Analysis I — Examples Sheet 4

Lent Term 2013 *V. Neale*

1. Show directly from the definition of an integral that $\int_0^a x^2 \, \mathrm{d}x = a^3/3$ for $a > 0$.

2. Let $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Does $\int_0^1 f$ exist?

3. Give an example of a continuous function $f : [0, \infty) \to [0, \infty)$ such that $\int_0^\infty f$ exists but f is unbounded.

4. Give an example of an integrable function $f : [0, 1] \to \mathbb{R}$ with $f \geqslant 0$ and $\int_0^1 f = 0$ and $f(x) > 0$ for some value of x.

   Show that this cannot happen if f is continuous.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic. Show that $\{x \in \mathbb{R} : f \text{ is discontinuous at } x\}$ is countable.

   Let $(x_n)_{n=1}^\infty$ be a sequence of distinct points in $(0, 1]$. Let $f_n(x) = 0$ if $0 \leqslant x < x_n$ and $f_n(x) = 1$ if $x_n \leqslant x \leqslant 1$. Let $f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$. Show that this series converges for every $x \in [0, 1]$.

   Show that f is increasing (and so is integrable).

   Show that f is discontinuous at every x_n.

6. Let $f(x) = \log(1 - x^2)$. Use the mean value theorem to show that $|f(x)| \leqslant 8x^2/3$ for $0 \leqslant x \leqslant 1/2$.

   Now let $I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \log x \, \mathrm{d}x - \log n$ for $n \in \mathbb{N}$. Show that $I_n = \int_0^{\frac{1}{2}} f(t/n) \, \mathrm{d}t$ and hence that $|I_n| \leqslant \dfrac{1}{9n^2}$.

   By considering $\sum_{j=1}^{n} I_j$, deduce that $\dfrac{n!}{n^{n+\frac{1}{2}} e^{-n}} \to \ell$ for some constant ℓ.

7. Let $I_n = \int_0^{\pi/2} \cos^n x \, \mathrm{d}x$. Prove that $nI_n = (n-1)I_{n-2}$, and hence that $\frac{2n}{2n+1} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$.

   Deduce Wallis's product:

   $$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)} = \lim_{n\to\infty} \frac{2^{4n}}{2n+1} \binom{2n}{n}^{-2}.$$

   By taking note of the previous exercise, prove that $\dfrac{n!}{n^{n+\frac{1}{2}} e^{-n}} \to \sqrt{2\pi}$ (Stirling's formula).

8. Do these improper integrals converge?

   (i) $\int_1^\infty \sin^2(1/x)\,\mathrm{d}x$.

   (ii) $\int_0^\infty x^p \exp(-x^q)\,\mathrm{d}x$ where $p,\ q > 0$.

9. Show that $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \to \log 2$ as $n \to \infty$, and find the limit as $n \to \infty$ of $\frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{(-1)^{n-1}}{2n}$.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $\int_a^b f(x)g(x)\,\mathrm{d}x = 0$ for every continuous function $g : [a, b] \to \mathbb{R}$ with $g(a) = g(b) = 0$. Must f vanish identically?

11. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative, that $f(0) = 0$, and that $|f'(x)| \leqslant M$ for $x \in [0, 1]$. Prove carefully that $|\int_0^1 f| \leqslant M/2$. Prove carefully that if, in addition, $f(1) = 0$, then $|\int_0^1 f| \leqslant M/4$. What could you say (and prove) if $|f'(x)| \leqslant Mx$ for all $x \in [0, 1]$?

12. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Let $G(x, t) = t(x - 1)$ for $t \leqslant x$ and $G(x, t) = x(t - 1)$ for $t \geqslant x$. Let $g(x) = \int_0^1 f(t)G(x, t)\,\mathrm{d}t$. Show that $g''(x)$ exists for $x \in (0, 1)$ and equals $f(x)$.

13. Let $I_n(\theta) = \int_{-1}^1 (1 - x^2)^n \cos(\theta x)\,\mathrm{d}x$. Prove that $\theta^2 I_n = 2n(2n - 1)I_{n-1} - 4n(n - 1)I_{n-2}$ for $n \geqslant 2$, and hence that $\theta^{2n+1} I_n(\theta) = n!\big(P_n(\theta)\sin\theta + Q_n(\theta)\cos\theta\big)$, where P_n and Q_n are polynomials of degree at most $2n$ with integer coefficients.

    Deduce that π is irrational.

14. Let $f_1,\ f_2 : [-1, 1] \to \mathbb{R}$ be increasing, and let $g = f_1 - f_2$. Show that there is some K such that for any dissection $\mathcal{D} = \{x_0 < x_1 < \cdots < x_n\}$ of $[-1, 1]$, we have $\sum_{j=1}^{n} |g(x_j) - g(x_{j-1})| \leqslant K$.

    Now let $g(x) = x\sin(1/x)$ for $x \neq 0$ and $g(0) = 0$. Show that g is integrable, but that it is not the difference of two increasing functions.

15. Show that if $f : [0, 1] \to \mathbb{R}$ is integrable then f has infinitely many points of continuity.

Please e-mail me with comments, suggestions and queries (v.r.neale@dpmms.cam.ac.uk).