Modular graph functions as iterated Eisenstein integrals

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Goal:

Understand modular functions arising from genus 1 superstring amplitudes.

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Modular graph functions are single-valued iterated Eisenstein integrals.

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Theorem

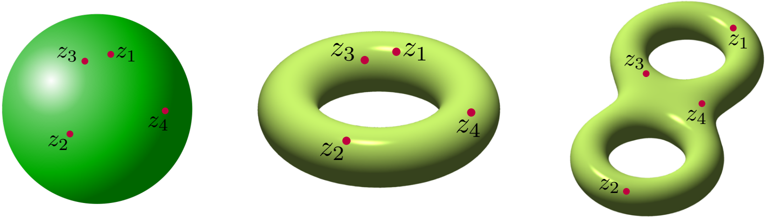
Modular graph functions are single-valued iterated Eisenstein integrals.

$$\mathsf{MGF} := \int_{\mathcal{E}_\tau} \mathsf{eMPL} \overset{\mathsf{sv}}{=} \int_{\gamma} \mathsf{eMPL} =: \mathsf{eMZV} \subset \mathsf{iEi} \subset \mathsf{MMV}$$

Players:

MGF	modular graph functions (Green, Vanhove, D'Hoker, Gürdoğan)
eMPL	elliptic multiple polylogarithms (Brown & Levin, Vanhove)
sv	single-valued integration (Schnetz)
eMZV	elliptic multiple zeta values (Enriquez, Matthes, Zerbini)
iEi	iterated Eisenstein integrals (Brown, Schlotterer, Brödel)
MMV	multiple modular values (Brown, Hain)

String worldsheets $\Rightarrow$ genus expansion:



$\cdots$

Amplitude:

$$\begin{aligned} \mathcal{A}_4 &= \mathcal{A}_4^{g=0} + \mathcal{A}_4^{g=1} + \mathcal{A}_4^{g=2} + \cdots \\ &= \int_{\mathfrak{M}_{0,4}} \Omega_0 + \int_{\mathfrak{M}_{1,4}} \Omega_1 + \int_{\mathfrak{M}_{2,4}} \Omega_2 + \cdots \end{aligned}$$

Tree level ($g = 0$): $\mathfrak{M}_{0,4} \cong \mathbb{C} \setminus \{0, 1\} \ni z_4$ via $(z_1, z_2, z_3) \mapsto (0, 1, \infty)$

$$\mathcal{A}_4^{g=0} \propto \int_{\mathbb{C}} \mathrm{d}^2 z_4 \cdot |z_4|^{s_{14}} \, |1 - z_4|^{s_{13}}$$
$$\propto \frac{\Gamma(1 - s_{12})\Gamma(1 - s_{13})\Gamma(1 - s_{14})}{\Gamma(1 + s_{12})\Gamma(1 + s_{13})\Gamma(1 + s_{14})}$$

where $s_{ij} = (k_i + k_j)^2 \cdot \alpha'/4$ and $s_{12} + s_{13} + s_{14} = 0$.

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One-loop ($g = 1$): Integrate over punctures first: $\mathfrak{M}_{1,4} \to \mathfrak{M}_{1,1}$

$$\mathcal{A}_4^{g=1}(s_{12}, s_{14}) = \int_{\mathfrak{M}_{1,1}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}_4(s_{12}, s_{14}|\tau)$$

$\Rightarrow$ modular functions on $\mathfrak{M}_{1,1}$ (non-holomorhpic)

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$$|z_i - z_j|^{s_{ij}} = \exp\left(s_{ij} \ln |z_i - z_j|\right)$$
$$\ln |z_i - z_j| = \text{Green's function on the sphere}$$

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Modular graph functions

The genus 1 contribution to the graviton amplitude of closed superstrings are integrals over the moduli $\tau = \tau_1 + i\tau_2 \in \mathfrak{M}_{1,1} \cong \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ of

$$\mathcal{B}_4(\{s_{ij}\}\,|\tau) = \left(\prod_{k=1}^{3} \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_k}{\tau_2}\right) \exp\left(\sum_{1\le i<j\le 4} s_{ij}\mathcal{G}(z_i - z_j|\tau)\right)\Bigg|_{z_4=0}.$$

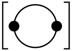
The Green's function on the torus $\mathcal{E}_\tau = \mathbb{C}/\Lambda_\tau$ with $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ is

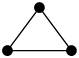
$$\mathcal{G}(z|\tau) = \frac{\tau_2}{\pi} \sum_{\omega\in\Lambda_\tau\setminus\{0\}} \frac{1}{|\omega|^2} \exp\left[\frac{\pi}{\tau_2}(\omega\bar{z} - \bar{\omega}z)\right].$$

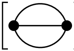
The low energy expansion is indexed by graphs G, with coefficients

$$\mathbf{D}\,[G]\,(\tau,\bar{\tau}) := \left(\prod_{v\in V(G)} \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_k}{\tau_2}\right) \prod_{i\to j\in E(G)} \mathcal{G}(z_i - z_j|\tau).$$

Examples:

$\mathbf{D}$ [] $= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2} \; \mathcal{G}(z_1|\tau)^2$

$\mathbf{D}$ [] $= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2} \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_2}{\tau_2} \; \mathcal{G}(z_1|\tau)\mathcal{G}(z_2|\tau)\mathcal{G}(z_1 - z_2|\tau)$

$\mathbf{D}$ [] $= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2} \; \mathcal{G}(z_1|\tau)^3$

MGFs are modular invariant, real analytic, with MZV coefficients $d_k^{(m,n)}$:

$$\mathbf{D}\,[G] = \sum_k (\pi\tau_2)^k \sum_{n,m\geq 0} q^m \bar{q}^n d_k^{(m,n)}$$

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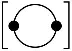
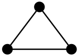
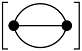
$\mathbf{D}\,[$  $]$ $$= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2}\, \mathcal{G}(z_1|\tau)^2 = E_2$$

$\mathbf{D}\,[$  $]$ $$= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2} \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_2}{\tau_2}\, \mathcal{G}(z_1|\tau)\mathcal{G}(z_2|\tau)\mathcal{G}(z_1 - z_2|\tau) = E_3$$

$\mathbf{D}\,[$  $]$ $$= \int_{\mathcal{E}_\tau} \frac{\mathrm{d}^2 z_1}{\tau_2}\, \mathcal{G}(z_1|\tau)^3$$

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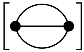
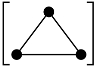
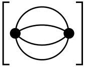
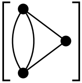
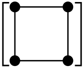
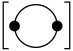
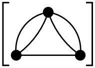
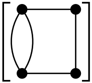
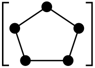
Real analytic Eisenstein series

$$E_k(\tau,\bar{\tau}) = \left(\frac{\tau_2}{\pi}\right)^k \sum_{\omega\in\Lambda_\tau\setminus\{0\}} \frac{1}{|\omega|^{2k}}$$

Many identities, for example

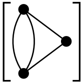
$\mathbf{D}$  $= \mathbf{D}$  $+\, \zeta_3$ $\quad$ (*Zagier*)

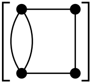
$\mathbf{D}$  $= 24\mathbf{D}$  $-\, 18\mathbf{D}$  $+\, 3\mathbf{D}$  ^{2}

$10\mathbf{D}$  $= 20\mathbf{D}$  $-\, 4\mathbf{D}$  $+\, 3\zeta_5$

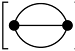
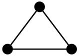
Eigenvalue equations with respect to $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar{\tau}}$, e.g.

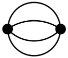
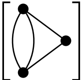
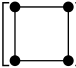
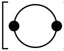
$$(\Delta - k(k-1))\, E_k = 0$$

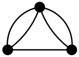
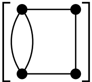
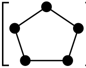
$(\Delta - 2)\, \mathbf{D}$  $= 9E_4 - E_2^2$

$(\Delta - 6)\, \mathbf{D}$  $= \dfrac{86}{5} E_5 - 4E_2 E_3 + \dfrac{1}{10}\zeta_5$

Many identities, for example

$\mathbf{D}$ [] $= \mathbf{D}$ [] $+ \zeta_3$ (*Zagier*)

$\mathbf{D}$ [] $= 24\mathbf{D}$ [] $- 18\mathbf{D}$ [] $+ 3\mathbf{D}$ []^2

$10\mathbf{D}$ [] $= 20\mathbf{D}$ [] $- 4\mathbf{D}$ [] $+ 3\zeta_5$

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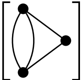
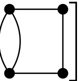
$(\Delta - 2)\, \mathbf{D}$ [] $= 9E_4 - E_2^2$

$(\Delta - 6)\, \mathbf{D}$ [] $= \frac{86}{5} E_5 - 4E_2 E_3 + \frac{1}{10}\zeta_5$

$\Rightarrow$ MGFs look like "integrals of Eisenstein series"

iterated integrals (Chen 1973)

Take a manifold X and differential forms $\omega_1, \ldots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0,1], X)$, we can construct functions (on γ):

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Examples:

1. hyperlogarithms (multiple polylogarithms):

$$X = \mathbb{C} \setminus \Sigma, \qquad \omega_i \in \left\{ \frac{\mathrm{d}z}{z - \sigma} \colon \ \sigma \in \Sigma \right\}$$

2. iterated integrals of modular forms:

$$X = \mathbb{H}, \qquad \omega_i \in \{ \mathrm{d}\tau f(\tau) \colon \ f \in M(\mathrm{SL}_2(\mathbb{Z})) \}$$

3. multiple elliptic polylogarithms:

$$X = \mathcal{E}_\tau \setminus \Sigma$$

Iterated integrals of Eisenstein series

Holomorphic Eisenstein series:

$$G_{2k}(\tau) = \sum_{\omega\in\Lambda_\tau^\times} \frac{1}{\omega^{2k}} = 2\zeta(2k) + \frac{2(2i\pi)^{2k}}{(2k-1)!}\sum_{n\geq 1}\sigma_{2k-1}(n)q^n$$

Iterated integrals:

$$\Gamma(f_1,\dots,f_k;q) := \int_0^q \frac{\mathrm{d}q'}{q'}\frac{f_1(q')}{4\pi^2}\Gamma(f_2,\dots,f_k;q') \qquad \Gamma(1) = \frac{\log q}{4\pi^2} = \frac{i\tau}{2\pi}$$

$$\mathrm{SV}\left(f_1,\dots,f_n\right) := \sum_{i=0}^{n}\Gamma(f_{i+1},\dots,f_n)\cdot\bar{\Gamma}(f_i,\dots,f_1) \qquad \mathrm{SV}(1) = -\frac{\tau_2}{\pi}$$

(all $f_k = 1$ but one $\mapsto$ Eichler integral)

Example (real analytic Eisenstein series)

$$E_s = \left(\frac{\tau_2}{\pi}\right)^s \sum_{\omega\in\Lambda_\tau^\times}\frac{1}{|\omega|^{2s}}$$

Iterated integrals of Eisenstein series

Holomorphic Eisenstein series:

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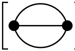
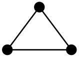
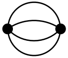
Example (real analytic Eisenstein series)

$$E_s = \frac{\pi^s}{\tau_2^{s-1}} \left\{ \frac{8\zeta_{2s-1}}{(2\pi)^{2s-1}} \binom{2s-3}{s-2} - \frac{(2s-1)!}{\pi} \mathrm{SV}\left(1^{(s-1)}, G_{2s}, 1^{(s-1)}\right) \right\}$$

Recall Zagier's result:

$\mathbf{D}\Big[$  $\Big] = \mathbf{D}\Big[$  $\Big] + \zeta_3 = E_3 + \zeta_3$

Higher loop sunrise integrals are Eisenstein integrals of higher depth:

$\mathbf{D}\Big[$  $\Big] = \frac{18}{5}E_4 - 3E_2^2 + \frac{10\zeta_5}{y} - \frac{432\pi^2}{y}\mathrm{SV}\left(1, G_4, 1, G_4, 1\right)$

$$+ \frac{72\zeta_3}{y}\left(\Gamma(1, G_4) + \bar{\Gamma}(1, G_4) - 2\zeta_4\Gamma(1)\bar{\Gamma}(1)\right)$$

Remarks

1. $\Gamma(a)\bar{\Gamma}(b)$ are linearly independent $\Rightarrow$ identities trivialize
2. straightforward q-expansion $\Gamma(\cdots) = \sum_{n=0}^{N} \log^n q \sum_{m\geq 0} a_{n,m}q^m$
3. action $\partial_\tau, \partial_{\bar{\tau}}$ and $\Delta = 4\tau_2^2\partial_\tau\partial_{\bar{\tau}}$ on Γ's easy to work out
4. rich theory developed recently by Francis Brown

$\Rightarrow$ modular polynomials in holomorphic and antiholomorphic iterated Eisenstein integrals explain the properties of MGFs

Elliptic polylogarithms (Brown & Levin)

Let $X = \mathcal{E}_\tau^\times = \mathcal{E}_\tau \setminus \{0\}$ and $\mathcal{E}_\tau = \mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$. The series

$$F(z,\alpha|\tau) = \frac{\vartheta'(0|\tau)\vartheta(z+\alpha|\tau)}{\vartheta(z|\tau)\vartheta(\alpha|\tau)} = \sum_{k\geq 0} \alpha^{k-1} g_k(z|\tau)$$

defines meromorphic functions $g_k(z)$ on $\mathbb{C}$ with

$$g_k(z+1|\tau) = g_k(z|\tau) \qquad g_k(z+\tau|\tau) = g_k(z|\tau) + \sum_{j=1}^{k} g_{k-j}(z|\tau)\frac{(-2i\pi)^j}{j!}.$$

Examples

$$g_0 = 1, \qquad g_1(z) = \frac{\vartheta'(z)}{\vartheta(z)} = \frac{1}{z} + \mathcal{O}(z) \qquad g_2(z) = \frac{\wp(z) - g_1^2(z)}{2}$$

- g_1 has first order poles (with unit residue) on Λ_τ
- g_k has no poles on $\mathbb{Z}$ (for any $k \neq 1$)

Fix a finite set $\Sigma \subset \mathbb{C}$ of punctures to define closed forms

$$\omega_{\sigma}^{(n)}(z) = g_n(z-\sigma)\,\mathrm{d}z \qquad \in \Omega^1\left(\mathbb{C} \setminus (\sigma + \Lambda_\tau)\right)$$

for each $n \geq 0$ and $\sigma \in \Sigma$. Elliptic MPL are their iterated integrals:

$$\int_0^z \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} = \tilde{\Gamma}\left(\begin{matrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{matrix}; z\right) = \int_0^z \mathrm{d}t\; g_{n_1}(t-z_1)\,\tilde{\Gamma}\left(\begin{matrix} n_2 \cdots n_r \\ z_2 \cdots z_r \end{matrix}; z\right)$$

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Remarks

1. holomorphic, homotopy invariant
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So: (MGF integrand) $\in$ algebra $\mathcal{A}_n$ generated by periods (eMPLs)

$$\int_{z_0}^{z_{r+1}} \omega_{z_1}^{(n_1)} \cdots \omega_{z_r}^{(n_r)} \quad \text{and their c.c.}$$

where $n_i \geq 0$ and $z_i \in \Sigma$. This $\mathcal{A}_n$ defines a subsheaf of $C^\omega(\mathrm{Conf}_n(\mathcal{E}_\tau))$.

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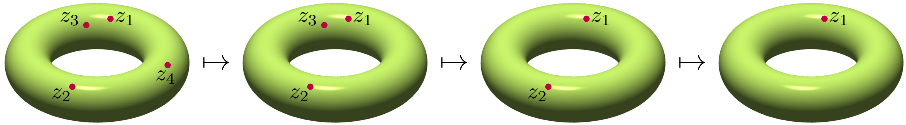
Approach: Integrate out each puncture sequentially along fibrations

$$\mathcal{E}_\tau \setminus \{z_1, \ldots, z_{n-1}\} \hookrightarrow \mathrm{Conf}_n(\mathcal{E}_\tau) \twoheadrightarrow \mathrm{Conf}_{n-1}(\mathcal{E}_\tau)$$



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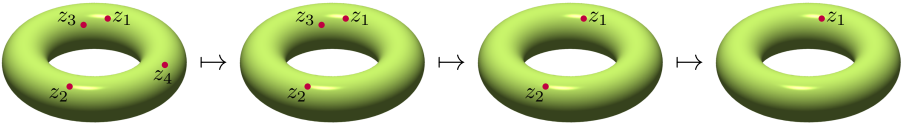

Lemma

Every period $f \in \mathcal{A}_n$ is an iterated integral on the fibre, e.g.

$$f = \sum_{u,v} \int_0^{z_n} u \cdot \left(\int_0^{z_n} v \right)^* \cdot f_{u,v}$$

where $f_{u,v} \in \mathcal{A}_{n-1}$ and u, v are forms independent of z_n.

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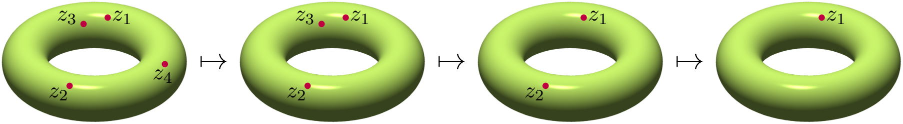


Example

$$\tilde{\Gamma}\begin{pmatrix} 1\,1 \\ z\,0 \end{pmatrix}; z\Big) = 2\tilde{\Gamma}\begin{pmatrix} 0\,2 \\ 0\,0 \end{pmatrix}; z\Big) + \tilde{\Gamma}\begin{pmatrix} 2\,0 \\ 0\,0 \end{pmatrix}; z\Big) - 2\tilde{\Gamma}\begin{pmatrix} 1\,1 \\ 0\,0 \end{pmatrix}; z\Big) + \zeta_2$$

Integration

Suppose we have written the integrand in the form

$$f = \left[\sum_{u,v} \int_0^{z_n} u \cdot \left(\int_0^{z_n} v \right)^* \cdot f_{u,v} \right] \cdot \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n,$$

Then we can easily find a primitive F with $\mathrm{d}F = f$ as

$$F = \left[\sum_{u,v} \int_0^{z_n} \omega_0^{(0)} u \cdot \left(\int_0^{z_n} v \right)^* \cdot f_{u,v} \right] \cdot \mathrm{d}\bar{z}_n.$$

Idea

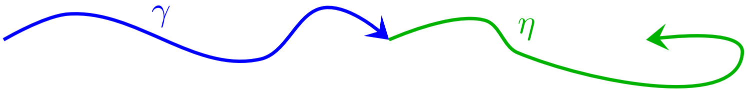
Apply Stokes to the fundamental domain $D = [0,1] \times [0,\tau] \setminus \Sigma$:

$$\int_D f = \int_{\partial D} F.$$

Problem: F does not extend to a smooth function on D°. In other words, F is not single-valued.

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

$$\int_{\gamma\star\eta} \omega_2\omega_1 = \int_{0\le t_1 \le t_2 \le 1} (\gamma \star \eta)^*(w_2)(t_2)(\gamma \star \eta)^*(w_1)(t_1),$$

split the interval

$$\underbrace{\{t_1 \le t_2\}}_{\int_{\gamma\star\eta}\omega_2\omega_1} = \{t_1 \le t_2 \le \tfrac{1}{2}\} \cup \{t_1 \le \tfrac{1}{2} \le t_2\} \cup \{\tfrac{1}{2} \le t_1 \le t_2\}$$

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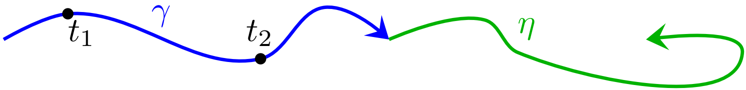

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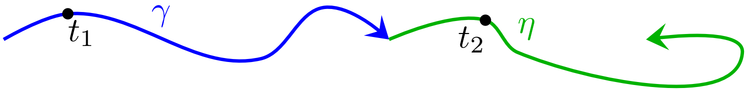

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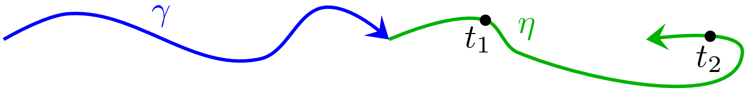

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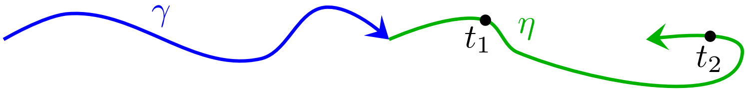

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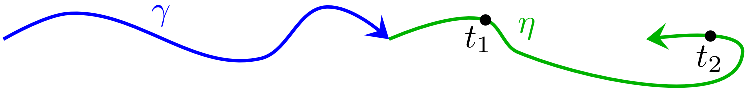

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More generally, the path concatenation formula reads

$$\int_{\gamma\star\eta} \omega_r \cdots \omega_1 = \sum_{k=0}^{r} \int_\eta \omega_r \cdots \omega_{k+1} \int_\gamma \omega_k \cdots \omega_1.$$

Monodromy

Analytic continuation $\mathcal{M}_\eta$ along a closed loop η with $\eta(0) = \eta(1) = 0$ is

$$\mathcal{M}_\eta \int_0^z \omega_r \cdots \omega_1 = \sum_{k=0}^{r} \int_0^z \omega_r \cdots \omega_{k+1} \underbrace{\int_\eta \omega_k \cdots \omega_1}_{\in \mathcal{A}_{n-1}} .$$

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Monodromy and derivatives commute

$$\partial_z \left(\mathcal{M}_\eta - \mathrm{id}\right) F = \left(\mathcal{M}_\eta - \mathrm{id}\right) \partial_z F = \left(\mathcal{M}_\eta - \mathrm{id}\right) f = 0$$

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$\Rightarrow$ the monodromies of F are antiholomorphic:

$$\left(\mathcal{M}_{\eta_\sigma} - \mathrm{id}\right) F = \sum_u \left(\int_0^{z_n} u \right)^* F_u^\sigma$$

for any basis $\eta_\sigma \in \pi_1(\mathcal{E}_\tau \setminus \Sigma)$ of loops. We can choose them such that

$$\int_{\eta_\sigma} \omega_z^{(n)} = (2i\pi)\delta_{\sigma,z}\delta_{1,n}$$

Note that the leading length of the monodromy is

$$(\mathcal{M}_\eta - \mathrm{id}) \int_0^z \omega_n \cdots \omega_1 = \int_0^z \omega_n \cdots \omega_2 \int_\eta \omega_1 + \text{lower length}$$

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So there is an antiholomorphic form with the opposite monodromies:

$$(\mathcal{M}_{\eta_\sigma} - \mathrm{id}) \left\{ \sum_{p \in \Sigma} \sum_u \left(\int_0^{z_n} u \omega_p^1 \right) \frac{F_u^\sigma}{2i\pi} \right\} = -\sum_u \left(\int_0^{z_n} u \right)^* F_u^\sigma + \text{lower length}$$

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Corollary: Existence of single-valued primitives

By adding antiholomorphic functions, we can find a primitive $F \in \mathcal{A}_n$ with

$$\mathrm{d}F = f \quad \text{and} \quad (\mathcal{M}_{\eta_\sigma} - \mathrm{id}) F = 0 \quad \text{for all } \sigma \in \Sigma$$

Stokes' theorem $\int_D f = \int_{\partial D} F$ gets contributions from

1. the punctures $\sigma \in \Sigma$:

$$\lim_{r\to 0} \oint_{|z-\sigma|=r} F = 0$$

2. the sides of D:

$$\int_0^1 F + \int_{1+\tau}^{\tau} F = -\int_0^1 \left(\mathcal{M}_{[0,\tau]} - \mathrm{id}\right) F$$
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Recall

The monodromies

$$\left(\mathcal{M}_{[0,\tau]} - \mathrm{id}\right) F \quad \text{and} \quad \left(\mathcal{M}_{[0,1]} - \mathrm{id}\right) F$$

are antiholomorphic iterated integrals.

Summary

Given a function $f \in \mathcal{A}_n$ single-valued on $\mathrm{Conf}_n(\mathcal{E}_\tau)$:

1. There is a function $\mathcal{F} \in \mathcal{A}_n$ that is single-valued on D° with $\partial_{z_n} F = f$.
2. We can apply Stokes' theorem to $\mathrm{d}(F \mathrm{d}\bar{z}_n) = f \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$.
3. All contributions are eMPL on the base $\mathcal{A}_{n-1}$.
4. Due to convergence, the result is necessarily single-valued and descends to $\mathrm{Conf}_{n-1}(\mathcal{E}_\tau)$.

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2. We can apply Stokes' theorem to $\mathrm{d}(F \mathrm{d}\bar{z}_n) = f \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$.
3. All contributions are eMPL on the base $\mathcal{A}_{n-1}$.
4. Due to convergence, the result is necessarily single-valued and descends to $\mathrm{Conf}_{n-1}(\mathcal{E}_\tau)$.

Corollary

After integrating out all but one puncture, a MGF is thus expressed in terms of iterated integrals on $\mathcal{E}_\tau^\times$, that is, eMZV and their c.c.

$$\omega_A(n_1, \cdots, n_r) = \int_0^1 \omega_0^{(n_1)} \cdots \omega_0^{(n_r)}, \quad \omega_B(n_1, \cdots, n_r) = \int_0^\tau \omega_0^{(n_1)} \cdots \omega_0^{(n_r)}$$

Iterated Eisenstein integrals

Theorem (Enriquez)

eMZV can be written (uniquely) as iterated Eisenstein integrals, with coefficients that are multiple zeta values.

$$2\pi i\partial_\tau \int_a^b g_{n_1}\cdots g_{n_r} = n_1 g_{1+n_1}(b)\int_a^b g_{n_2}\cdots g_{n_r} - n_r g_{1+n_r}(a)\int_a^b g_{n_1}\cdots g_{n_{r-1}}$$

$$+\sum_{\mu=1}^{r-1}\sum_{k=0}^{n_\mu+n_{\mu+1}+1}(n_\mu+n_{\mu+1}-k)\left[\binom{k-1}{n_{\mu+1}-1}-\binom{k-1}{k-n_\mu}\right]G_{n_\mu+n_{\mu+1}+1-k}$$

$$\times\int_a^b g_{n_1}\cdots g_{n_{\mu-1}}g_k g_{n_{\mu+2}}\cdots g_{n_r}$$

Example

$$\omega_A(0,1,0,0) = \frac{3\zeta_3}{4\pi^2} - 36\zeta_4\Gamma(1,1,1) + 18\Gamma(1,1,G_4)$$

Summary

- modular graph functions are real analytic modular functions for $\mathrm{SL}_2(\mathbb{Z})$
- they are bilinear in holomorphic and antiholomorphic iterated integrals of Eisenstein series
- such representations can be computed algorithmically

Corollary

The coefficients of the q-expansions are rational linear combinations of multiple zeta values.

Summary

- modular graph functions are real analytic modular functions for $\mathrm{SL}_2(\mathbb{Z})$
- they are bilinear in holomorphic and antiholomorphic iterated integrals of Eisenstein series
- such representations can be computed algorithmically

Corollary

The coefficients of the q-expansions are rational linear combinations of multiple zeta values.

Conjecture (Zerbini, Schlotterer, Brödel)

The coefficients of the q-expansions are rational linear combinations of *single-valued* multiple zeta values.

extras

Shuffle product

The shuffle product of two words

$$w_{n+m}\cdots w_{n+1} \sqcup\!\sqcup\, w_n \cdots w_1 = \sum_{\sigma} w_{\sigma(n+m)} \cdots w_{\sigma(1)}$$

is the sum of all their shuffles σ, i.e. permutations which preserve the relative order of letters in both factors:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \quad \text{and} \quad \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

For arbitrary words u and v, we find that ($\int_\gamma$ is linearly extended)

$$\left(\int_\gamma u\right) \cdot \left(\int_\gamma v\right) = \int_\gamma (u \sqcup\!\sqcup\, v).$$

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Example

$$\int_\gamma \omega_3 \cdot \int_\gamma \omega_2\omega_1 = \int_\gamma \omega_3\omega_2\omega_1 + \int_\gamma \omega_2\omega_3\omega_1 + \int_\gamma \omega_2\omega_1\omega_3$$

$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

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Example

$$\int_\gamma \omega_3 \cdot \int_\gamma \omega_2\omega_1 = \int_\gamma (\omega_3\omega_2\omega_1 + \omega_2\omega_3\omega_1 + \omega_2\omega_1\omega_3)$$

$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

q-expansion of real analytic Eisenstein series

$$E_s = \frac{4(2s-3)!}{(s-1)!(s-2)!}\zeta_{2s-1}(4y)^{1-s} + (-4\pi y)^s \frac{\zeta_{2s}}{(2i\pi)^{2s}}$$
$$+ \frac{2}{(s-1)!}\sum_{N\geq 1} N^{s-1}\sigma_{1-2s}(N)\left(q^N + \bar{q}^N\right)\sum_{m=0}^{n-1}\frac{(n+m-1)!}{m!(n-m-1)!}(4Ny)^{-m}$$

KZB connection

$$
\begin{aligned}
\mathrm{d}\tilde{\Gamma}\begin{pmatrix} n_1 \cdots n_r \\ z_1 \cdots z_r \end{pmatrix}; z\Big) &= \sum_{p=1}^{k-1} (-1)^{n_p+1} \tilde{\Gamma}\begin{pmatrix} \cdots n_{p-1} & 0 & n_{p+1} \cdots \\ \cdots z_{p-1} & 0 & z_{p+1} \cdots \end{pmatrix} \omega_{p,p+1}^{n_p+n_{p+1}} \\
&+ \sum_{p=1}^{k} \sum_{r=0}^{n_p+1} \left[\binom{n_{p-1}+r-1}{n_{p-1}-1} \tilde{\Gamma}\begin{pmatrix} \dots & n_{p-1}+r & n_{p+1} & \dots \\ & z_{p-1} & z_{p+1} & \end{pmatrix} \omega_{p,p-1}^{n_p-r} \right. \\
&\left. - \binom{n_{p+1}+r-1}{n_{p+1}-1} \tilde{\Gamma}\begin{pmatrix} \dots & n_{p-1} & n_{p+1}+r & \dots \\ & z_{p-1} & z_{p+1} & \end{pmatrix} \omega_{p,p+1}^{n_p-r} \right]
\end{aligned}
$$

where

$$
\omega_{ij}^{n} = (\mathrm{d}z_j - \mathrm{d}z_i) g_n(z_j - z_i; \tau) + \frac{n\mathrm{d}\tau}{2i\pi} g_{n+1}(z_j - z_i; \tau)
$$