Feynman integrals, graph polynomials and zeta values

Erik Panzer

All Souls College

North meets South Colloquium
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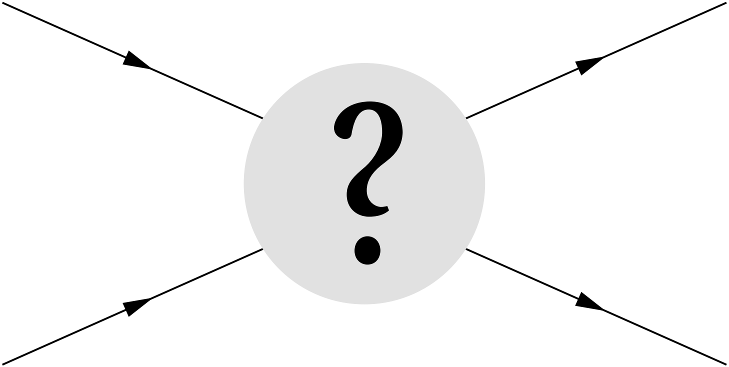
Oxford

Perturbative Quantum Field Theory (QFT)



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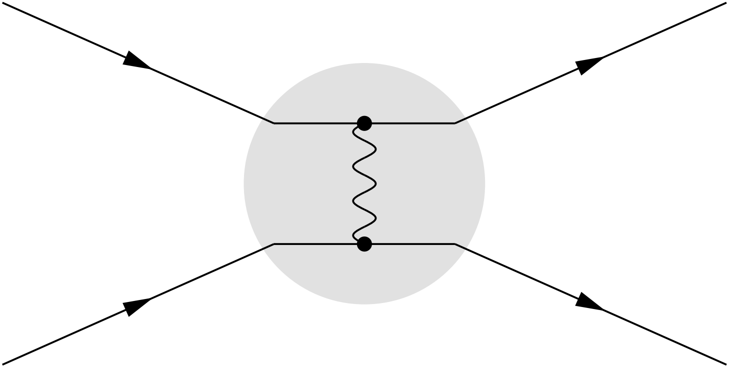

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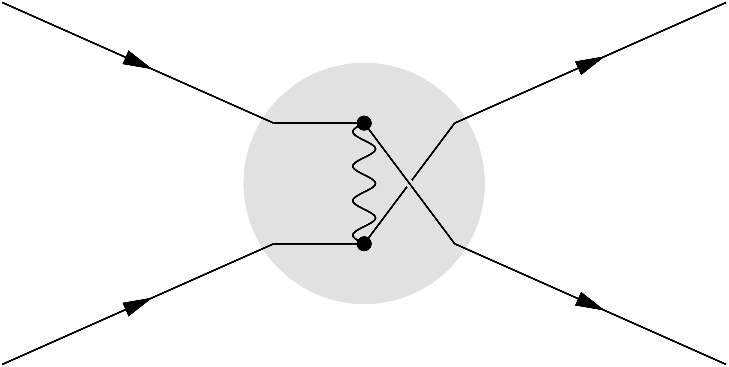

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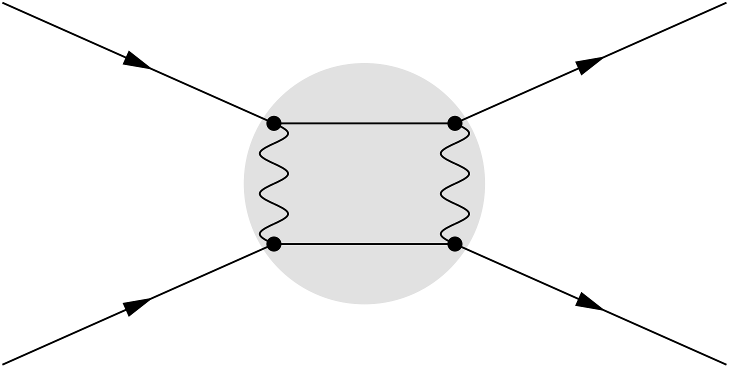

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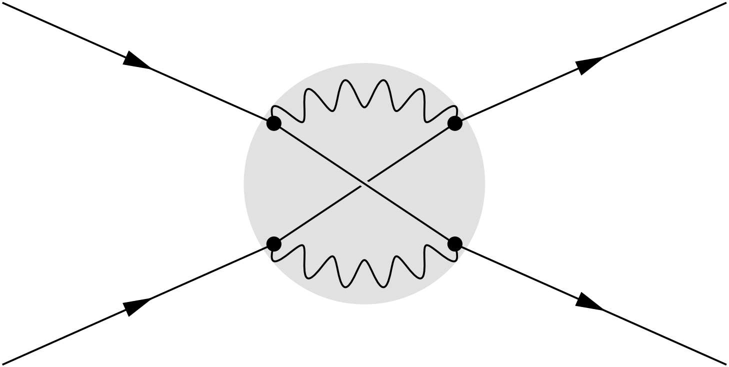

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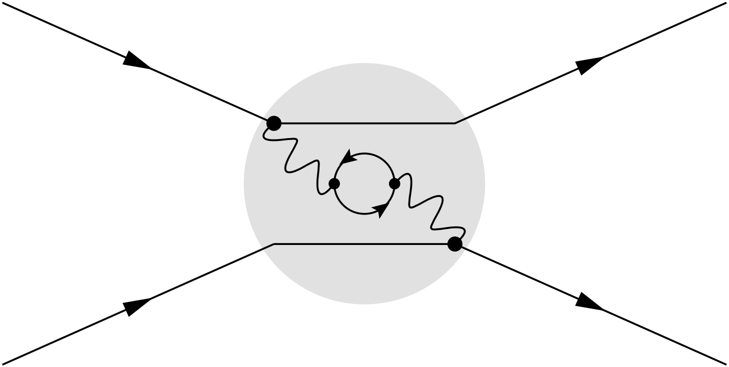

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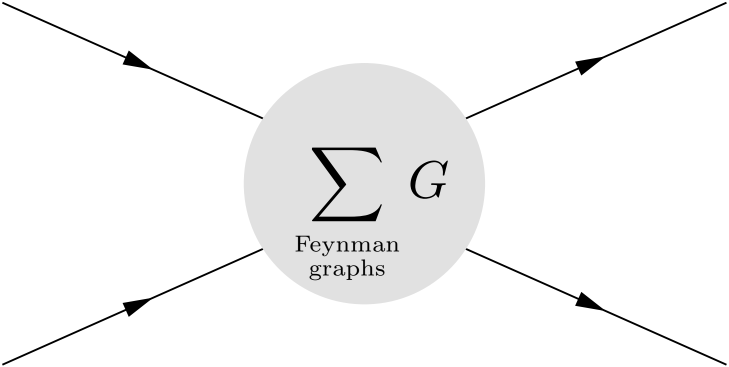

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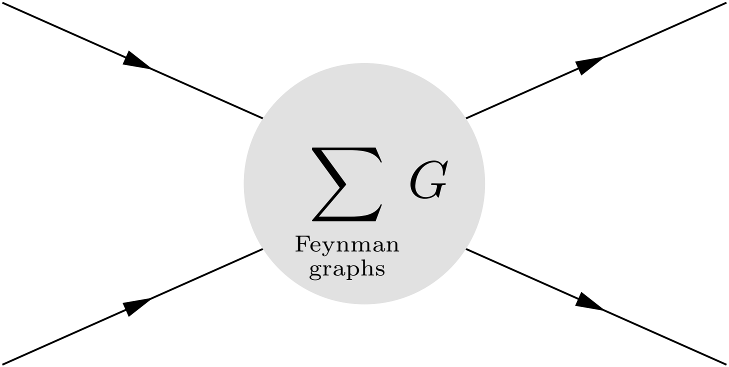

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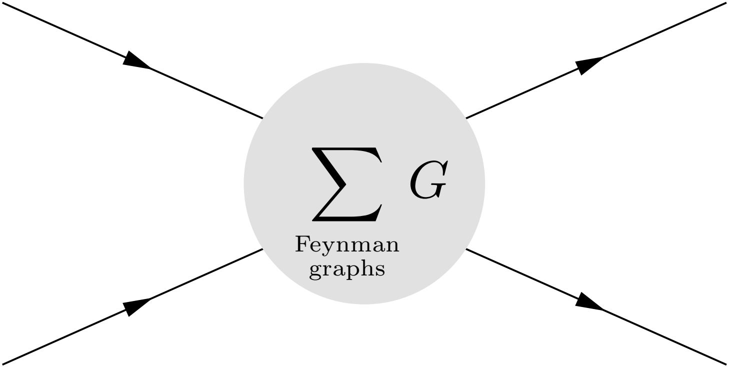
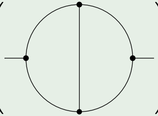
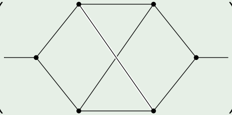


- each Feynman graph represents a Feynman integral $\Phi(G)$
- truncated sum $\sum_G \Phi(G)$ approximates the process
- very accurate measurements demand precise theory predictions
  $\Rightarrow$ many graphs have to be included

Example

$\Phi\Big($  $\Big) = 6\zeta_3$ $\quad\Phi\Big($  $\Big) = 20\zeta_5$

Riemann zeta function: $$\zeta_n = \sum_{0<k} \frac{1}{k^n}$$

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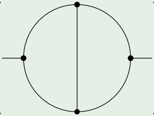
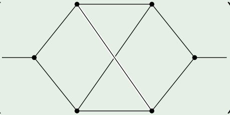
$\Phi\Bigg($  $\Bigg) = 6\zeta_3$ $\qquad$ $\Phi\Bigg($  $\Bigg) = 20\zeta_5$

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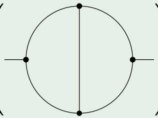
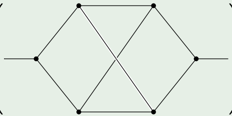
$\Phi\Bigg($  $\Bigg) = 6\zeta_3$ $\qquad$ $\Phi\Bigg($  $\Bigg) = 20\zeta_5$

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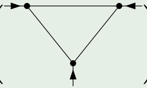
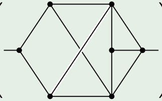
$\Phi\Bigg($  $\Bigg) = \frac{2\,\mathrm{Im}\,[\mathrm{Li}_2(z) + \log(1-z)\log|z|]}{\mathrm{Im}\,z} = \frac{2D_2(z)}{\mathrm{Im}\,z}$

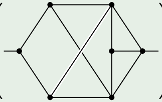
Polylogarithms: $\mathrm{Li}_n = \sum_{0<k} \frac{z^k}{k^n} \quad \Rightarrow \quad \zeta_n = \mathrm{Li}_n(1)$

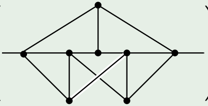
Example

$$\Phi\left(\right.$$  $$\left.\right) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{1\,044}{5}\zeta_8$$

Double zeta value: $$\zeta_{3,5} = \sum_{0<k<m} \frac{1}{k^3 m^5}$$

Example

$\Phi\Big($  $\Big) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{1\,044}{5}\zeta_8$

$\Phi\Big($  $\Big) = \frac{92\,943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_3\zeta_{3,5}\right) - \frac{1155}{4}\zeta_3^2\zeta_5$

$$+\, 896\zeta_3\left(\frac{27}{80}\zeta_{3,5} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8\right)$$

Multiple zeta values:

$$\zeta_{n_1,\ldots,n_d} = \sum_{0<k_1<\ldots<k_d} \frac{1}{k_1^{n_1}\cdots k_d^{n_d}}$$

(one) definition of the Feynman integrals

Introduce variables α_e for each edge e, and let $\mathsf{sdd} = |E(G)| - 2 \cdot \mathsf{loops}(G)$:

$$\Phi(G) = \int_{(0,\infty)^E} \frac{\Omega}{\psi^{2-\mathsf{sdd}} \varphi^{\mathsf{sdd}}}, \qquad \Omega = \delta(1 - \alpha_N) \prod_{e \in E} \mathrm{d}\alpha_e$$

Graph polynomials:

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e$$

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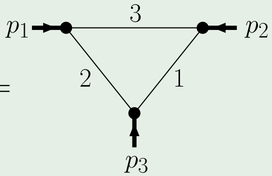
$G =$ 

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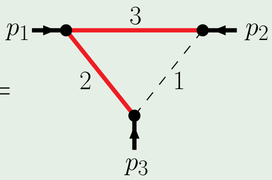
$G =$ 

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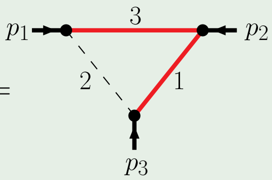

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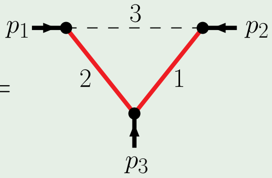
$G =$ 

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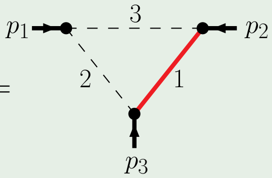
$G =$ 

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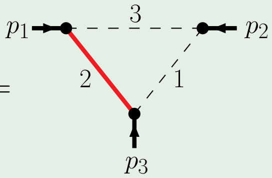
$G =$ 

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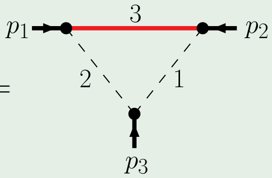
$G =$ 

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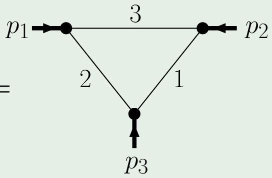
$G =$ 

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Comments

- Feynman integrals are periods à la [Kontsevich,Zagier]
- underlying geometry:

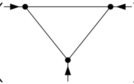
$$\Phi(G) = \int_\sigma \omega$$

$$\sigma = \{[\alpha_1, \ldots, \alpha_E] \colon \ \alpha_i \geq 0\} \in H_{E-1}(\mathbb{P}^{E-1}(\mathbb{R}), \cup \{\alpha_i = 0, \infty\})$$

$$\omega = \frac{\Omega}{\psi^{2-\mathrm{sdd}} \varphi^{\mathrm{sdd}}} \in H_{\mathrm{dR}}^{E-1}(\mathbb{P}^{E-1} \setminus \{\psi \cdot \varphi = 0\})$$

- after desingularization, Feynman integrals become *motivic periods* [Francis Brown]
- graph hypersurfaces like $\{\psi = 0\}$ tend to be very singular

Example: massless triangle

$\Phi\Bigg($  $\Bigg) = \displaystyle\int\int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$

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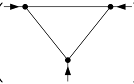
$\Phi\Big($  $\Big)$

$$= \int\int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$$

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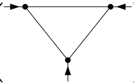
$\Phi\Bigg($  $\Bigg)$

$$
\begin{aligned}
&= \int\int \frac{\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)} \\
&= \int \frac{\mathrm{d}\alpha_2}{(z+\alpha_2)(\bar{z}+\alpha_2)} \log \frac{(1+\alpha_2)(z\bar{z}+\alpha_2)}{(1-z)(1-\bar{z})\alpha_2} \\
&= \frac{1}{z-\bar{z}} \int \left(\frac{\mathrm{d}\alpha_2}{\alpha_2+\bar{z}} - \frac{\mathrm{d}\alpha_2}{\alpha_2+z} \right) \log \frac{(\alpha_2+1)(\alpha_2+z\bar{z})}{(1-z)(1-\bar{z})\alpha_2}
\end{aligned}
$$

Polylogarithms are iterated integrals:

$$\mathrm{Li}_1(z) = \sum_{0<k} \frac{z^k}{k} = -\log(1-z) = \int_0^z \frac{\mathrm{d}t}{1-t}$$

$$\mathrm{Li}_2(z) = \sum_{0<k} \frac{z^k}{k^2} = \int_0^z \frac{\mathrm{d}t}{t}\,\mathrm{Li}_1(t)$$

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Insert these equations into each other:

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$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{d}t_1}{t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2} \int_0^{t_2} \cdots \int_0^{t_{n-2}} \frac{\mathrm{d}t_{n-1}}{t_{n-1}} \mathrm{Li}_1(t_{n-1})$$

Polylogarithms are iterated integrals:

$$\mathrm{Li}_1(z) = \sum_{0<k} \frac{z^k}{k} = -\log(1-z) = \int_0^z \frac{\mathrm{d}t}{1-t}$$

$$\mathrm{Li}_2(z) = \sum_{0<k} \frac{z^k}{k^2} = \int_0^z \frac{\mathrm{d}t}{t}\,\mathrm{Li}_1(t)$$

$$\vdots$$

$$\mathrm{Li}_n(z) = \sum_{0<k} \frac{z^k}{k^n} = \int_0^z \frac{\mathrm{d}t}{t}\,\mathrm{Li}_{n-1}(t)$$

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Insert these equations into each other:

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Definition (Hyperlogarithms)

$$G(\underbrace{\sigma_1, \ldots, \sigma_w}_{\vec{\sigma}}; z) := \int_0^z \frac{\mathrm{d}z_1}{z_1 - \sigma_1} \int_0^{z_1} \frac{\mathrm{d}z_2}{z_2 - \sigma_2} \cdots \int_0^{z_{w-1}} \frac{\mathrm{d}z_w}{z_w - \sigma_w}$$

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Example

$$G(\sigma_3; z) \cdot G(\sigma_2, \sigma_1; z) =$$
$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

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- multivalued, monodromies, path concatenation

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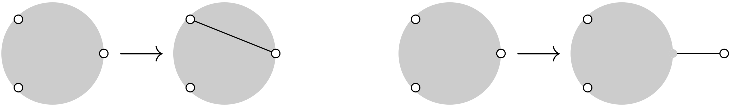
- multivalued, monodromies, path concatenation
- algebraic description $G : T(\Sigma) \rightarrow \{\text{transcendental functions}\}$

Tensor algebra $\quad T(\Sigma) := \mathbb{Q}\langle \omega_\sigma : \sigma \in \Sigma \rangle = \mathrm{lin}_{\mathbb{Q}} \Sigma^*$

Linearly reducible families (infinite)

- 3-constructible graphs (3-point functions) [Brown, Schnetz, Panzer]



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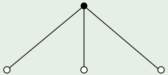

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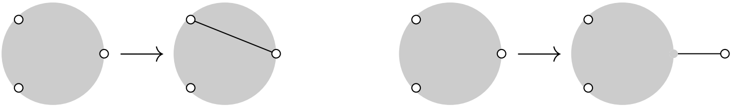

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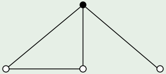

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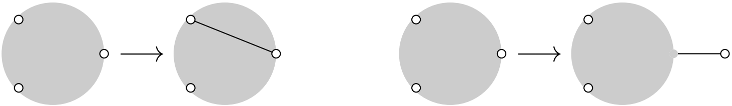

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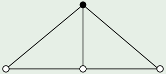

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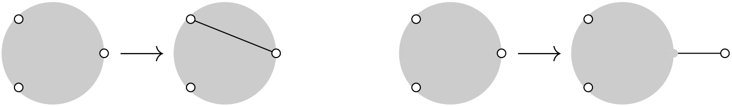

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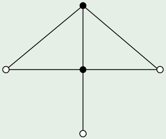

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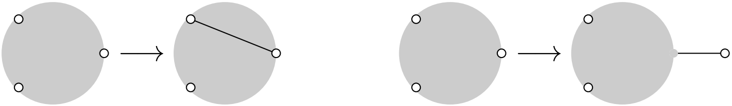

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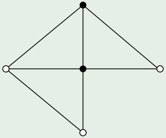

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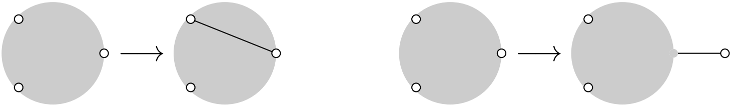

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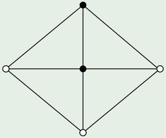

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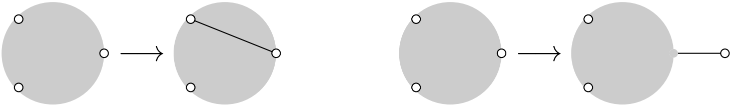

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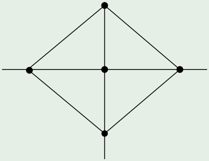

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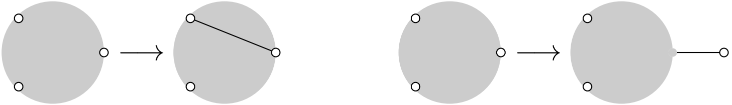

Theorem (Panzer)

All such Feynman integrals are MPL over the alphabet $\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, 1-z\bar{z}, 1-z-\bar{z}, z\bar{z}-z-\bar{z}\}$.

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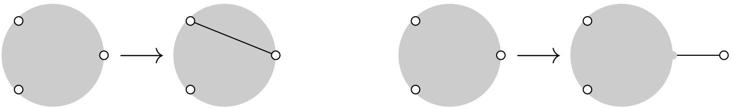

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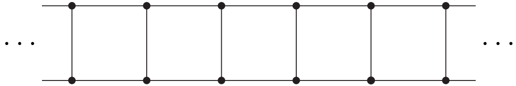
- minors of ladder-boxes (up to 2 legs off-shell)



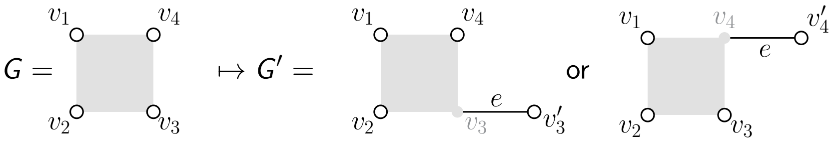
Theorem (Panzer)

These are MPL with alphabet $\{x, 1+x\}$ *for* $x = s/t$.

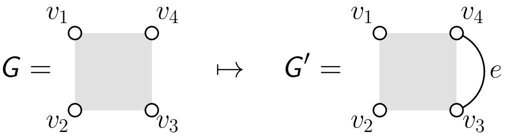
4-point recursions

Start with the box and repeat, in any order:

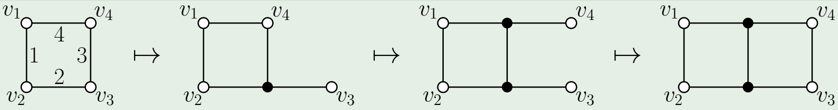
- Appending a vertex:



- Adding an edge:



Example

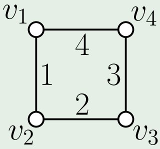


Forest polynomials

Let f_3, f_4, f_{12} and f_{14} denote the spanning forest polynomials such that

$$\varphi = \mathcal{F} = (p_1 + p_2)^2 f_{12} + (p_1 + p_4)^2 f_{14} + p_3^2 f_3 + p_4^2 f_4$$

Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad f_{12} = \alpha_2\alpha_4 \quad f_3 = \alpha_2\alpha_3$$
$$f_{14} = \alpha_1\alpha_3 \quad f_4 = \alpha_3\alpha_4$$

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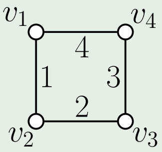
Definition

$$F(G; z) := \int_{\mathbb{R}_+^E} \psi_G^{-D/2} \cdot \delta^{(4)}\left(\frac{f}{\psi} - z\right) \prod_{e \in E} \mathrm{d}\alpha_e^{a_e - 1} \alpha_e \qquad (\mathbb{R}_+^4 \longrightarrow \mathbb{R}_+)$$

Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$f_{12} = \alpha_2 \alpha_4 \qquad f_3 = \alpha_2 \alpha_3$$

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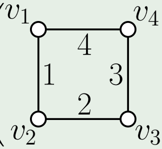

$$F\left(G ; z \right) = \begin{cases} \dfrac{1}{z_3 z_4} & (D = 4) \\[2ex] \dfrac{z_{12}}{[\underbrace{z_{12}(z_{14} + z_3 + z_4) + z_3 z_4}_{Q}]^2} & (D = 6) \end{cases}$$

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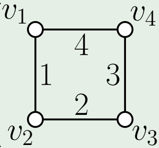
Definition

$$\Phi(G) = \frac{\Gamma(\mathrm{sdd})}{\prod_e \Gamma(a_e)} \int_0^\infty \frac{F(G;z)\,\Omega}{\left[(p_1+p_2)^2 z_{12} + (p_1+p_4)^2 z_{14} + p_3^2 z_3 + p_4^2 z_4\right]^{\mathrm{sdd}}}$$

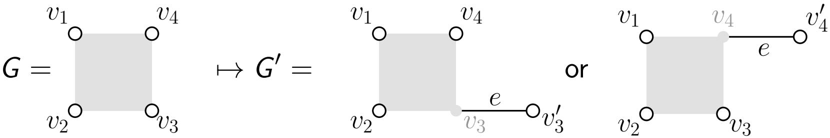
Example

$F\Big($



$;z\Big) = \begin{cases} \dfrac{1}{z_3 z_4} & (D=4) \\[2ex] \dfrac{z_{12}}{\big[\underbrace{z_{12}\,(z_{14}+z_3+z_4) + z_3 z_4}_{Q}\big]^2} & (D=6) \end{cases}$

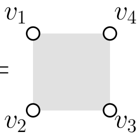
Appending a vertex

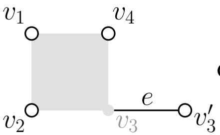


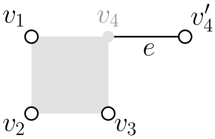
Using $(f'_{12}, f'_{14}, f'_3, f'_4, \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e \psi, \psi)$,

$$F(G'; z) = \int_0^{z_4} F(G; z_{12}, z_{14}, z_3, z_4 - \alpha_e)\ \alpha_e^{a_e - 1} \mathrm{d}\alpha_e$$

Appending a vertex

$G =$ 

$\mapsto G' =$ 

or 

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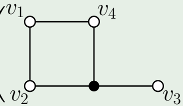
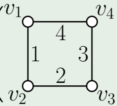
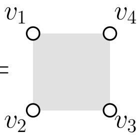
Example ($D = 6$ and $a_e = 1$)

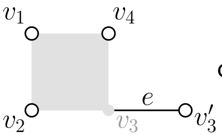
$F\Bigg($ 

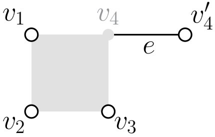
$; z\Bigg) = \int_0^{z_3} F\Bigg($ 

$; z_{12}, z_{14}, z_3', z_4\Bigg)\ \mathrm{d}z_3'$

Appending a vertex

$G =$ 

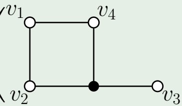
$\mapsto G' =$ 

or 

Using $(f'_{12}, f'_{14}, f'_3, f'_4, \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e\psi, \psi)$,

$$F(G'; z) = \int_0^{z_4} F(G; z_{12}, z_{14}, z_3, z_4 - \alpha_e)\ \alpha_e^{a_e-1}\mathrm{d}\alpha_e$$

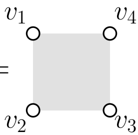
Example ($D = 6$ and $a_e = 1$)

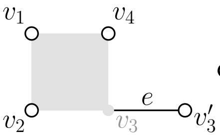
$F\Big($ 

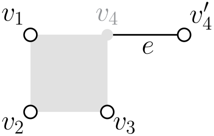
$; z\Big)$

$$= \int_0^{z_3} \frac{z_{12}\ \mathrm{d}z'_3}{[z_{12}(z_{14} + z'_3 + z_4) + z'_3 z_4]^2} = \frac{z_3}{(z_{14}+z_4)\cdot Q}$$

Appending a vertex

$G =$ 

$\mapsto G' =$ 

or 

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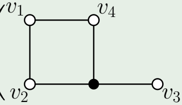
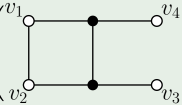
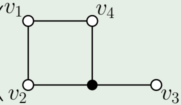
Example ($D = 6$ and $a_e = 1$)

$F\Big($ 

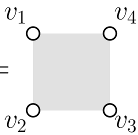
$; z\Big) = \int_0^{z_3} \frac{z_{12}\ \mathrm{d}z'_3}{[z_{12}(z_{14} + z'_3 + z_4) + z'_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$

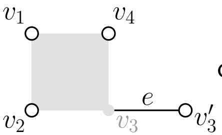
$F\Big($ 

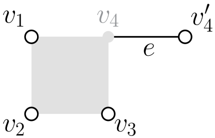
$; z\Big) = \int_0^{z_4} F\Big($ 

$; z_{12}, z_{14}, z_3, z'_4\Big)\ \mathrm{d}z'_4$

Appending a vertex

$G =$ 

$\mapsto G' =$ 

or 

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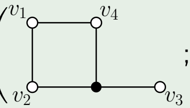
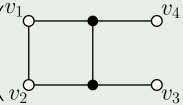
Example ($D = 6$ and $a_e = 1$)

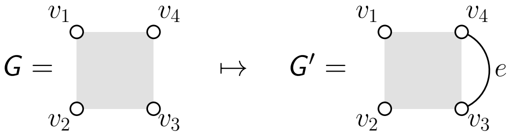
$F\Bigg($ 

$; z \Bigg) = \int_0^{z_3} \frac{z_{12}\ \mathrm{d}z'_3}{[z_{12}(z_{14} + z'_3 + z_4) + z'_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$

$F\Bigg($ 

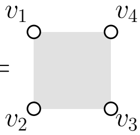
$; z \Bigg) = \frac{1}{z_{12} - z_{14}} \ln \frac{z_{12}(z_3 + z_{14})(z_4 + z_{14})}{z_{14} \cdot Q}$

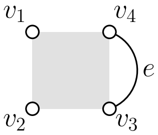
Adding an edge



$$F_{G'}(z) = Q^{a_e + \mathsf{sdd} - D} \int_0^{z_{12}} x^{D/2-2} \left[Q^{D/2-\mathsf{sdd}} \cdot F_G \right]_{z_{12} = z_{12} - x} \mathrm{d}x$$

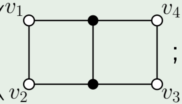
Adding an edge

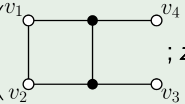
$G =$ 

$\mapsto$ $G' =$ 

$$F_{G'}(z) = Q^{a_e + \mathrm{sdd} - D} \int_0^{z_{12}} x^{D/2-2} \left[Q^{D/2-\mathrm{sdd}} \cdot F_G \right]_{z_{12}=z_{12}-x} \mathrm{d}x$$

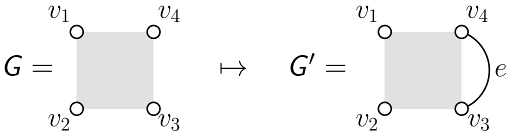
Example ($D = 6$ and $a_e = 1$)

$F\Big($ 

$; z \Big) = \dfrac{1}{Q^2} \displaystyle\int_0^{z_{12}} F \Big($ 

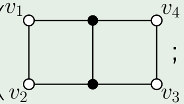
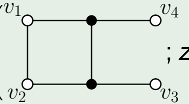
$; z_{12} - x, z_{14}, z_3, z_4 \Big) \, x \mathrm{d}x$

Adding an edge



$$F_{G'}(z) = Q^{a_e + \mathrm{sdd} - D} \int_0^{z_{12}} x^{D/2-2} \left[Q^{D/2 - \mathrm{sdd}} \cdot F_G \right]_{z_{12} = z_{12} - x} \mathrm{d}x$$

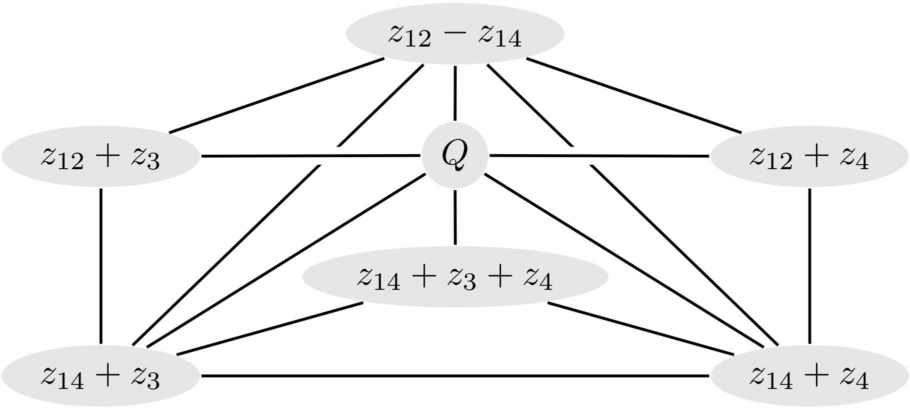
Example ($D = 6$ and $a_e = 1$)

$F\Big($  $; z \Big) = \frac{1}{Q^2} \int_0^{z_{12}} F \Big($  $; z_{12} - x, z_{14}, z_3, z_4 \Big) \, x \, \mathrm{d}x$

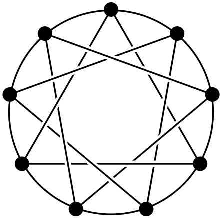
$$= \frac{z_{12} - z_{14}}{Q^2} \left[\ln \frac{Q}{z_3 z_4} \ln \frac{(z_{14} + z_3)(z_{14} + z_4)}{z_{14}(z_{14} + z_3 + z_4)} - \mathrm{Li}_2 \left(\frac{z_3 z_4 (z_{14} - z_{12})}{z_{14} Q} \right) \right]$$

$$+ \frac{z_{12} - z_{14}}{Q^2} \mathrm{Li}_2 \left(\frac{z_3 z_4}{Q} \right) + \frac{z_{12}}{Q^2} \ln \frac{z_{14} z_3 z_4}{z_{12}(z_{14} + z_3)(z_{14} + z_4)} - \frac{\ln(z_3 z_4 / Q)}{Q(z_{14} + z_3 + z_4)}$$

Compatibility graph of box-ladders



You made it! The talk is over!



Thank you for your attention!

A Galois coaction on ϕ^4 periods?

Theorem (Deligne)

For $N \in \{1, 2, 3, 4, '6', 8\}$, the algebra of motivic MPL at Nth roots of unity is isomorphic to a freely generated shuffle algebra. Example:

$$MZV \cong \mathbb{Q}[\pi^2] \otimes \mathbb{Q}\langle f_3, f_5, f_7, \ldots \rangle \qquad MDV \cong \mathbb{Q}[i\pi] \otimes \mathbb{Q}\langle f_2, f_3, f_4, f_5, \ldots \rangle$$

Example

$$\zeta_{2n+1} \mapsto f_{2n+1} \qquad \zeta_{3,5} \mapsto -5 f_5 f_3$$

Message: periods are not just numbers, but have a structure! Consider the map δ_k which clips off the first letter:

$$\delta_k(f_{n_1} \ldots f_{n_r}) := \begin{cases} f_{n_2} \ldots f_{n_r} & \text{if } k = n_1 \\ 0 & \text{else} \end{cases}$$

Example

$$\delta_3(\zeta_3) = 1 \qquad \delta_3(\zeta_{3,5}) = 0 \qquad \delta_5(\zeta_{3,5}) = -5\zeta_3 \qquad \delta_k \zeta_{2n} = 0$$

Coaction conjecture (O. Schnetz)

The periods of primitive log.-div. ϕ^4 graphs are closed under the action of the operators δ_k.
Other words: The cosmic Galois group acts on ϕ^4 periods.

Example

$$P_{7,11} = -\frac{332262}{43} f_8 f_3 + \frac{54918}{55} f_6 f_5 + \frac{1134}{13} f_4 f_7 - \frac{1874502}{3485} f_2 f_9 + 5\,670 f_2 f_3 f_3 f_3 - \frac{3\,216\,912\,825\,399\,005\,402\,331\,281\,812\,377\,062\,149}{14\,080\,217\,073\,343\,074\,027\,422\,017\,273\,458\,000} \left(\frac{\pi}{\sqrt{3}}\right)^{11}.$$

Note: After δ_{2k}, only odd letters survive $\Rightarrow$ MZV, in ϕ^4.

- highly non-trivial constraint on ϕ^4 periods
- proven for generalized periods [F. Brown]