

## Analysis I — Intermediate value theorem proofsorter

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We saw one proof of the Intermediate value theorem in lectures, and mentioned another approach as an exercise. I encourage you to try to produce a second proof along these lines for yourself. When you have done so, or when you are revising the course, you might like to try this ‘proofsorter’ activity. I have written out the two proofs and then jumbled them, so your job is to separate out the two proofs and within each to put the statements in the correct order. You might want to print the sheet of statements so that you can cut them up.

Here is a reminder of the statement of the theorem.

**Theorem.** (*Intermediate Value Theorem*) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function that is continuous on  $[a, b]$  with  $f(a) < 0 < f(b)$ . Then there is  $c \in (a, b)$  such that  $f(c) = 0$ .

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Then  $b_n = a_n + \frac{1}{2^n}(b_0 - a_0) \rightarrow A$  as  $n \rightarrow \infty$ .

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Since  $f(z_n) < 0$  for all  $n$ , we have  $f(s) \leq 0$ .

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Since  $f$  is continuous at  $A$ , we have  $f(a_n) \rightarrow f(A)$  and  $f(b_n) \rightarrow f(A)$  as  $n \rightarrow \infty$ .

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Now  $(a_n)_{n=1}^\infty$  is increasing (by construction) and bounded above (e.g. by  $b$ ), so converges, say  $a_n \rightarrow A$  as  $n \rightarrow \infty$ . Note that  $a \leq A \leq b$ .

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If  $s = b$ , then there is a sequence  $(x_n)_{n=1}^\infty$  in  $S$  with  $x_n \rightarrow b$  as  $n \rightarrow \infty$ .

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Then  $a \in S$  so  $S \neq \emptyset$ .

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Let  $S = \{x \in [a, b] : f(x) < 0\}$ .

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So  $f(s) = 0$ .

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If  $f(x_n) > 0$ , then set  $a_{n+1} = a_n$  and  $b_{n+1} = x_n$ .

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Then  $f(a_{n+1}) < 0 < f(b_{n+1})$  and  $b_{n+1} - a_{n+1} = \frac{1}{2^{n+1}}(b_0 - a_0)$ .

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If  $f(x_n) = 0$ , then take  $c = x_n$  — done.

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If  $f(x_n) < 0$ , then set  $a_{n+1} = x_n$  and  $b_{n+1} = b_n$ .

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We see that  $S$  is bounded above (e.g. by  $b$ ).

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Then  $z_n \rightarrow s$  as  $n \rightarrow \infty$ .

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So  $s > a$ .

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Given  $a \leq a_n < b_n \leq b$  with  $f(a_n) < 0 < f(b_n)$  and  $b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$ , put  $x_n = (a_n + b_n)/2$ .

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Since  $f(x_n) < 0$  for all  $n$ , we have  $f(b) \leq 0$ , which is a contradiction.

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But  $f$  is continuous at  $s$ , so  $f(z_n) \rightarrow f(s)$  as  $n \rightarrow \infty$ .

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Since  $s = \sup S$  and  $s > a$ , for each natural number  $n$  there is some  $z_n$  in  $S$  such that  $s - \frac{1}{n} < z_n < s$ .

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So  $s < b$ .

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Also,  $f(b_n) > 0$  for all  $n$ , so  $f(A) \geq 0$ .

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But  $f(a_n) < 0$  for all  $n$ , so  $f(A) \leq 0$ .

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So  $S$  has a supremum. Let  $s = \sup S$ .

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Let  $a_0 = a$  and  $b_0 = b$ . Then  $f(a_0) < 0 < f(b_0)$ .

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Then  $f(w_n) \geq 0$  for all  $n$  and  $f(w_n) \rightarrow f(s)$  as  $n \rightarrow \infty$ , so  $f(s) \geq 0$ .

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If  $s = a$ , then there is a sequence  $(y_n)_{n=1}^\infty$  in  $[a, b]$  with  $y_n \rightarrow a$  as  $n \rightarrow \infty$  and  $f(y_n) \geq 0$  for all  $n$ , so  $f(a) \geq 0$ . This is a contradiction.

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Since  $s < b$ , there is a sequence  $(w_n)_{n=1}^\infty$  with  $w_n \rightarrow s$  as  $n \rightarrow \infty$  and  $s < w_n \leq b$  for all  $n$ .

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So  $f(A) = 0$ , so can take  $c = A$ .

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