Higher Green's Function and Arithmetic Application

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Number Fields.

In number theory, it is an interesting question to explicitly generate number fields. Special values of special functions are useful.
Let

$$f(x) := \mathbf{e}(x) := e^{2\pi i x}.$$

Then

$$x \in \mathbb{Q} \Longrightarrow f(x) \in \mathbb{Q}^{\mathrm{ab}} \subset \overline{\mathbb{Q}}.$$

In fact, every abelian extension of $\mathbb{Q}$ is contained in a field generated by $f(x)$ for some $x \in \mathbb{Q}$ (Kronecker-Weber Theorem).

Number Fields.

Consider the upper half-plane

$$\mathcal{H} := \{z = x + iy \in \mathbb{C} : y > 0\},$$

which is acted on by $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Denote $Y := \Gamma\backslash\mathcal{H}$.

Definition 1

A point $z \in \mathcal{H}$ is called a CM point if it satisfies a quadratic equation $Az^2 + Bz + C = 0$ with $A, B, C \in \mathbb{Z}$.
The quantity $\Delta := B^2 - 4AC < 0$ is called the discriminant of z.

The following set is finite

$$Z(\Delta) := \{[z] \in Y : \mathrm{disc}(z) = \Delta\}.$$

with a faithfully transitive action by the class group of the order $\mathcal{O}_\Delta := \mathbb{Z} + \mathbb{Z}\frac{\Delta+\sqrt{\Delta}}{2}$ in the imaginary quadratic field $K := \mathbb{Q}(\sqrt{\Delta})$.

Modular Function.

Recall the Klein j-invariant $j : Y \to \mathbb{C}$

$$j(z) := q^{-1} + 744 + 196884q + \dots, \quad q := e^{2\pi i z}.$$

If z is a CM point, $j(z)$ is called a singular moduli. The theory of complex multiplication implies

- $z \in Z(\Delta) \Rightarrow j(z) \in K(\mathcal{O}_\Delta) \subset K^{\mathrm{ab}}$ with $K(\mathcal{O}_\Delta)$ the ring class field for $\mathcal{O}_\Delta$.
- The set $\{j(z) : z \in Z(\Delta)\}$ are all Galois conjugates over $\mathbb{Q}$.

For example:

Δ	$Z(\Delta)$	$j(z)$ for $z \in Z(\Delta)$
-7	$\left\{\frac{1+\sqrt{-7}}{2}\right\}$	$-3375 = -3^3 \cdot 5^3$
-23	$\left\{\frac{1+\sqrt{-23}}{2}, \frac{\pm 1+\sqrt{-23}}{4}\right\}$	roots of $x^3 + 3491750x^2 - 5151296875x + 5^9 \cdot 11^3 \cdot 17^3 = 0$

Remark: all abelian extensions of K can be constructed using values of Weber functions at CM points.

Norms of Singular Moduli

Suppose $\Delta_1, \Delta_2 < 0$ are co-prime, fundamental discriminants. Denote

$$F := \mathbb{Q}(\sqrt{\Delta}) \subset \mathbb{R}, \quad \Delta := \Delta_1 \Delta_2,$$
$$K_i := \mathbb{Q}(\sqrt{\Delta_i}), \quad K := K_1 K_2 \supset F,$$
$$\chi : \text{genus character corresponding to } K/F,$$
$$\mathcal{O}_F \subset F : \text{ring of integers of } F,$$
$$w_i : \text{number of roots of unities in } K_i,$$
$$Z_\chi := \frac{4}{w_1 w_2} \cdot \sum_{z_1 \in Z(\Delta_1), z_2 \in Z(\Delta_2)} (z_1, z_2) \in \mathrm{Div}(Y^2).$$

Define a harmonic function on $Y^2 \backslash \mathrm{Diagonal}$

$$G_1(z_1, z_2) := \log |j(z_1) - j(z_2)|^2$$

with singularity along the diagonal. It's a Green's function for Y.

Norms of Singular Moduli

Theorem (Gross-Zagier, 1985)

In the notations above, we have

$$G_1(Z_\chi) = \sum_{\lambda \in S_1} \sigma_\chi^+(\lambda),$$

where S_1 is the finite set

$$S_m := \left\{ \lambda \in \mathcal{O}_F : \lambda\sqrt{\Delta} \gg 0, \lambda \in \frac{1}{2}\mathbb{Z} + \frac{m}{2}\sqrt{\Delta} \right\}, \ m \geq 1,$$

and for an ideal $\mathfrak{a} \subset \mathcal{O}_F$, we have

$$\sigma_\chi^+(\mathfrak{a}) := -\sum_{\mathfrak{b}|\mathfrak{a}} \chi(\mathfrak{b}) \log \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{b}) \geq 0.$$

Norms of Singular Moduli

Example ($\Delta_1 = -7, \Delta_2 = -23$)

.

$$\frac{1}{2} G_1(Z_\chi) \overset{GZ}{=} -\frac{1}{2} \sum_{a \in \mathbb{Z},\ |a| < \sqrt{7 \cdot 23}} \sigma_\chi^+ \left(\frac{a + \sqrt{7 \cdot 23}}{2} \right)$$

$$= \sigma_\chi^+ \left(\tfrac{1+\sqrt{161}}{2} \right) + \sigma_\chi^+ \left(\tfrac{3+\sqrt{161}}{2} \right) + \sigma_\chi^+ \left(\tfrac{5+\sqrt{161}}{2} \right)$$

$$\sigma_\chi^+ \left(\tfrac{7+\sqrt{161}}{2} \right) + \sigma_\chi^+ \left(\tfrac{9+\sqrt{161}}{2} \right) + \sigma_\chi^+ \left(\tfrac{11+\sqrt{161}}{2} \right)$$

$$= 4 \log 5 + 2 \log 19 + 2 \log 17 + 3 \log 7 + 3 \log 5 + 2 \log 5$$

$$= 9 \log 5 + 3 \log 7 + 2 \log 17 + 2 \log 19.$$

Sketch of (Analytic) Proof.

1. Realize $G_1(z_1, z_2)$ as a regularized theta integral (Borcherds 1997)

$$G_1(z_1, z_2) = \int_Y^{\mathrm{reg}} (j(z) - 744)\Theta(z, (z_1, z_2)) \frac{d\overline{z} \wedge dz}{2iy^2}.$$

2. Apply the Siegel-Weil formula to conclude

$$\Theta(z, Z_\chi) = E_\chi(z)$$

with E_χ a real-analytic modular function.

3. Construct a modular form $\widetilde{E}_\chi(z)$ of weight 2 such that

$$L_2 \widetilde{E}_\chi(z) = E_\chi(z)$$

with $L_2 := -2iy^2 \partial_{\overline{z}}$ the weight lowering operator.

4. Apply Stokes' theorem to express $G_1(Z_\chi)$ in terms of the Fourier coefficients of $\widetilde{E}_\chi(z)$.

Higher Green's function.

Denote

- $Q_{s-1}(t) := \int_0^\infty (t + \sqrt{t^2-1}\cosh v)^{-s} dv$ the Legendre function of the second kind,
- $d(\cdot,\cdot)$ the hyperbolic distance on $\mathcal{H}$.

Consider the Poincaré series

$$G_s(z_1, z_2) := -\sum_{\gamma\in\Gamma} Q_{s-1}(\cosh d(z_1, \gamma z_2)), \ \mathrm{Re}(s) > 1,$$

which is an eigenfunction under the Laplacian of z_1 (and of z_2) with eigenvalue $s(1-s)$.

$$G_1(z_1, z_2) = \text{``}\lim_{s\to 1} G_s(z_1, z_2)\text{''}.$$

When $s = k \geq 2$ is an integer, $G_k(z_1, z_2)$ is called the higher Green's function (Gross-Kohnen-Zagier 1987, Zhang 1997).

Conjecture of Gross-Zagier.

Let $f \in M^!_{2-2k}$ be a modular form with integral Fourier coefficients $c_f(m)$'s. Define

$$G_{k,f}(z_1, z_2) := \sum_{m \geq 1} c_f(-m) m^{k-1} G_k(z_1, z_2) \mid T_m$$

with T_m the m^{th} Hecke operator. If $(z_1, z_2) \in Y^2 \setminus \cup_{m, c_f(-m) \neq 0} T_m$ is a pair of CM points, then

$$G_{1,f}(z_1, z_2) = \log |\alpha|$$

for some $\alpha \in \overline{\mathbb{Q}}$.

Conjecture. (Gross-Zagier)

Let $k \geq 2$ and $f \in M^!_{2-2k}$ be as above. For distinct CM points $z_i \in Z(\Delta_i)$, there exists $\alpha \in \overline{\mathbb{Q}} \subset \mathbb{C}$ such that

$$G_{k,f}(z_1, z_2) = \Delta^{(1-k)/2} \log |\alpha|.$$

Conjecture of Gross-Zagier.

Progress on original conjecture.

- $\Delta_1 = \Delta_2$ (Zhang 1997, Viazovska 2011).
- $\Delta_1 = -4$, Δ_2 arbitrary and $k = 2$ (A. Mellit's thesis 2008).

Progress on averaged versions.

- Averaged over one Galois orbit, k odd and even (Bruinier-Ehlen-Yang in progress).
- Averaged over both Galois orbits, k odd (Gross-Kohnen-Zagier 1987), in which case $\alpha \in \mathbb{Q}$.

Conjecture of Gross-Zagier.

Furthermore, an explicit factorization is given.

Theorem (Gross-Kohnen-Zagier)

For fundamental Δ_1, Δ_2 and odd $k \in \mathbb{N}$, we have

$$G_{k,f}(Z_\chi) = \sum_{m \geq 1} \frac{c_f(-m)}{\Delta^{(k-1)/2}} \sum_{\lambda \in S_m} C_k(\lambda, m) \sigma_\chi^+(\lambda),$$

where $C_k(\lambda, m) := (m\sqrt{\Delta})^{k-1} P_{k-1}\left(\frac{\mathrm{Tr}(\lambda)}{m\sqrt{\Delta}}\right) \in \mathbb{Z}$ with $P_{k-1}(t)$ the $(k-1)^{th}$ Legendre polynomial.

Example when k is even (Mellit)

$$G_2\left(i, \frac{1+i\sqrt{11}}{2}\right) = \frac{4}{\sqrt{11}} \log \left| \frac{(\sqrt{11}+3)\cdot(\sqrt{11}+2)^2}{(-\sqrt{11}+3)\cdot(-\sqrt{11}+2)^2} \right|.$$

Main Results.

Theorem 1 (L, 2018)

Let $k \in \mathbb{Z}_{\geq 2}$ be an even integer, $\Delta_1, \Delta_2 < 0$ co-prime fundamental discriminants, and $F := \mathbb{Q}(\sqrt{\Delta})$ with $\Delta = \Delta_1 \Delta_2$. For $f \in M^!_{2-2k}$ with $c_f(m) \in \mathbb{Z}$ for all $m \in \mathbb{Z}$, let $G_{k,f}(z_1, z_2)$ be as above. Then there exists $\alpha^- \in F^\times$ and $\kappa \in \mathbb{N}$ such that

- $\mathrm{Nm}_{F/\mathbb{Q}}(\alpha^-) = 1$
- $\kappa \cdot G_{k,f}(Z_\chi) = \Delta^{(1-k)/2} \log(\alpha^-)$.

Remark

The constant κ can be bounded explicitly and only depends on F and k.

Factorizations.

For each ideal $\mathfrak{a} \subset \mathcal{O}_F$, we can rewrite

$$\sigma_\chi^+(\mathfrak{a}) = \log \alpha^+(\mathfrak{a})$$

with $\alpha^+(\mathfrak{a}) \in \mathbb{N}$ *the* generator of the ideal

$$I^+(\mathfrak{a}) := \prod_{\mathfrak{b}|\mathfrak{a}} (\mathfrak{b} \cdot \mathfrak{b}')^{\chi(\mathfrak{b})}.$$

Then the result of Gross-Kohnen-Zagier becomes

Theorem (Gross-Kohnen-Zagier)

Let $\alpha^+ \in \mathbb{Q}_{>0}$ be the generator of the ideal

$$\prod_{m \geq 1} \prod_{\lambda \in S_m} I^+(\lambda)^{-c_f(-m)C_k(\lambda, m)}.$$

Then $G_{k,f}(Z_\chi) = \Delta^{(1-k)/2} \log(\alpha^+)$ for k odd.

Factorizations.

For each ideal $\mathfrak{a} \subset \mathcal{O}_F$, define the fractional ideal

$$I^-(\mathfrak{a}) := \prod_{\mathfrak{b}|\mathfrak{a}} \left(\frac{\mathfrak{b}}{\mathfrak{b}'}\right)^{\chi(\mathfrak{b})},$$

which is not necessarily principal.

Theorem 2 (L, 2018)

Let $\kappa \in \mathbb{N}$ and $\alpha^- \in F$ be as in Theorem 1, i.e.

$$\kappa \cdot G_{k,f} = \Delta^{(1-k)/2} \log(\alpha^-)$$

with k even and $f \in M^!_{2-2k}$. Then α^- generates the ideal

$$\prod_{m\geq 1} \prod_{\lambda \in S_m} I^-(\lambda)^{-c_f(-m)C_k(\lambda,m)\cdot\kappa}.$$

Example

For $\Delta_1 = -7, \Delta_2 = -23$, we have $\Delta = 161$ and

$$G_4(Z_\chi) \overset{num}{=} -4.157888612785\ldots \overset{\text{Thm 1}}{=} \frac{1}{\kappa \cdot 161^{3/2}} \log(\alpha^-).$$

Theorem 2 says $\alpha^- \in F$ generates the ideal

$$\left(\frac{\mathfrak{p}_5}{\mathfrak{p}_5'}\right)^{-2878\kappa} \left(\frac{\mathfrak{p}_{17}}{\mathfrak{p}_{17}'}\right)^{-3580\kappa} \left(\frac{\mathfrak{p}_{19}}{\mathfrak{p}_{19}'}\right)^{2628\kappa}, \ \mathfrak{p}_\ell = (\pi_\ell),$$

where $\pi_5 = 38 + 3\sqrt{\Delta}, \pi_{17} = 12 + \sqrt{\Delta}, \pi_{19} = 25 + 2\sqrt{\Delta}$. With bound on κ and the numerical calculation of $G_4(Z_\chi)$ above, we conclude $\kappa = 1$ and

$$\alpha^- = \left(\frac{\pi_5}{\pi_5'}\right)^{-2878} \cdot \left(\frac{\pi_{17}}{\pi_{17}'}\right)^{-3580} \cdot \left(\frac{\pi_{19}}{\pi_{19}'}\right)^{2628} \cdot \varepsilon_F^{1168}$$

where $\varepsilon_F = 11775 + 928\sqrt{\Delta}$ is the fundamental unit of F.

Sketch of Proof (GKZ).

1. Realize $G_{k,f}(z_1, z_2)$ as a regularized theta integral
$$G_{k,f}(z_1, z_2) = \int_{\Gamma\backslash\mathcal{H}}^{\mathrm{reg}} f(z) R_z^{k-1} \Theta(z, (z_1, z_2)) d\mu(z).$$
with R_z the raising operator (Bruinier, Viazovska).
2. Apply the Siegel-Weil formula to conclude
$$\Theta(z, Z_\chi) = E_\chi(z)$$
with E_χ an Eisenstein series.
3. Apply Cohen's operator to construct a real-analytic modular form $\widetilde{E}_{\chi,k}(z)$ of weight $2k$ such that
$$L_2 \widetilde{E}_{\chi,k}(z) = R_z^{k-1} E_\chi(z).$$
This only works for k odd.
4. Apply Stokes' theorem to express $G_{f,k}(Z_\chi)$ in terms of the Fourier coefficients of $\widetilde{E}_{\chi,k}$.

Modification for even k.

For even k, we use theta lifts to construct a real-analytic modular form $\widetilde{E}_{\chi,k}(z)$ of weight $2k$ such that

$$L_2\widetilde{E}_{\chi,k}(z) = R_z^{k-1}E_\chi(z).$$

Roughly speaking, we consider

$$\mathrm{O}(1,1) \xrightarrow{\Theta-\text{lift}} \mathrm{SL}_2 \xrightarrow{\Theta-\text{lift}} \mathrm{O}(2,2)$$

over $\mathbb{Q}$ and lift $\log(t)$, then restrict the resulting Hilbert modular form to the diagonal to obtain $\tilde{E}_{\chi,k}$. For odd k, one can consider

$$\mathrm{O}(2) \xrightarrow{\Theta-\text{lift}} \mathrm{SL}_2 \xrightarrow{\Theta-\text{lift}} \mathrm{O}(2,2).$$

Application to Singular Moduli

Question by Maaser (2011).

Are there finitely many singular moduli that are units?

Answer by Habeggar (2014).

Yes. But not only ineffectively.

Bilu, Habeggar, Kühne (2018).

No singular moduli is a unit.

Natural Question.

Can the difference of two singular moduli be a unit?

Application to Singular Moduli

Theorem 3 (L, 2019)

Let $z_1, z_2 \in Y$ be distinct CM points. Then the algebraic integer $j(z_1) - j(z_2)$ is never a unit.

Remark

We recover the result of Bilu-Habegger-Kühne from the result above since

$$j\left(\frac{1+\sqrt{-3}}{2}\right) = 0,$$

Idea of Proof.

Habegger

- Bound the height of a singular moduli $j(z)$ from below and above in terms of discriminant Δ.
- Lower bound exceeds upper bound for $|\Delta| \geq N$ with N ineffective since upper bound is ineffective.

Bilu-Habegger-Kühne

- Replace with effective upper bound.
- Figure out $N(\sim 10^{15})$, and check remaining cases by computer.

Idea of Proof.

Our approach: assume $j(z_1) - j(z_2)$ is a unit

- By Gross-Zagier, we have

$$0 = \log(\mathrm{Nm}(j(z_1) - j(z_2)) = G_1(Z_\chi) = \sum_{\lambda \in S_1} \sigma_\chi^+(\lambda),$$

  with S a finite set and $\sigma_\chi^+(\lambda) \geq 0$.
  Assumption implies $\sigma_\chi^+(\lambda) = 0$ for all $\lambda \in S_1$.
- By Gross-Kohnen-Zagier, we have

$$G_3(Z_\chi) = \sum_{\lambda \in S_1} P_2(\lambda - \lambda')\sigma_\chi^+(\lambda)$$

  with P_2 the 2nd Legendre polynomial.
- By definition,

$$G_3(z_1, z_2) < 0$$

  for any $z_j \in Y$. Contradiction!