Analysis I — Examples Sheet 1

Lent Term 2013 *V. Neale*

1. Let (a_n) be a sequence of real numbers. We say that $a_n \to \infty$ as $n \to \infty$ if given any K we can find an N (depending on K) such that $a_n \geqslant K$ for all $n \geqslant N$.

   (i) Write down a similar definition for $a_n \to -\infty$ as $n \to \infty$.

   (ii) Show that $a_n \to -\infty$ as $n \to \infty$ if and only if $-a_n \to \infty$ as $n \to \infty$.

   (iii) Suppose that $a_n \neq 0$ for all n. Show that if $a_n \to \infty$ as $n \to \infty$ then $\frac{1}{a_n} \to 0$ as $n \to \infty$.

   (iv) Suppose that $a_n \neq 0$ for all n. Is it true that if $\frac{1}{a_n} \to 0$ as $n \to \infty$ then $a_n \to \infty$ as $n \to \infty$? Give a proof or a counterexample.

2. Sketch the graphs of $y = x$ and $y = (x^4 + 1)/3$, and thereby illustrate the behaviour of the real sequence (a_n) where $a_{n+1} = (a_n^4 + 1)/3$. For which of the three starting cases $a_1 = 0$, $a_1 = 1$, $a_1 = 2$ does the sequence converge? Prove your assertions (rigorously — a picture is useful for intuition but insufficient for a proof).

3. Let $a_1 > b_1 > 0$ and let $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = 2a_n b_n/(a_n + b_n)$ for $n \geqslant 1$. Show that $a_n > a_{n+1} > b_{n+1} > b_n$. Deduce that the two sequences converge to a common limit. What is that limit?

4. Let $[a_n, b_n]$, $n = 1, 2, ...,$ be closed intervals with $[a_n, b_n] \cap [a_m, b_m] \neq \emptyset$ for all n, m. Prove that $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

5. The real sequence (a_n) is bounded but does not converge. Prove that it has two convergent subsequences with different limits.

6. Investigate the convergence of the following series. For each expression containing the complex number z, find all z for which the series converges.

$$\sum_n \frac{\sin n}{n^2} \qquad \sum_n \frac{n^2 z^n}{5^n} \qquad \sum_n \frac{(-1)^n}{4 + \sqrt{n}} \qquad \sum_n \frac{z^n(1-z)}{n}$$

7. Consider the two series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$, having the same terms but taken in a different order. Let s_n and t_n be the corresponding partial sums to n terms. Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$. Show that $s_{2n} = H_{2n} - H_n$ and $t_{3n} = H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n$. Show that (s_n) converges to a limit, say s, and that (t_n) converges to $3s/2$.

8. Let (a_n) be a sequence of complex numbers. Define $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$ for all $n \geqslant 1$. Show that if $a_n \to a$ as $n \to \infty$ then $b_n \to a$ as $n \to \infty$ also.

9. Show that $\sum_n \frac{1}{n(\log n)^\alpha}$ converges if $\alpha > 1$ and diverges otherwise.

   Does $\sum_n \frac{1}{n \log n \log\log n}$ converge?

10. Prove the *root test*, which says the following.

    Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geqslant 0$ for all n. Suppose that there is some a such that $a_n^{1/n} \to a$ as $n \to \infty$. If $a < 1$, then the series converges. If $a > 1$, then the series diverges.

    What happens if $a = 1$?

    Add this test to your series grid.

11. Let z be a complex number such that $z^{2^j} \neq 1$ for every positive integer j. Show that the series

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \cdots$$

    converges to $\frac{z}{1-z}$ if $|z| < 1$ and converges to $\frac{1}{1-z}$ if $|z| > 1$. What happens if $|z| = 1$?

12. Let (a_n) be a sequence of positive real numbers such that $\sum_n a_n$ diverges. Show that there exist b_n with $\frac{b_n}{a_n} \to 0$ as $n \to \infty$ and $\sum_n b_n$ divergent.

13. Can we write the open interval $(0, 1)$ as a disjoint union of closed intervals of positive length?

Please e-mail me with comments, suggestions and queries (v.r.neale@dpmms.cam.ac.uk).