Analysis I — Bolzano-Weierstrass theorem proofsorter

Lent Term 2013 *V. Neale*

We saw one proof of the Bolzano-Weierstrass theorem in lectures, and mentioned another approach as an exercise. I encourage you to try to produce a second proof along these lines for yourself. When you have done so, or when you are revising the course, you might like to try this 'proofsorter' activity. I have written out the two proofs and then jumbled them, so your job is to separate out the two proofs and within each to put the statements in the correct order. You might want to print the sheet of statements so that you can cut them up.

Here is a reminder of the statement of the theorem.

Theorem. *(Bolzano-Weierstrass Theorem) Let* $(x_n)_{n=1}^{\infty}$ *be a bounded sequence of real numbers, say* $|x_n| \leqslant K$ *for all* n*. Then there is a convergent subsequence: that is, there are* $n_1 < n_2 < n_3 < \cdots$ *and a real number* x *such that* $x_{n_j} \to x$ *as* $j \to \infty$.

Please e-mail me with comments, suggestions and queries (v.r.neale@dpmms.cam.ac.uk). You are also welcome to leave a comment on the course blog.

If x_{n_i} and x_{n_j} are summits and $n_j > n_i$, then we must have $x_{n_i} \geqslant x_{n_j}$.
If there are only finitely many summits, then there is a term beyond which there are no summits, say x_m.
Since a_j, $b_j \to a$ as $n \to \infty$, we also have $x_{n_j} \to a$ as $j \to \infty$, so this is a convergent subsequence.
Consider the subsequence of summits x_{n_1}, x_{n_2}, ...
This gives an interval $[a_{n+1}, b_{n+1}]$ of length $\frac{1}{2}(b_n - a_n) = \frac{1}{2^{n+1}}(b_0 - a_0)$ that contains infinitely many terms of the sequence.
If not, then set $a_{n+1} = c_n$, $b_{n+1} = b_n$.
Having obtained $n_0 < n_1 < \cdots < n_j$ such that $x_{n_i} \in [a_i, b_i]$ for $0 \leqslant i \leqslant j$, choose $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$.
Each interval $[a_n, b_n]$ contains infinitely many of the x_i.
This gives an increasing subsequence $(x_{n_i})_{i=1}^{\infty}$.
The sequence $(a_n)_{n=1}^{\infty}$ is increasing (by construction) and is bounded above (e.g. by b_0), so converges, say $a_n \to a$ as n_∞.
Say that x_n is a *summit* if it is at least as high as all subsequent points. That is, x_n is a summit exactly when $x_n \geqslant x_m$ for all $m \geqslant n$.
Given x_{n_j} that is not a summit, there is some $n_{j+1} > n_j$ with $x_{n_{j+1}} > x_{n_j}$.
Having obtained an interval $[a_n, b_n]$ of length $\frac{1}{2^n}(b_0 - a_0)$ that contains infinitely many terms of the sequence, set $c_n = (a_n + b_n)/2$.
Now $b_n = a_n + \frac{1}{2^n}(b_0 - a_0) \to a$ as $n \to \infty$ too.
Since x_{n_1} is not a summit, there is some $n_2 > n_1$ with $x_{n_2} > x_{n_1}$.
If there are infinitely many summits, then they form a decreasing subsequence.
Either way, we have a bounded monotone subsequence, which must converge.
If $[a_n, c_n]$ contains infinitely many terms of the sequence, then set $a_{n+1} = a_n$, $b_{n+1} = c_n$.
Let $a_0 = -K$, $b_0 = K$. The interval $[a_0, b_0]$ contains infinitely many terms of the sequence.
Choose n_0 such that $x_{n_0} \in [a_0, b_0]$.
Let $n_1 = m + 1$.
This gives a subsequence $(x_{n_j})_{j=0}^{\infty}$ with $a_j \leqslant x_{n_j} \leqslant b_j$ for all j.