Some questions to get you thinking about Analysis I

Here are some questions to get you thinking about some of the ideas that you will cover in the Analysis I course in the Lent Term.

Limits and convergence [6]

- Remind yourself of the definition of convergence: what does it mean to say that a real sequence $(x_n)_{n\geq 1}$ tends to a limit as n tends to infinity? Pick some examples of sequences that display different behaviours, and check that you can analyse them using the definition of convergence.
- Remind yourself of the definition of convergence of a series $\sum_{n=1}^{\infty} x_n$. Again, pick some examples and check that you understand them.
- If I give you a series $\sum_{n=1}^{\infty} x_n$, and I tell you that $\sum_{n=1}^{\infty} |x_n|$ converges, does that necessarily mean that $\sum_{n=1}^{\infty} x_n$ converges? If I give you a series such that $\sum_{n=1}^{\infty} x_n$ converges, does that necessarily mean that $\sum_{n=1}^{\infty} |x_n|$ converges?
- Pick some examples of convergent sequences and some examples of non-convergent sequences. For each one, look at some subsequences. Can you find one that converges? One that does not converge? Several that converge? Several that do not converge? If you can find several that converge, do they necessarily tend to the same limit?

  (If the sequence is 1, 2, 3, 4, 5, 6, ..., then here are some *subsequences*:
  – 1, 3, 5, 7, 9, 11, ...;
  – 2, 3, 5, 7, 11, 13, ...;
  – 1, 2, 3, 10, 10001, 201924792, ...;
  – 1, 2, 3, 4, 5, 6, ...)

  Now try the same questions with some sequences that are *bounded* (there is some constant K so that $|x_n| \leq K$ for all $n \geq 1$).
- I give you two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, and I tell you that $\sum_{n=1}^{\infty} b_n$ converges, and that $0 \leq a_n \leq b_n$ for all $n \geq 1$. Must the series $\sum_{n=1}^{\infty} a_n$ converge?
- You should know from Numbers and Sets that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (can you prove this?). What about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ — does this converge?

Continuity [3]

- What does it mean to say that a function is continuous? Try some examples of functions that (intuitively) are continuous and functions that aren't, to get a feel for what the concept means. Can you formulate a definition phrased in terms of limits? Can you formulate a definition using ϵ and δ?
- Why does every cubic polynomial with real coefficients have (at least) one real root? (Look at the graph.)
- Can you extend the idea of the previous point to other continuous functions? Make a conjecture — and try to prove it rigorously!
- Here are some functions defined on subsets of the real line $\mathbb{R}$. For each function $f : I \to \mathbb{R}$, decide whether f is bounded (that is, whether there is some constant K so that $|f(x)| \leq K$ for all $x \in I$), and if so whether f attains its bounds.
  – $f : (0,1) \to \mathbb{R}$, $f(x) = 1/x$.
  – $f : (1,2) \to \mathbb{R}$, $f(x) = 1/x$.
  – $f : [1,2] \to \mathbb{R}$, $f(x) = 1/x$.

- $f : [-1,1] \to \mathbb{R}$, with $f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$

Can you generalise this? Look for necessary and sufficient conditions for $f : I \to \mathbb{R}$ to be bounded and to attain its bounds.

Differentiability [5]

- How is the derivative defined?
- When is it possible to differentiate a (real-valued) function? Here are some examples of functions $f : \mathbb{R} \to \mathbb{R}$ — are they differentiable everywhere, or at just some points, or nowhere?
  - $f(x) = x^2$.
  - $f(x) = |x|$.
  - $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
- Try to formulate a precise definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable at a point $x \in \mathbb{R}$. Phrase your definition using limits, and then using ϵ and δ.
- What does the chain rule say? Can you prove it rigorously?
- Take a differentiable function $f : [a,b] \to \mathbb{R}$ that has $f(a) = f(b)$. What can you say about the gradient of f in between a and b? Can you prove a theorem?
- Take a differentiable function $f : [a,b] \to \mathbb{R}$. Try to generalise/adapt what you found in the last bullet point to say something about the gradient of f in between a and b.

Power series [4]

- How do the definitions of convergence for real sequences and series extend to complex sequences and series?
- If I give you a convergent series $\sum_{n=1}^{\infty} a_n z^n$ (where $z \in \mathbb{C}$), what can you say about $\sum_{n=1}^{\infty} a_n w^n$ for $|w| < |z|$? If I give you a series $\sum_{n=1}^{\infty} a_n z^n$ that does not converge, what can you say about $\sum_{n=1}^{\infty} a_n w^n$ for $|w| > |z|$?
- How do we define the exponential sine and cosine functions? What properties do we expect these functions to have? Can you deduce the properties from your definitions?

Integration [6]

- How would you rigorously define integration?
- What properties do you expect the integral to have?
- Do you expect to be able to integrate every function? Give examples of integrable and non-integrable functions.
- How are differentiation and integration linked? Can you prove anything about this?

Please send comments and corrections to me.

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