Symbolic integration of multiple polylogarithms

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Applications of Computer Algebra
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Problem: Multiple integrals of rational functions

$$\int_0^\infty \int_0^\infty \frac{\log(1+1/(xz))\,\mathrm{d}x\,\mathrm{d}y}{(1+y)(1+x+y+1/z)} = ?$$

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Sometimes such integrals are expressible as multiple polylogarithms (MPL)

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and their special values, like multiple zeta values (MZV)

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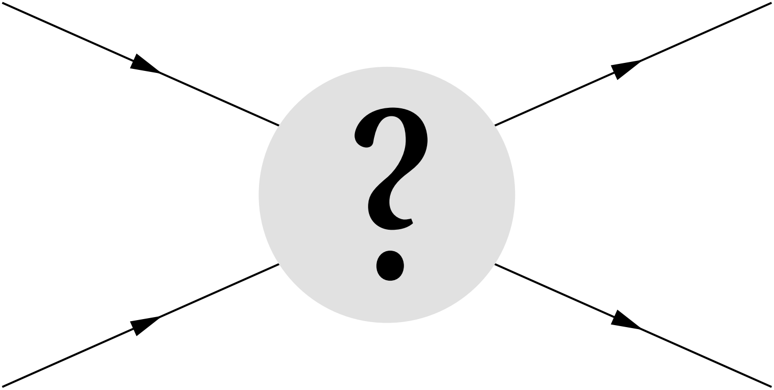
Example

$$\int_0^\infty \cdots \int_0^\infty \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_4}{[\alpha_1+\alpha_4+(\alpha_1+\alpha_4+1)(\alpha_2+\alpha_3)][(\alpha_1+\alpha_2)(\alpha_3+\alpha_3\alpha_4+\alpha_4)+\alpha_1\alpha_2(\alpha_3+\alpha_4)]}$$

$$= 6\zeta_3 = 6\sum_{k=1}^\infty \frac{1}{k^3}$$

Motivation

Perturbative Quantum Field Theory



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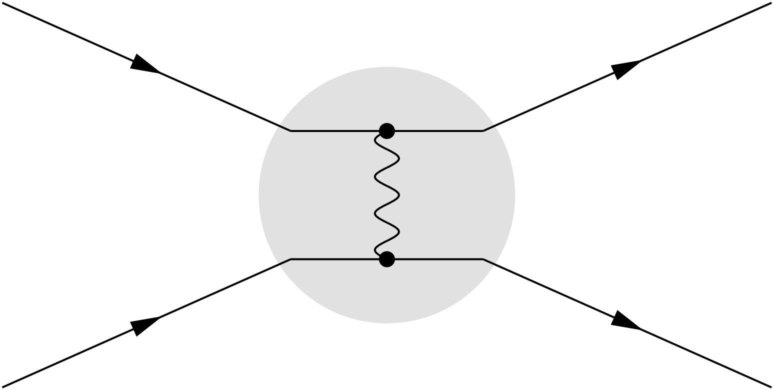

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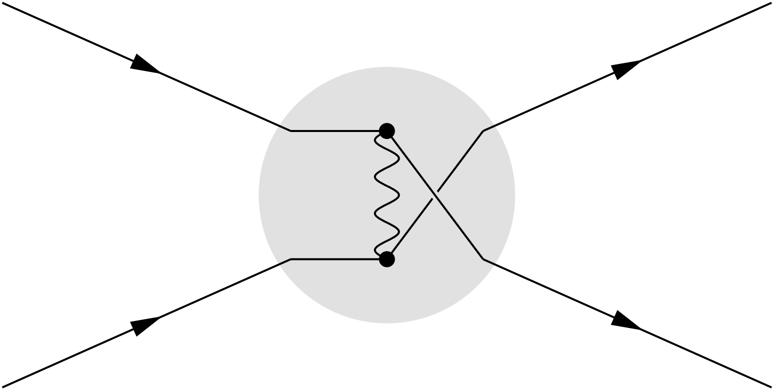

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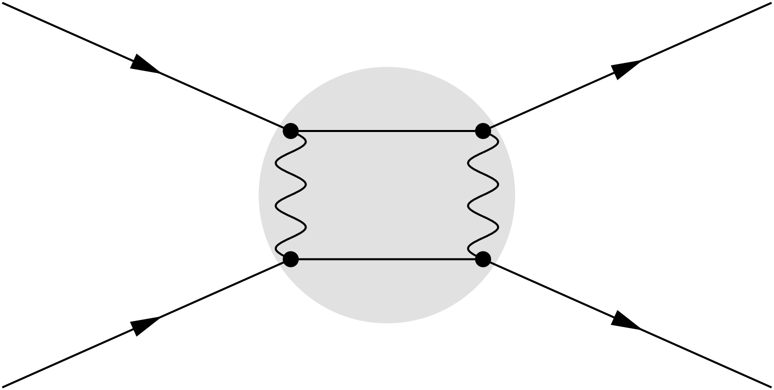

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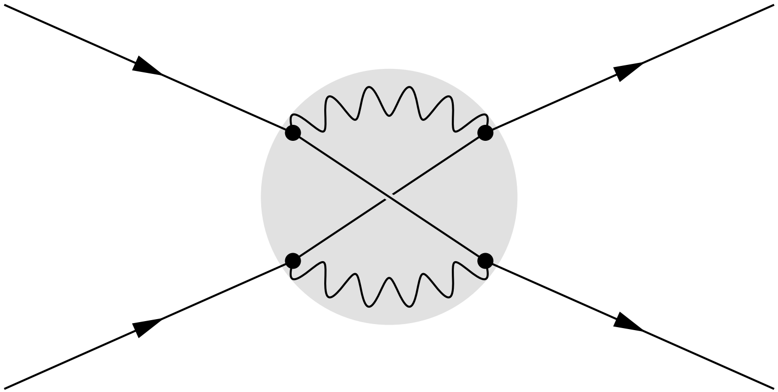

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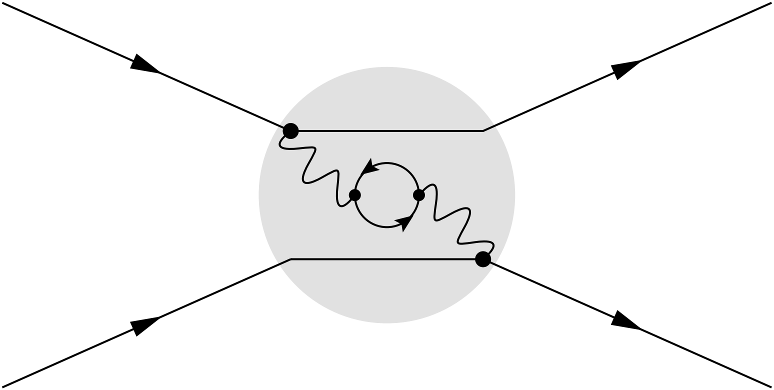

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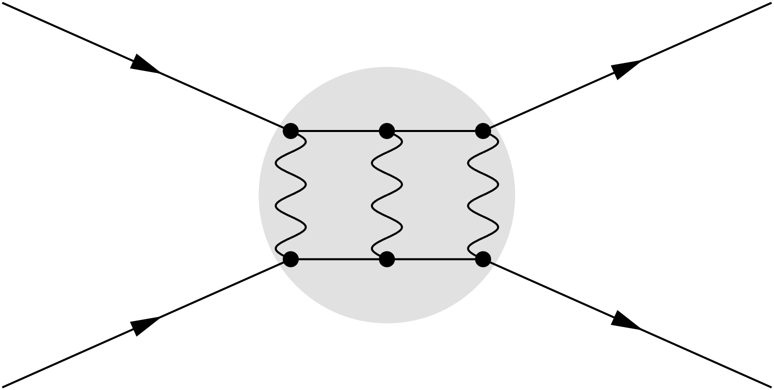

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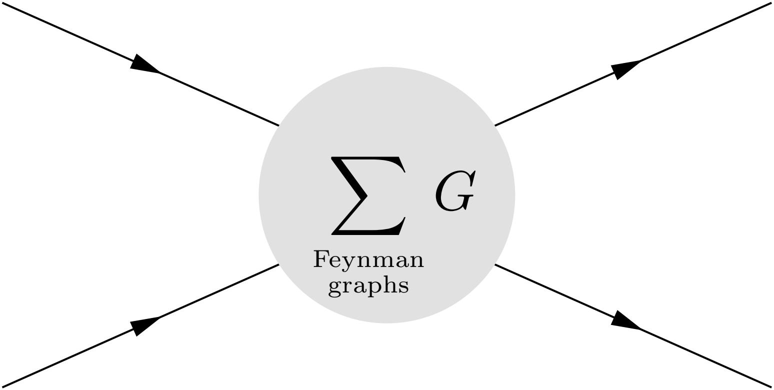

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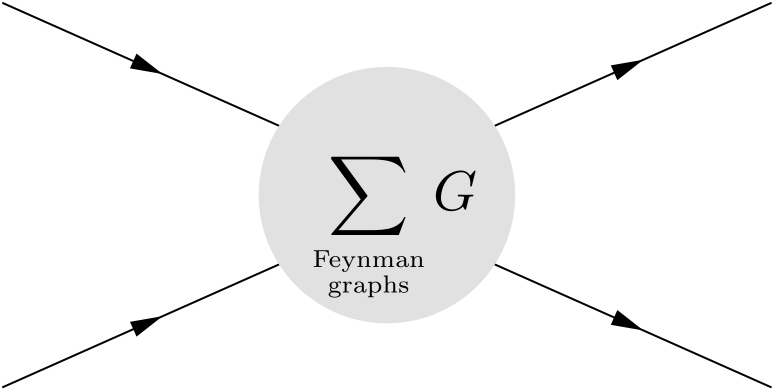

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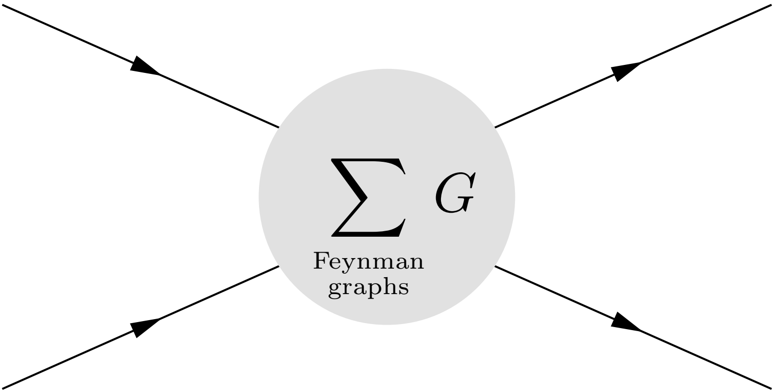
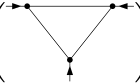


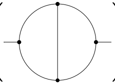
- each Feynman graph represents a Feynman integral (FI), $\Phi(G)$
- truncated sum $\sum_G \Phi(G)$ approximates the process
- very accurate measurements demand precise theoretical predictions
  Challenges: number of graphs & complexity of integrals

Some FI are expressible as MPL, depending on momenta and masses like



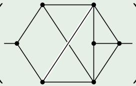
$$\Phi\left(\quad\right) = \frac{4i\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{z-\bar{z}}$$

and also including constants (like MZV) as in



$$\Phi\left(\quad\right) = 6\zeta_3.$$

Many conjectures by PSLQ, for example



$$\Phi\left(\quad\right) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$$

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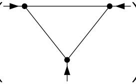
$\Phi\Big($  $\Big) = \frac{4i\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{z-\bar{z}}$

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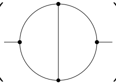
$\Phi\Big($  $\Big) = 6\zeta_3.$

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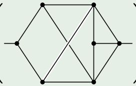
$\Phi\Big($  $\Big) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$

Questions

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2. How to compute it explicitly?

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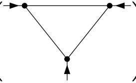
$$\Phi\left(\right.$$  $$\left. \right) = \frac{4i\,\mathrm{Im}\left[\mathrm{Li}_2(z) + \log(1-z)\log|z|\right]}{z-\bar{z}}$$

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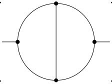
$$\Phi\left(\right.$$  $$\left. \right) = 6\zeta_3.$$

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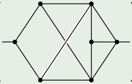
$$\Phi\left(\right.$$  $$\left. \right) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{25056}{875}\zeta_2^4$$

Questions

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2. How to compute it explicitly in an efficient and automated way?

Schwinger parameters

$$\Phi(G) = \int_{(0,\infty)^E} \frac{1}{\psi^{D/2}} \left(\frac{\psi}{\varphi}\right)^{E - h_1(G)D/2} \delta(1-\alpha_N) \prod_e \mathrm{d}\alpha_e$$

Graph polynomials:

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e$$

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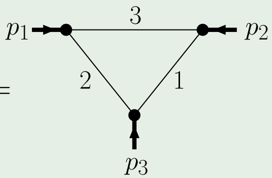
$G =$ 

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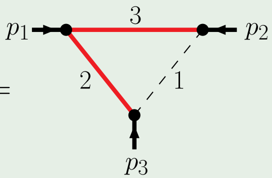
$G =$ 

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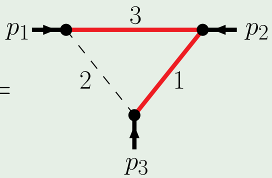
$G =$ 

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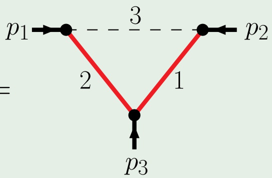
$G =$ 

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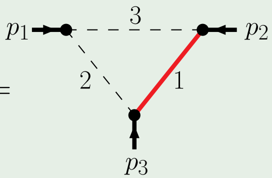
$G =$ 

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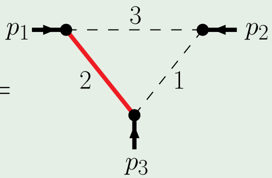
$G =$ 

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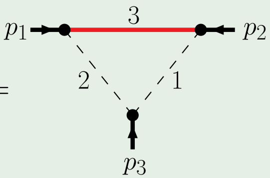

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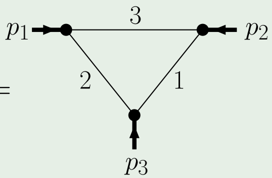

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Some other definite integral representations

- hypergeometric functions ${}_pF_q$ (expansion in a, b, ...)

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t $$

- Appell's functions F_1, F_2, F_3, F_4

$$ F_3\left(\begin{matrix} a, a' \\ b, b' \end{matrix}\middle| c \middle| x\ y\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \times \int_0^1 \int_0^{1-v} u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux)^{-a}(1-vy)^{-a'}\mathrm{d}u\mathrm{d}v $$

- Phase-space integrals
- periods of moduli spaces, string amplitudes

$$ \int_{0<t_1<\cdots<t_5<1} \frac{\mathrm{d}t_1\cdots\mathrm{d}t_5}{(1-t_1)(1-t_2)t_3(t_4-t_2)(t_5-t_3)t_5} = \frac{6}{5}\zeta_2^2 = \frac{\pi^4}{30} $$

Techniques

We can represent MPL as iterated path integrals in one dimension:

$$G(\underbrace{0\cdots 0,\sigma_d}_{n_d},\ldots,\underbrace{0\cdots 0,\sigma_1}_{n_1};z) = (-1)^d \operatorname{Li}_{n_1,\ldots,n_d}\left(\frac{\sigma_2}{\sigma_1},\cdots,\frac{\sigma_d}{\sigma_{d-1}},\frac{z}{\sigma_d}\right)$$

Hyperlogarithms [Poincaré 1884, Lappo-Danilevsky 1927]

$$G(\sigma_1,\ldots,\sigma_w;z) := \int_0^z \frac{\mathrm{d}z_1}{z_1-\sigma_1}\int_0^{z_1}\frac{\mathrm{d}z_2}{z_2-\sigma_2}\cdots\int_0^{z_{w-1}}\frac{\mathrm{d}z_w}{z_w-\sigma_w}$$

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- locally analytic, multivalued functions on $\mathbb{C}\setminus\{0,\sigma_1,\ldots,\sigma_w\}$
- divergences at $z\to\sigma_i$ (or ∞) are logarithmic
- span an algebra via the shuffle product

$$G(a,b;z)\cdot G(c;z) = G(a,b,c;z)+G(a,c,b;z)+G(c,a,b;z)$$

- path concatenation (analytic continuation) via coproduct

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Multiple integration with hyperlogarithms

Idea: Use Fubini's theorem to compute

$$f_n = \int_0^\infty f_{n-1}\, \mathrm{d}\alpha_n = \int_0^\infty \cdots \int_0^\infty f_0\, \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n.$$

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$$f_n := \int_0^\infty f_{n-1}\, \mathrm{d}\alpha_n = \lim_{\alpha_n \to \infty} F(\alpha_n) - \lim_{\alpha_n \to 0} F(\alpha_n).$$

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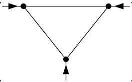
Corollary (algorithmic)

If all $\sigma_i \in \mathbb{Q}(\alpha)$, then $G(\vec{\sigma}; \sigma_0)$ is a hyperlogarithm in α with alphabet

$$\Sigma_\alpha = \{\text{zeros and poles of } \sigma_i(\alpha) - \sigma_j(\alpha) \colon\ 0 \leq i < j \leq w+1\}$$

Example: massless triangle

$\Phi\Big($  $\Big)$

$$= \int_0^\infty \int_0^\infty \frac{\mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$$

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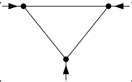
Φ () $= \int_0^\infty \int_0^\infty \frac{\mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$

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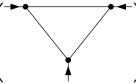
$\Phi\Bigg($  $\Bigg)$

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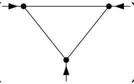
$\Phi\Big($  $\Big)$

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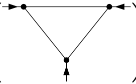

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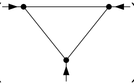
Φ () $= \int_0^\infty \int_0^\infty \frac{\mathrm{d}\alpha_2\, \mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$

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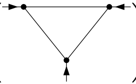
$$\Phi\left(\right.$$  $$\left.\right) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}\alpha_2\, \mathrm{d}\alpha_3}{(1+\alpha_2+\alpha_3)(\alpha_2\alpha_3 + z\bar{z}\alpha_3 + (1-z)(1-\bar{z})\alpha_2)}$$

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Definition (Polynomial reduction [Brown])

Let S denote a set of polynomials, then S_e are the irreducible factors of

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Lemma (approximation of Landau varieties)

If the singularities of F are cointained in S, then the singularities of $\int_0^\infty F\,\mathrm{d}\alpha_e$ are contained in S_e. Goal: bounds as tight as possible

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This gives only very coarse upper bounds. For example, $z\bar{z}-1$ is spurious: It drops out in $S_{2,3} \cap S_{3,2} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}\}$ because

$$S_{2,3} = \{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, z\bar{z}-z-\bar{z}\}\,.$$

Improvements

- Fubini algorithm [Brown]: intersect over different orders
- Compatibility graphs [Brown, Panzer]

Compatibility graphs

Keep track of compatibilities $C \subset \binom{S}{2}$ between polynomials:

- start with the complete graph ψ —— φ
- in S_e, only take resultants $[f, g]_e$ for compatible $\{f, g\} \in C$
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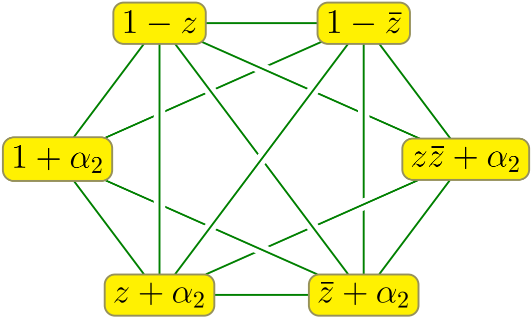

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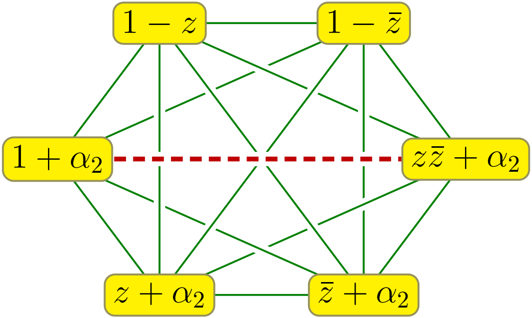

$z\bar{z}\alpha_1 + \alpha_2$ and $\alpha_1 + \alpha_2$ not compatible $\Rightarrow$ no resultant $1 - z\bar{z}$ in $(S, C)_{3,2}$

Problem for multiple integrals

If some $\sigma_i(\alpha) - \sigma_j(\alpha)$ does not factorize linearly in α, the transformation to $G(\cdots;\alpha)$ introduces algebraic letters.

Definition

If for some order of variables (edges), all $S_{1,\ldots,k}$ are linear in α_{k+1}, then S (the Feynman graph G with $S = \{\psi, \varphi\}$) is called linearly reducible.

Write $\mathcal{O}(S) = \mathbb{Q}[\vec{\alpha}, f^{-1}\colon f \in S]$ and

$$\mathrm{MPL}(S) = \mathcal{O}(S) \otimes \mathrm{lin}_{\mathbb{Q}} \{\text{iterated integrals of } \mathrm{d}\log(f)\text{'s } (f \in S)\}.$$

Lemma (algorithmic)

If S is linearly reducible and $f_0 \in MPL(S)$, then

$$\int_0^\infty \cdots \int_0^\infty f_0 \, \mathrm{d}\alpha_1 \cdots \alpha_N \in MPL(S_{1,\ldots,N}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

MPL have plenty of relations, like

$$-G(0,-\alpha;1) = G(0,0;\alpha) - G(0,-1;\alpha) - \zeta_2$$

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Lemma

All hyperlogarithms $G(\vec{\sigma};\alpha)$, $\partial_\alpha \vec{\sigma} = 0$, are linearly independent over $\overline{\mathbb{C}(\alpha)}$.

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Problem: Bases for MPL

Given some MPL $G(\vec{\sigma}(\vec{\alpha}), z(\vec{\alpha}))$ or $\mathrm{Li}_{\vec{n}}(\vec{z}(\vec{\alpha}))$ whose arguments ($\vec{\sigma}, z$ or $\vec{z}$) are rational functions of variables $\alpha_1, \ldots, \alpha_n$, write it in the basis

$$\sum_{\vec{\sigma}_1,\ldots,\vec{\sigma}_n} G(\vec{\sigma}_1(\alpha_2,\ldots,\alpha_n);\alpha_1) G(\vec{\sigma}_2(\alpha_3,\ldots,\alpha_n);\alpha_2) \cdots G(\vec{\sigma}_n;\alpha_n).$$

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- completely symbolic (no numerics)
- dependens on order of the variables $\alpha_1, \ldots, \alpha_n$
- in general not the shortest or "simplest" representation
- allows for symbolic verification of MPL identities

HyperInt

- `Maple`
- `Manual.ws`
- open source: `https://bitbucket.org/PanzerErik/hyperint`
- polynomial reduction
- integration of hyperlogarithms
- transformations of MPL to $G(\cdots;\alpha_1)\cdots G(\cdots;\alpha_N)$-basis
- symbolic computation of constants (MZV and alternating sums)
- Feynman graph polynomials

Example

```
> read "HyperInt.mpl":
> hyperInt(polylog(2,-x)*polylog(3,-1/x)/x,x=0..infinity):
> fibrationBasis(%);
```

$$\frac{8}{7}\zeta_2^3$$

computes $\int_0^\infty \mathrm{Li}_2(-x)\,\mathrm{Li}_3(-1/x)\mathrm{d}x = \frac{8}{7}\zeta_2^3$.

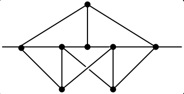
Sometimes a linearly reducible order is obvious, like for

$$\int_{0<t_1<\cdots<t_5<1} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_5}{(1-t_1)(1-t_2)t_3(t_4-t_2)(t_5-t_3)t_5} = \frac{6}{5}\zeta_2^2 = \frac{\pi^4}{30}$$

```
> hyperInt(1/(1-t1)/(1-t2)/t3/(t4-t2)/(t5-t3)/t5,
  [t1=0..t2,t2=0..t3,t3=0..t4,t4=0..t5,t5=0..1]):
> fibrationBasis(%);
```

$$\frac{6}{5}\zeta_2^2$$

In complicated cases, one first computes a polynomial reduction to check if a linearly reducible order exists. Both, polynomial reduction and integration can be parallelized manually.

$\Phi\Bigg($  $\Bigg)$

$$= \frac{92943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_{3,5}\zeta_3\right) - \frac{1155}{4}\zeta_3^2\zeta_5 + 896\zeta_3\left(\frac{27}{80}\zeta_{3,5} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8\right)$$

HyperInt: triangle

Graph polynomials:

```
> E:=[[1,2],[2,3],[3,1]]:
> M:=[[3,1],[1,z*zz],[2,(1-z)*(1-zz)]]:
> psi:=graphPolynomial(E):
> phi:=secondPolynomial(E,M):
```

Integration:

```
> hyperInt(eval(1/psi/phi,x[3]=1),[x[1],x[2]]):
> factor(fibrationBasis(%,[z,zz]));
```

$$(G(z;1)\,G(zz;0) - G(z;0)\,G(zz;1) + G(zz;0,1) - G(zz;1,0) + G(z;1,0) - G(z;0,1))/(z - zz)$$

Polynomial reduction:

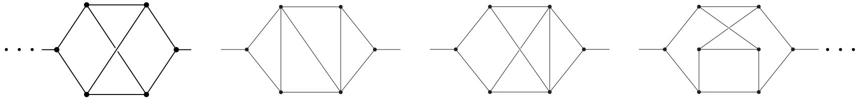
```
> L[{}]:=[{psi,phi}, {{psi,phi}}]:
> cgReduction(L):
> L[{x[1],x[2]}][1];
```

$$\{-1+z, -1+zz, -zz+z\}$$

Linearly reducible Feynman graphs

Linearly reducible families (fixed loop order)

1. all ≤ 4 loop massless propagators [Panzer]



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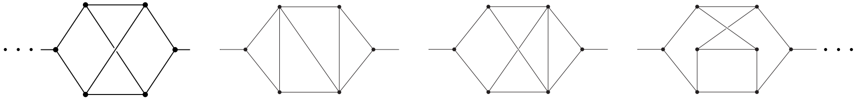

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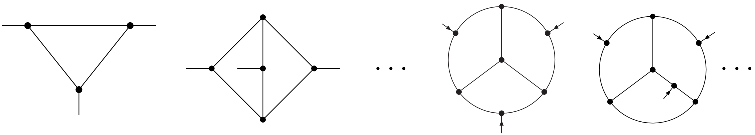

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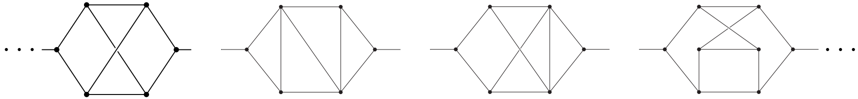

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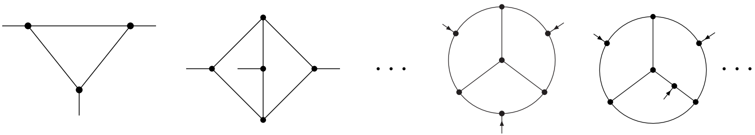
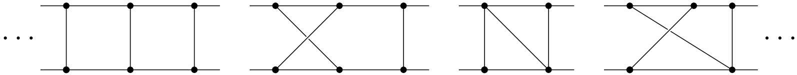
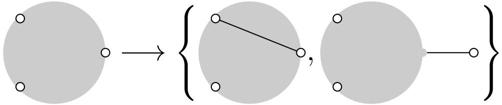


3. all ≤ 2 loop massless on-shell 4-point [Lüders]



Linearly reducible families (infinite)

- 3-constructible graphs [Brown, Schnetz, Panzer]



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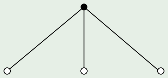

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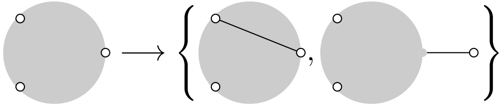

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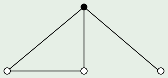

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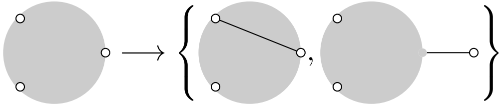

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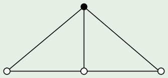

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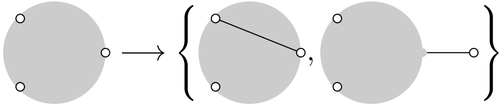

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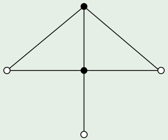

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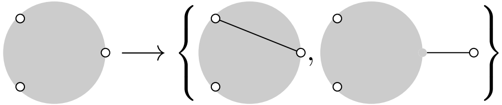

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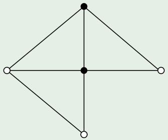

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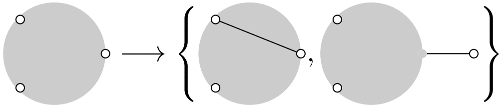

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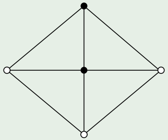

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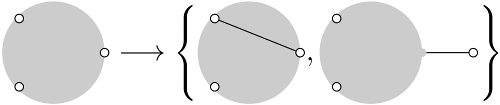

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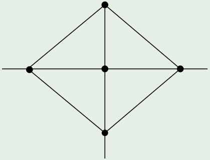

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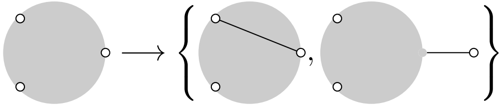

Theorem [Panzer]

All ϵ-coefficients of these graphs (off-shell) are MPL over the alphabet $\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, 1-z\bar{z}, 1-z-\bar{z}, z\bar{z}-z-\bar{z}\}$.

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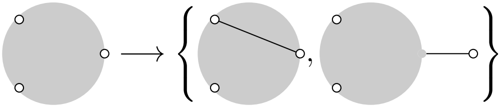

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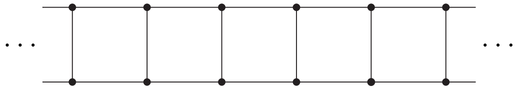
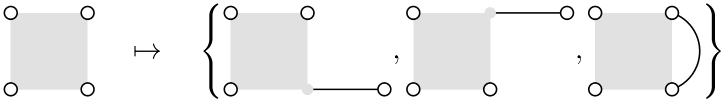
- minors of ladder-boxes (≤ 2 legs massive)



Theorem [Panzer]

All ϵ-coefficients of these graphs are MPL. For the massless case, the alphabet is just $\{x, 1+x\}$ for $x = s/t$.

Minors of ladder boxes are closed under the operations



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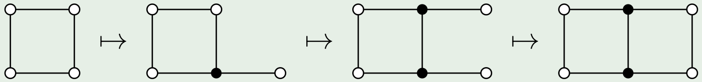

Theorem

All minors of ladder boxes (with $p_1^2 = p_2^2 = 0$) evaluate to MPL.

Summary

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