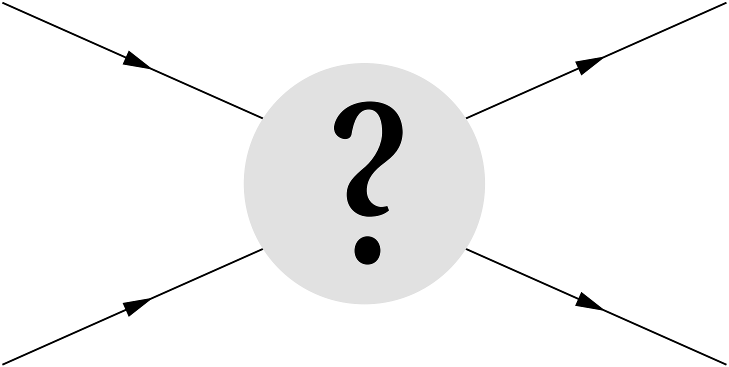
Parametric Feynman integrals with hyperlogarithms

Erik Panzer

All Souls College

February 15th
Trinity College Dublin

Perturbative Quantum Field Theory (**old fashioned**)



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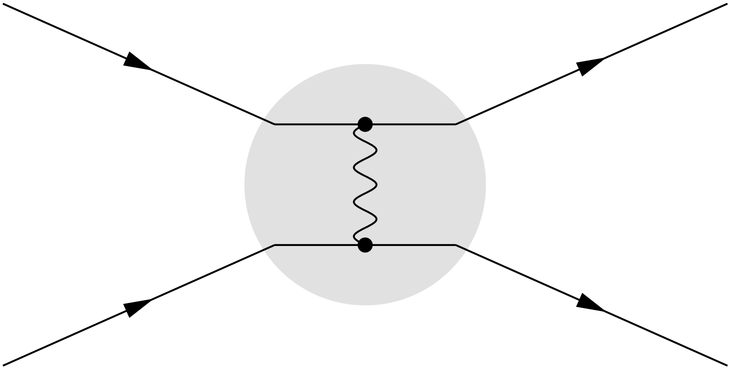

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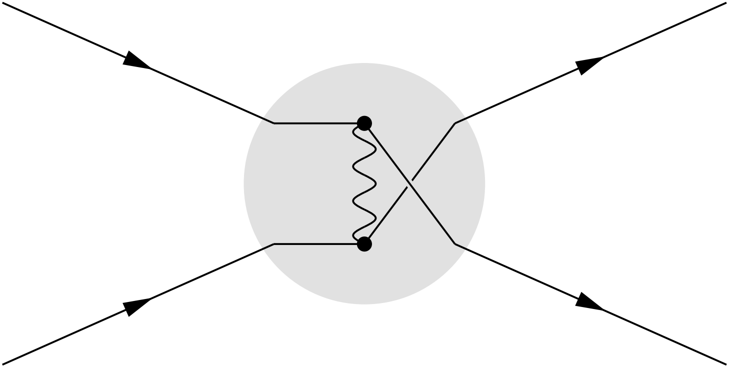

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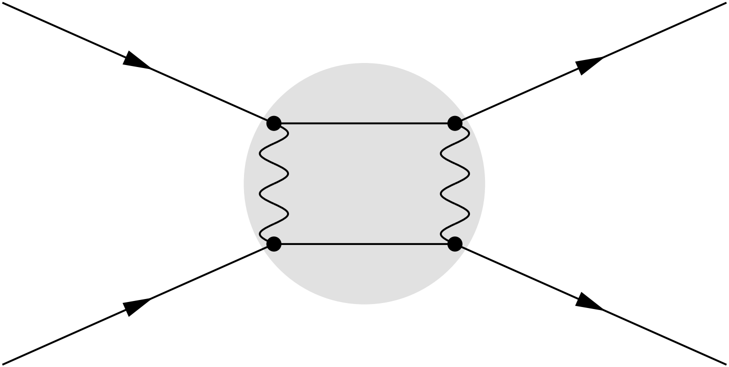

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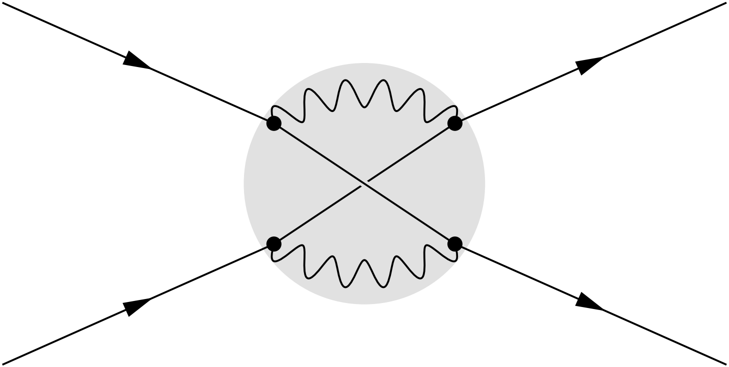

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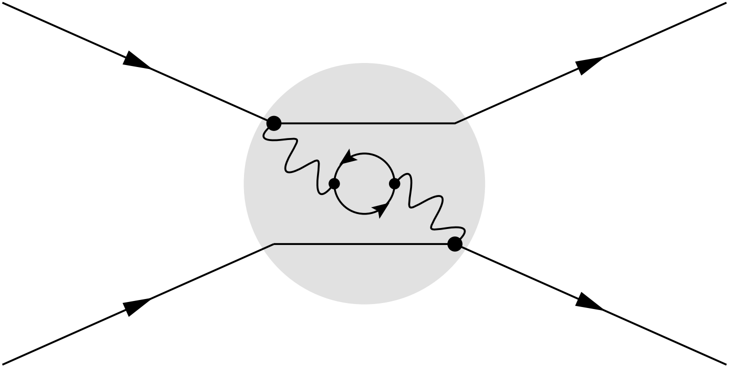

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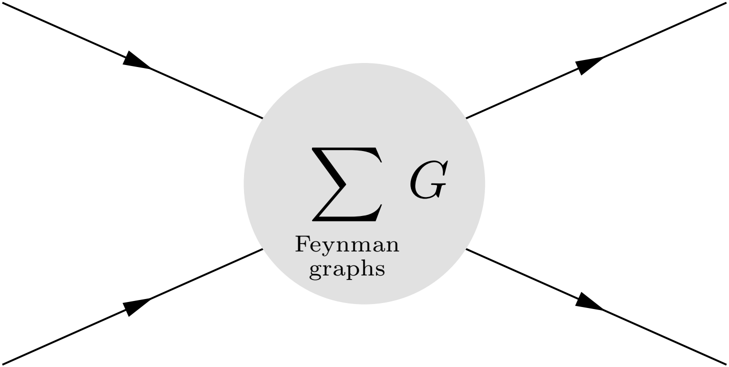

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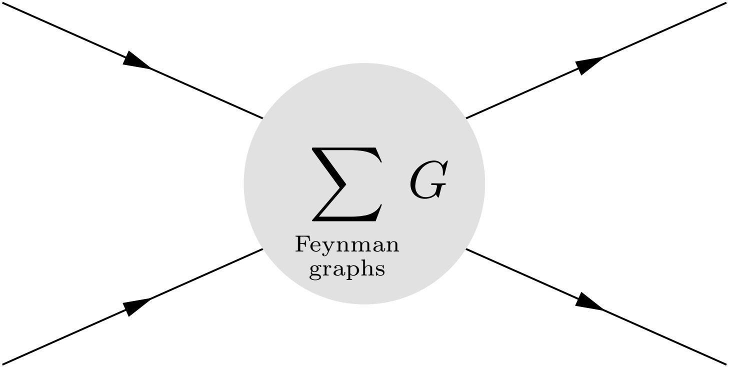

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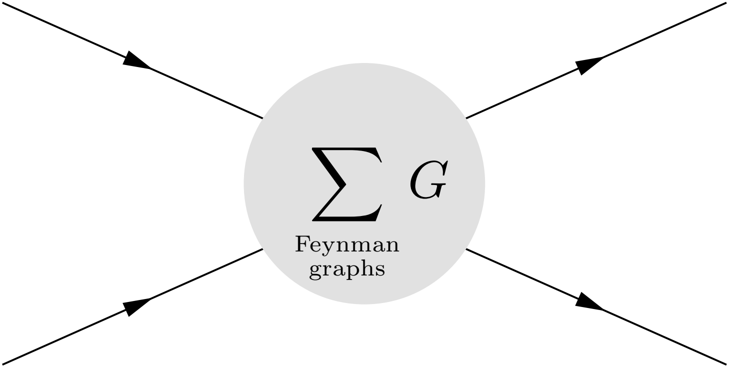

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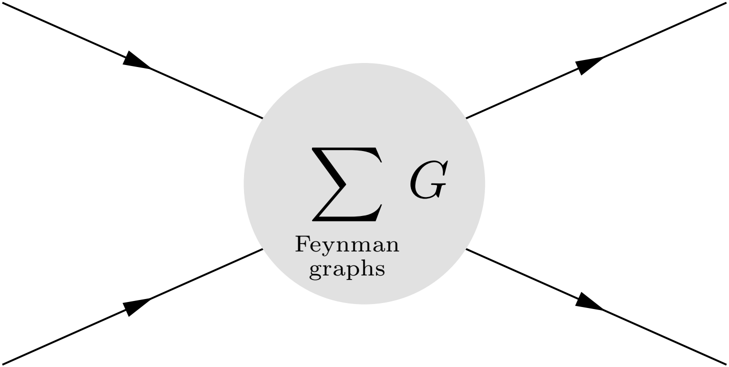

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Some FI are expressible with

- logarithms: $$-\log(1-z) = \sum_{0<k} \frac{z^k}{k}$$
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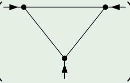
$\Phi\Bigg($  $\Bigg) = \dfrac{2\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{\mathrm{Im}\, z}$

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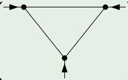
Φ () $$= \frac{2\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{\mathrm{Im}\, z}$$

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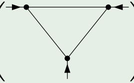
$\Phi\Big($  $\Big) = \dfrac{2\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{\mathrm{Im}\, z}$

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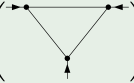
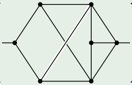
$\Phi\Big($  $\Big) = \dfrac{2\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{\mathrm{Im}\, z}$

$\Phi\Big($  $\Big) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$

Schwinger parameters

With the *superficial degree of divergence* $\mathsf{sdd} = |E(G)| - D/2 \cdot \mathsf{loops}(G)$,

$$\Phi(G) = \Gamma(\mathsf{sdd}) \int_{(0,\infty)^E} \frac{\Omega}{\psi^{D/2}} \left(\frac{\psi}{\varphi}\right)^{\mathsf{sdd}}, \qquad \Omega = \delta(1-\alpha_N) \prod_{e \in E} \mathrm{d}\alpha_e$$

Graph polynomials:

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e$$

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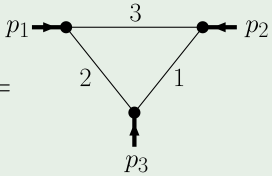
Example ($D=4$)

$G =$ 

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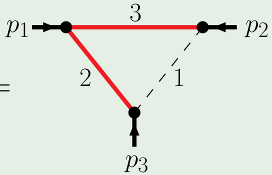

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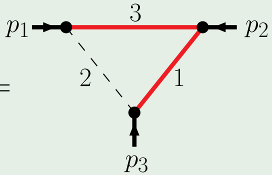
$G =$ 

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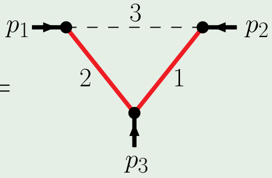

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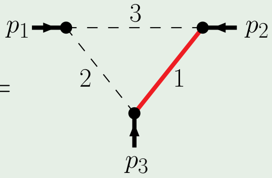
$G =$ 

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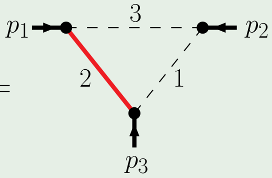
$G =$ 

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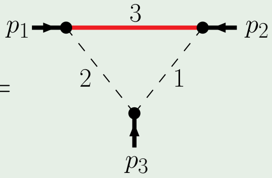

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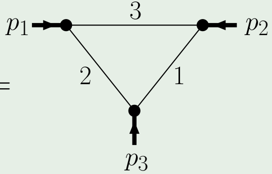
$G =$ 

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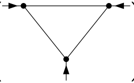
$\Phi\Bigg($  $\Bigg)$

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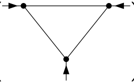

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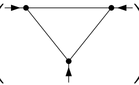
Φ ()

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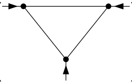
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We need a class of transcendental functions, closed under $\int \frac{\mathrm{d}z}{z-\sigma}$.

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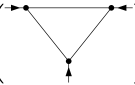
$\Phi\Big($  $\Big)$

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Definition (Hyperlogarithms, Lappo-Danilevsky 1927)

$$G(\underbrace{\sigma_1,\ldots,\sigma_w}_{\vec{\sigma}}; z) := \int_0^z \frac{\mathrm{d}z_1}{z_1-\sigma_1} G(\sigma_2,\ldots,\sigma_w; z_1)$$

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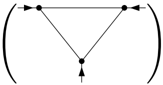
Φ 

$$= \int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3+z\bar{z}\alpha_3+(1-z)(1-\bar{z})\alpha_2)}$$

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$$= \frac{1}{z-\bar{z}} \int \left(\frac{\mathrm{d}\alpha_2}{\alpha_2+\bar{z}} - \frac{\mathrm{d}\alpha_2}{\alpha_2+z}\right) \log\frac{(\alpha_2+1)(\alpha_2+z\bar{z})}{(1-z)(1-\bar{z})\alpha_2}$$

We need a class of transcendental functions, closed under $\int \frac{\mathrm{d}z}{z-\sigma}$.

Definition (Hyperlogarithms, Lappo-Danilevsky 1927)

$$G(\underbrace{\sigma_1,\ldots,\sigma_w}_{\vec{\sigma}};z) := \int_0^z \frac{\mathrm{d}z_1}{z_1-\sigma_1}\int_0^{z_1}\frac{\mathrm{d}z_2}{z_2-\sigma_2}G(\sigma_3,\ldots,\sigma_w;z_2)$$

- $\Sigma = \{-1,0,1\}$ harmonic polylogarithms (HPL) [Remiddi & Vermaseren]
- $\Sigma = \{0,1,1-y,-y\}$ 2-dimensional HPL [Gehrmann & Remiddi]

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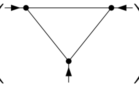
$\Phi\Big($  $\Big)$

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Example

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$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

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- multivalued, monodromies, path concatenation
- represent all MPL:

$$G(0^{n_d-1},\sigma_d,\ldots,0^{n_1-1},\sigma_1;z) = (-1)^d\,\mathrm{Li}_{n_1,\ldots,n_d}\left(\frac{\sigma_2}{\sigma_1},\cdots,\frac{\sigma_d}{\sigma_{d-1}},\frac{z}{\sigma_d}\right)$$

Symbolic integration: Rewriting hyperlogarithms

Problem: Given $G(\vec{\sigma}(\alpha); z)$, write it as hyperlogarithm with constant letters and final argument α. Recursive solution via

$$\mathrm{d}\, G(\vec{\sigma}; z) = \sum_{i=1}^{n} G(\cdots, \not{\sigma}_i, \cdots; z)\, \mathrm{d}\log \frac{\sigma_i - \sigma_{i-1}}{\sigma_i - \sigma_{i+1}} \qquad \sigma_0 := z, \sigma_{n+1} := 0$$

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Computation of integration constants relies on shuffle algebra, rescalings

$$G(\lambda \vec{\sigma}; \lambda z) = G(\vec{\sigma}; z)$$

and Möbius transformations.

All this is implemented in the Maple program `HyperInt`.

Linear reducibility

We need that all partial integrals

$$f_n := \int_0^\infty f_{n-1}\ \mathrm{d}\alpha_n = \int_{(0,\infty)^n} f_0\ \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n \qquad \left(f_0 = \frac{1}{\psi^{D/2-\mathrm{sdd}} \varphi^{\mathrm{sdd}}} \right)$$

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- condition on the polynomials ψ and φ only
- sufficient criteria: polynomial reduction algorithms (Brown)

Polynomial reduction

Denote alphabets (divisors) by sets S of irreducible polynomials.

Definition

Let S denote a set of polynomials $f = f^e \alpha_e + f_e$ linear in α_e. Then with $[f,g]_e := f^e g_e - f_e g^e$, S_e shall be the set of irreducible factors of

$$\{f^e, f_e \colon\ f \in S\} \quad \text{and} \quad \{[f,g]_e \colon\ f,g \in S\}\,.$$

Example (massless triangle)

$$S = \{\psi, \varphi\} = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2\alpha_3 + z\bar{z}\alpha_1\alpha_3 + (1-z)(1-\bar{z})\alpha_1\alpha_2\}$$

$$[\varphi, \psi]_3 = (\alpha_1 + \alpha_2)(\alpha_2 + \alpha_1 z\bar{z}) - (1-z)(1-\bar{z})\alpha_1\alpha_2$$

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Lemma

If the singularities of F are cointained in S, then the singularities of $\int_0^\infty F \mathrm{d}\alpha_e$ are contained in S_e.

Polynomial reduction

Corollary (linear reducibility)

If all $S^k := (S^{k-1})_k$ are linear in α_{k+1}, then any MPL F with alphabet in S^0 integrates to a MPL $\int_0^\infty F \prod_{e=1}^n \mathrm{d}\alpha_e$ with alphabet in S^n.

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This gives only very coarse upper bounds, for example $z\bar{z}-1$ is spurious: It drops out in $S_{2,3} \cap S_{3,2} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}\}$ because

$$S_{2,3} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, z\bar{z}-z-\bar{z}\}.$$

Note that $z\bar{z}-z-\bar{z}$ is spurious.

Compatibility graphs

Keep track of compatibilities $C \subset \binom{S}{2}$ between polynomials:

- start with the complete graph ψ —— φ
- in S_e, only take resultants $[f, g]_e$ for compatible $\{f, g\} \in C$
- in C_e, only pairs $[f, g]_e$ —— $[g, h]_e$ become compatible

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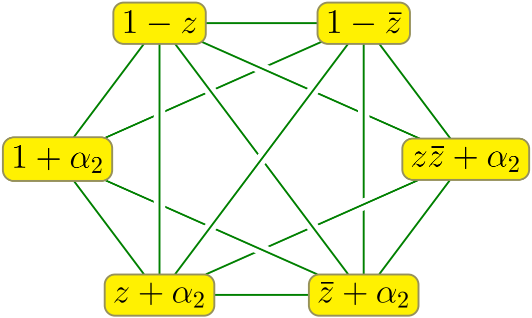

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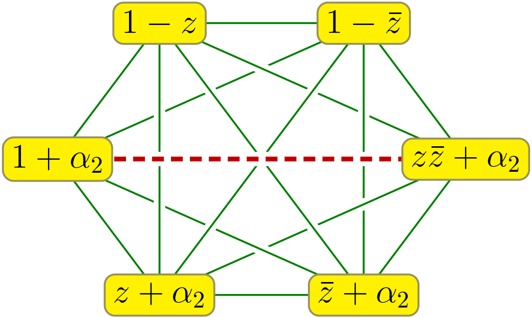
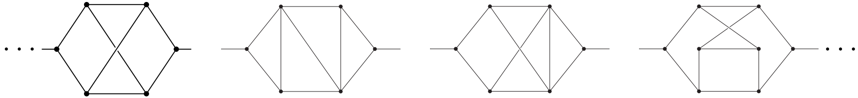


$z\bar{z}\alpha_1 + \alpha_2$ and $\alpha_1 + \alpha_2$ not compatible $\Rightarrow$ no resultant $1 - z\bar{z}$ in $(S,C)_{3,2}$

Linearly reducible families (fixed loop order)

1. all ≤ 4 loop massless propagators (Panzer)



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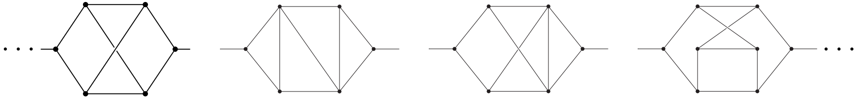
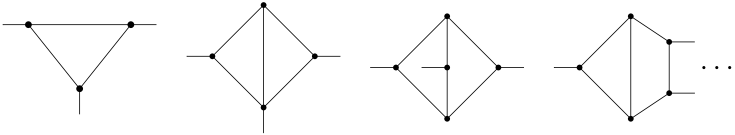


2. all ≤ 3 loop massless off-shell 3-point (Chavez & Duhr, Panzer)



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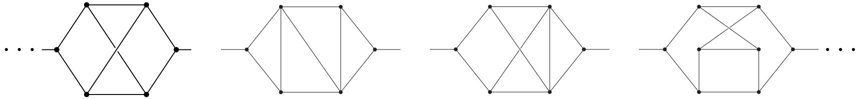

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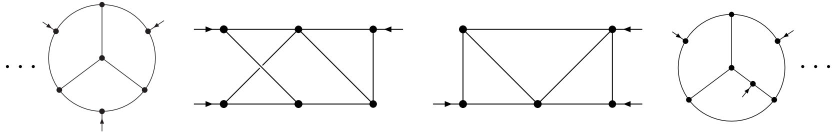

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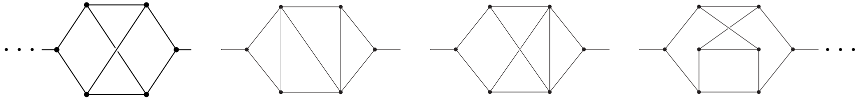

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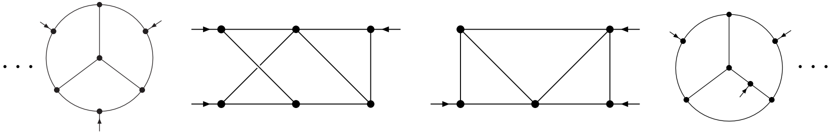
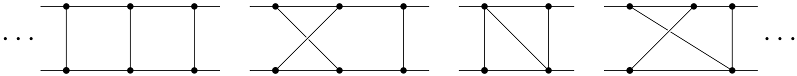
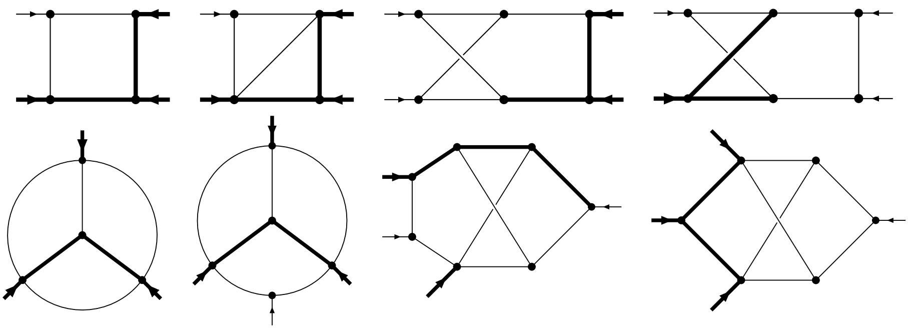


3. all ≤ 2 loop massless on-shell 4-point (Lüders)

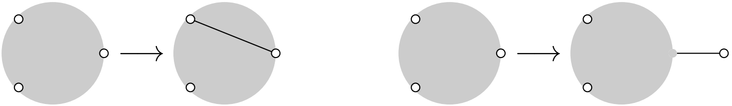


Linearly reducible massive graphs (some examples)

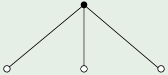


Linearly reducible families (infinite)

- 3-constructible graphs (3-point functions) [Brown, Schnetz, Panzer]



Example



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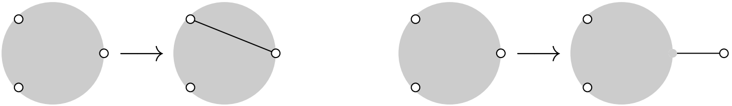

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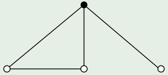

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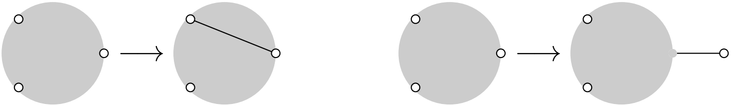

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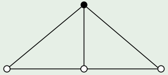

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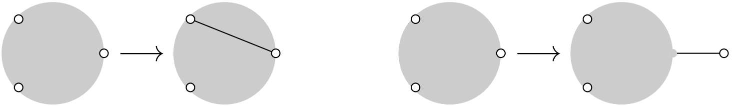

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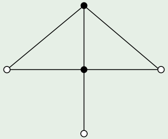

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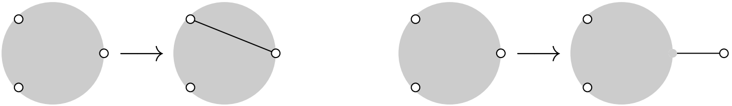

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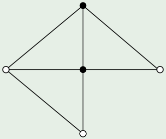

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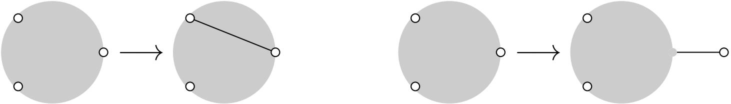

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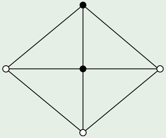

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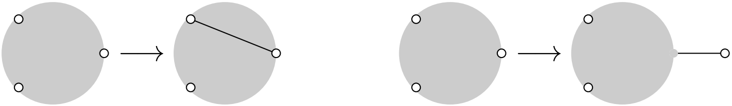

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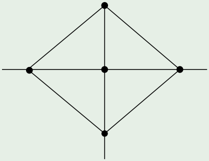

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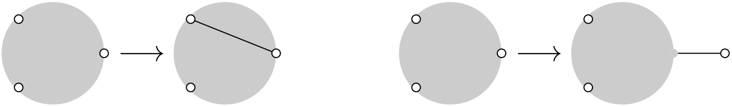

Theorem (Panzer)

All ϵ-coefficients of these graphs (off-shell) are MPL over the alphabet $\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, 1-z\bar{z}, 1-z-\bar{z}, z\bar{z}-z-\bar{z}\}$.

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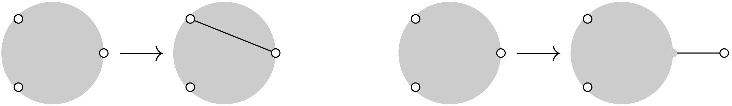

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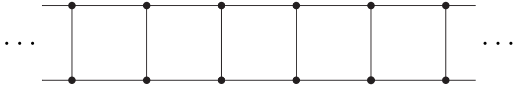
- minors of ladder-boxes (up to 2 legs off-shell)



Theorem (Panzer)

All ϵ-coefficients of these graphs are MPL. For the massless case, the alphabet is just $\{x, 1+x\}$ *for* $x = s/t$.