Analysis I — Examples Sheet 2

Lent Term 2013 *V. Neale*

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ if x is rational and $f(x) = 1 - x$ otherwise. Find $\{a : f \text{ is continuous at } a\}$.

2. Prove that $f(x) \to \infty$ as $x \to \infty$ if and only if $f(x_n) \to \infty$ for every sequence such that $x_n \to \infty$.

3. Suppose that $f(x) \to \ell$ as $x \to a$ and $g(y) \to k$ as $y \to \ell$. Give an example to show that it is not necessarily true that $g(f(x)) \to k$ as $x \to a$.

4. Let $f_n : [0,1] \to [0,1]$ be continuous, for each natural number n. Let $h_n(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$. Show that h_n is continuous on $[0,1]$ for each natural number n. Must $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous?

5. Let $g : [0,1] \to [0,1]$ be a continuous function. By considering $f(x) = g(x) - x$, or otherwise, show that there is some c in $[0,1]$ such that $g(c) = c$. [So every continuous function from $[0,1]$ to itself has a fixed point.]

   Give an example of a bijective function $h : [0,1] \to [0,1]$ such that $h(x) \neq x$ for all $x \in [0,1]$.

   Give an example of a continuous function $p : (0,1) \to (0,1)$ such that $p(x) \neq x$ for all $x \in (0,1)$.

6. The unit circle in $\mathbb{C}$ is mapped to $\mathbb{R}$ by a map $e^{i\theta} \to f(\theta)$, where $f : [0, 2\pi] \to \mathbb{R}$ is continuous and $f(0) = f(2\pi)$. Show that there exist two diametrically opposite points that have the same image.

7. Let $f : [0,1] \to \mathbb{R}$ be continuous, with $f(0) = f(1) = 0$. Suppose that for every $x \in (0,1)$ there exists δ with $0 < \delta < \min\{x, 1-x\}$ and $f(x) = \big(f(x-\delta) + f(x+\delta)\big)/2$. Show that $f(x) = 0$ for all x.

8. Let $f : [a,b] \to \mathbb{R}$ be bounded. Suppose that $f\big((x+y)/2\big) \leqslant \big(f(x) + f(y)\big)/2$ for all x, $y \in [a,b]$. Prove that f is continuous on (a,b). Must it be continuous at a and b too?

9. Prove that $2x^5 + 3x^4 + 2x + 16 = 0$ has no real solutions outside $[-2,-1]$ and exactly one inside.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Which of the following statements must be true?

   (i) If f is increasing then $f'(x) \geqslant 0$ for all $x \in (a, b)$.

   (ii) If $f'(x) \geqslant 0$ for all $x \in (a, b)$ then f is increasing.

   (iii) If f is strictly increasing then $f'(x) > 0$ for all $x \in (a, b)$.

   (iv) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing.

11. (i) Let $f : [0, 1] \to \mathbb{R}$ be twice differentiable with $f''(t) \geqslant 0$ for all $t \in [0, 1]$.
   If $f'(0) > 0$ and $f(0) = 0$, then explain why $f(t) > 0$ for $t > 0$.
   If $f'(0) \geqslant 0$ and $f(0) = f(1) = 0$, then what can you say about f and why?
   If $f'(1) \leqslant 0$ and $f(0) = f(1) = 0$, then what can you say about f and why?

   (ii) Let $f : [0, 1] \to \mathbb{R}$ be twice differentiable with $f''(t) \geqslant 0$ for all $t \in [0, 1]$ and with $f(0) = f(1) = 0$. Show that $f(t) \leqslant 0$ for all $t \in [0, 1]$.

   (iii) Let $g : [a, b] \to \mathbb{R}$ be twice differentiable with $g''(t) \geqslant 0$ for all $t \in [a, b]$. By considering the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(t) = g\big((1 - t)a + tb\big) - (1 - t)g(a) - tg(b),$$

   show that

$$g\big((1 - t)a + tb\big) \leqslant (1 - t)g(a) + tg(b)$$

   for all $t \in [0, 1]$.

   [So a twice-differentiable function with everywhere positive second derivative is *convex*. See Jensen's inequality in the Probability course for more about convex functions. But note that not all convex functions are twice differentiable.]

12. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable for all x. Prove that if $f'(x) \to \ell$ as $x \to \infty$ then $f(x)/x \to \ell$. If $f(x)/x \to \ell$ as $x \to \infty$, must $f'(x)$ tend to a limit?

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has the intermediate value property: if $f(a) < c < f(b)$ then $c = f(x)$ for some x between a and b. Suppose also that for every rational r, the set S_r of all x with $f(x) = r$ is closed (that is, if (x_n) is any sequence in S_r with $x_n \to a$ as $n \to \infty$ then $a \in S_r$). Prove that f is continuous.

Please e-mail me with comments, suggestions and queries (v.r.neale@dpmms.cam.ac.uk).