\omega, t)$

$S = \{(\omega, s) : \omega : [0,s] \to \mathbb{R}$ is continuous with $\omega(0) = 0\}$ are stopped paths.

(OptSEP) $\quad P_\gamma(\mu) = \inf\{ \mathbb{E}[\gamma((B_t)_{t \le \tau}, \tau)] : \tau$ solves SEP$\}$ $\quad B_\tau \sim \mu$

<u>Rk</u> $\mu \in \mathcal{P}_2(\mathbb{R})$ s (B^τ) uI iff $\mathbb{E}[\tau] = \int x^2 d\mu$.

<u>Thm</u> Suppose γ is lsc & bdd from below. Then (OptSEP) admits a solution.

<u>Proof</u> Show the set of RST ξ with $B_\xi = \mu$ is compact.

<u>Thm</u> Suppose γ is lsc & bdd from below. Let

$$D_\gamma(\mu) = \sup\Big\{ \int \psi \, d\mu : \psi \in C(\mathbb{R}),\ |\psi(y)| \le a + by^2,\ \exists M \text{ a cont. } (\mathcal{F}_t)\text{-martingale } M_0 = 0,\ |M_t| \le a + bt + cB_t^2,\ M_t + \psi(B_t) \le \gamma((B_s)_{s \le t}, t)\ \forall t \ge 0 \Big\}$$

Then $P_\gamma(\mu) = D_\gamma(\mu)$.

Now we combine p-tic tools with the geometric intuition from OT.

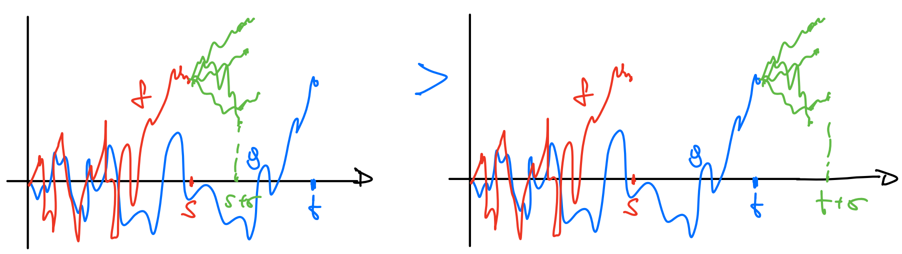
For $(f, s), (h, u) \in S$ let $(f \oplus h)(r) := \begin{cases} f(r) & : r \le s \\ f(s) + h(r - s) & , s \le r \le s + u \end{cases}$

$$\gamma^{(f,s) \oplus}(h, u) := \gamma(f \oplus h, s + u)$$

<u>Def</u> The pair $((f, s), (g, t)) \in S \times S$ is a stop-go pair if $f(s) = g(t)$ and

$$\mathbb{E}\left[\gamma^{(f,s)\oplus}((B_u)_{u \le \sigma}, \sigma)\right] + \gamma(g, t) > \gamma(f, s) + \mathbb{E}\left[\gamma^{(g,t)\oplus}((B_u)_{u \le \sigma}, \sigma)\right] \quad \forall (\mathcal{F}^B) \text{ s.t. } \sigma.$$

(provided all defined & finite).



Akin to c-cm but with just two pairs of points...

Def A set $\Gamma \subseteq S$ is called γ-monotone if

$$SG \cap (\Gamma^{<} \times \Gamma) = \emptyset$$

where $SG \subseteq S \times S$ are stop-go pairs & $\Gamma^{<} = \{(f,s) : \exists (\tilde{f},\tilde{s}) \in \Gamma : s < \tilde{s}, f \equiv \tilde{f} \text{ on } [0,s]\}$

Thm $\gamma : S \to \mathbb{R}$ is Borel measurable s.t. (OptSEP) is well posed & has an optimiser τ. Then $\exists \Gamma \subset S$ γ-monotone s.t. $((B_t)_{t \le \tau}, \tau) \in \Gamma$ a.s.

Prop Let $\gamma(f,t) = h(t)$ for a strictly convex $h : \mathbb{R} \to \mathbb{R}$ for which (OptSEP) is well posed. Then a minimiser exists & is a Root stopping time $\tau = \tau_R$ for a barrier R.

Proof By above $\exists \tilde{\tau}$ & $\exists \Gamma$ $((B_t)_{t \le \tilde{\tau}}, \tilde{\tau}) \in \Gamma$ a.s. & $(\Gamma^{<} \times \Gamma) \cap SG = \emptyset$.

We have $((f,s),(g,t)) \in SG$ if $f(s) = g(t)$ and

$$\mathbb{E}[h(s+\sigma)] + h(t) > h(s) + \mathbb{E}[h(t+\sigma)]$$ i.e. $h(t) - h(s) > \mathbb{E}[h(t+\sigma) - h(s+\sigma)]$

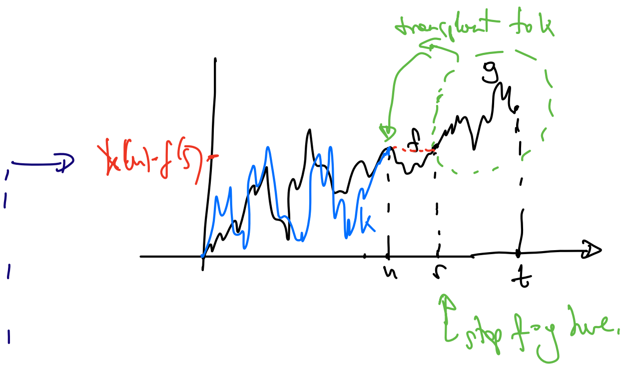
& strict convexity of h $\Rightarrow$ iff $t < s$.

Let $R_{cl} = \{(s,x) : \exists (g,t) \in \Gamma, g(t) = x, t \le s\}$

$R_{op} =$ — " — , — " — , — $t < s\}$

Take $(g,t) \in \Gamma \Rightarrow (t, g(t)) \in R_{cl}$ by definition. Also if $\inf\{s \le t : (s, g(s)) \in R_{op}\} < t$ then $(f,s) = (g|_{[0,s]}, s) \in \Gamma^{<}$ & $(s, f(s)) \in R_{op}$ for some $s < t$. By def of R_{op}, but $\exists (k,u) \in \Gamma : k(u) = f(s)$ & $u < s$. Then $((f,s),(k,u)) \in SG \cap (\Gamma^{<} \times \Gamma)$ a contradiction.

$\Rightarrow$ $(g,t) \in \Gamma \Rightarrow \inf\{s \le t : (s, g(s)) \in R_{cl}\} \le t \le \inf\{s \le t : (s, g(s)) \in R_{op}\}$



$h(u) + h(t) > h(u+t-s) + h(s)$

$\Updownarrow$ $h(t) - h(s) > h(u+t-s) - h(u)$ & $u < s$. ✓

$\Rightarrow$ $\tau_{R_{cl}} \le \tilde{\tau} \le \tau_{R_{op}}$ but the two Roots are $=$ by strong Markov & $\inf\{t > 0 : B_t = 0\} = 0$ a.s.

∎

$\Rightarrow$ minimizer $\tilde{\tau}$ is unique. If two τ_1, τ_2 then also $\tau_1 \mathbb{1}_{u \leq 1/2} + \tau_2 \mathbb{1}_{u \geq 1/2} =: \hat{\tau}$ & by the above $\hat{\tau} = \tau_{Rot}$ $\Rightarrow$ $Rot_1 = Rot_2$.

Rk Here we only showed opt $\Rightarrow$ Γ-monotone. The reverse is not conjectured. Probably requires n-tuples of paths?

Rk

$$\mathcal{M}^{cont}(\mu, \nu) = \{ P \in C([0,1], \mathbb{R}) : \text{continuous martingales with } X_0 \sim \mu,\ X_1 \sim \nu \}$$

is not compact $\Rightarrow$ things break down big time.