Analysis I — Examples Sheet 3

Lent Term 2013 *V. Neale*

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leqslant |x - y|^2$ for all $x, y \in \mathbb{R}$. Show that f is constant.

2. Given $\alpha \in \mathbb{R}$, define $f_\alpha : [-1, 1] \to \mathbb{R}$ by $f_\alpha(x) = |x|^\alpha \sin(1/x)$ for $x \neq 0$ and $f_\alpha(0) = 0$. Is f_0 continuous? Is f_1 differentiable? Draw a table, with nine columns labelled $-\frac{1}{2}$, 0, $\frac{1}{2}$, $\ldots$, $\frac{7}{2}$ and with six rows labelled "f_α bounded", "f_α continuous", "f_α differentiable", "f'_α bounded", "f'_α continuous", "f'_α differentiable". Place ticks and crosses at appropriate places in the table, and give justifications.

   (If you wish, start by considering $g_\alpha(x) = x^\alpha \sin(1/x)$ for $x \neq 0$ and $g_\alpha(0) = 0$, for $\alpha \in \{0, 1, 2, 3\}$.)

3. By applying the mean value theorem to $\log(1 + x)$ on $[0, a/n]$ with $n > |a|$, prove carefully that $(1 + a/n)^n \to e^a$ as $n \to \infty$.

4. Find $\lim\limits_{n\to\infty} n(a^{1/n} - 1)$, where $a > 0$.

5. "Let f' exist on (a, b) and let $c \in (a, b)$. If $c + h \in (a, b)$ then $\big(f(c + h) - f(c)\big)/h = f'(c + \theta h)$. Let $h \to 0$; then $f'(c + \theta h) \to f'(c)$. Thus f' is continuous at c." Is this argument correct?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(0) = 0$. Show that f is infinitely differentiable (including at 0), and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Comment, in the light of what you know about Taylor series.

7. Find the radius of convergence of each of these power series.

$$\sum_{n\geqslant 0} \frac{2 \cdot 4 \cdot 6 \cdots (2n + 2)}{1 \cdot 4 \cdot 7 \cdots (3n + 1)} z^n \qquad \sum_{n\geqslant 1} \frac{z^{3n}}{n 2^n} \qquad \sum_{n\geqslant 0} \frac{n^n z^n}{n!} \qquad \sum_{n\geqslant 1} n^{\sqrt{n}} z^n$$

8. Find the derivative of $\tan x$. How do you know that there is a differentiable inverse function $\arctan x$ for $x \in \mathbb{R}$? What is its derivative?

   Now let $g(x) = x - x^3/3 + x^5/5 - \cdots$ for $|x| < 1$. By considering $g'(x)$, explain carefully why $\arctan x = g(x)$ for $|x| < 1$.

9. (L'Hôpital's rule.) Let $f,\ g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f(a) = g(a) = 0$, that $g'(x)$ does not vanish near a, and that $f'(x)/g'(x) \to \ell$ as $x \to a$. Show that $f(x)/g(x) \to \ell$ as $x \to a$.

   Use the rule with $g(x) = x - a$ to show that if $f'(x) \to \ell$ as $x \to a$, then f is differentiable at a with $f'(a) = \ell$.

   Find a pair of functions f and g satisfying the conditions above for which $\lim_{x \to a} f(x)/g(x)$ exists, but $\lim_{x \to a} f'(x)/g'(x)$ does not.

   Investigate the limit as $x \to 1$ of $\dfrac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$.

10. The *infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ is said to *converge* if the sequence $p_n = (1+a_1)\cdots(1+a_n)$ converges. Suppose that $a_n \geqslant 0$ for all n. Write $s_n = a_1 + \cdots + a_n$. Prove that $s_n \leqslant p_n \leqslant e^{s_n}$, and deduce that $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

    Evaluate $\prod_{n=2}^{\infty}\bigl(1 + 1/(n^2-1)\bigr)$.

11. Let f be continuous on $[-1,1]$ and twice differentiable on $(-1,1)$. Let $\phi(x) = \bigl(f(x) - f(0)\bigr)/x$ for $x \neq 0$ and $\phi(0) = f'(0)$. Show that ϕ is continuous on $[-1,1]$ and differentiable on $(-1,1)$.

    By using a second-order mean value theorem for f, show that $\phi'(x) = f''(\theta x)/2$ for some $0 < \theta < 1$. Hence prove that there exists $c \in (-1,1)$ with $f''(c) = f(-1) + f(1) - 2f(0)$.

12. We say that f' *has the property of Darboux* if $a < b$ and $f'(a) < z < f'(b)$ means that there is some c with $a < c < b$ and $f'(c) = z$.

    Prove the theorem of Darboux: that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' has the property of Darboux.

    Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that there does not exist a differentiable function $F : \mathbb{R} \to \mathbb{R}$ with $F' = f$.

13. (i) Show that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence 1, that it converges for all z with $|z| = 1$ and $z \neq 1$, and that it diverges if $z = 1$.

    (ii) Let $|z_1| = |z_2| = \cdots = |z_m| = 1$. Find a power series $\sum_{n=0}^{\infty} a_n z^n$ that has radius of convergence 1, that converges for all z with $|z| = 1$ and $z \notin \{z_1, z_2, \cdots, z_m\}$, but that diverges if $z = z_j$ for some $1 \leqslant j \leqslant m$.

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