

Multiple polylogarithms and Feynman integrals

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- ① hyperlogarithms & iterated integrals
- ② multiple polylogarithms
- ③ functional equations, symbols
- ④ parameter integrals
- ⑤ other stuff
- ⑥ coaction (Brown, Goncharov)

Some FI/amplitudes are expressible via multiple polylogarithms (MPL)

$$\text{Li}_{n_1, \dots, n_d}(z_1, \dots, z_d) = \sum_{0 < k_1 < \dots < k_d} \frac{z_1^{k_1} \dots z_d^{k_d}}{k_1^{n_1} \dots k_d^{n_d}}, \quad |z_1 \dots z_d| < 1$$

and their special values, e.g. multiple zeta values (MZV)

$$\zeta_{n_1, \dots, n_d} = \text{Li}_{n_1, \dots, n_d}(1, \dots, 1).$$

Example ($p_1^2 = 1$, $p_2^2 = |z|^2$, $p_3^2 = |1 - z|^2$)

$$\Phi \left(\text{triangle diagram} \right) = \frac{4i \operatorname{Im} [\text{Li}_2(z) + \log(1 - z) \log |z|]}{z - \bar{z}}$$

$$\Phi \left(\text{hexagon diagram} \right) = 252 \zeta_3 \zeta_5 + \frac{432}{5} \zeta_{3,5} - \frac{25056}{875} \zeta_2^4$$

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iterated integrals (Chen 1973)

Take a manifold X and differential forms $\omega_1, \dots, \omega_n \in \Omega^1(X)$. Integrating these along a path $\gamma \in C^1([0, 1], X)$, we can construct functions (on γ):

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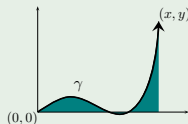
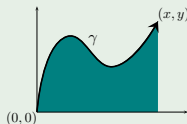
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- 1 If $\omega = df$ is exact, $\int_{\gamma} \omega = f(\gamma(1)) - f(\gamma(0))$ is boring.
- 2 Not all iterated integrals are homotopy invariant.

Example

Take $\omega = ydx \in \Omega^1(\mathbb{R}^2)$, then $\int_{\gamma} \omega$ is the area between γ and the x-axis.



$\Rightarrow$ integrability condition (Chen), simplest case:

$$\int_{\gamma} \omega \text{ homotopy invariant} \Leftrightarrow d\omega = 0$$

Hyperlogarithms (Poincaré 1884, Lappo-Danilevsky 1927)

Let $X = \mathbb{C} \setminus \Sigma$ for a finite set of points Σ . The regular, non-exact forms

$$\omega_\sigma = d \log(z - \sigma) = \frac{dz}{z - \sigma}$$

generate homotopy invariant iterated integrals, called **hyperlogarithms**.

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All MPL can be written this way:

$$\int_0^z \omega_0^{n_d-1} \omega_{\sigma_d} \cdots \omega_0^{n_1-1} \omega_{\sigma_1} = (-1)^d \operatorname{Li}_{n_1, \dots, n_d} \left(\frac{\sigma_2}{\sigma_1}, \dots, \frac{\sigma_d}{\sigma_{d-1}}, \frac{z}{\sigma_d} \right)$$

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Notation and some special cases:

- $\int_0^z \omega_{\sigma_n} \cdots \omega_{\sigma_1} = I(0; \sigma_1, \dots, \sigma_n; z) = G(\sigma_n, \dots, \sigma_1; z)$ [Goncharov]
- $\Sigma = \{-1, 0, 1\}$ harmonic polylogarithms (HPL) [Remiddi & Vermaseren]
- $\Sigma = \{0, 1, 1-y, -y\}$ 2 dimensional HPL [Gehrmann & Remiddi]

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

$$\int_{\gamma \star \eta} \omega_2 \omega_1 = \int_{0 \leq t_1 \leq t_2 \leq 1} (\gamma \star \eta)^*(\omega_2)(t_2) (\gamma \star \eta)^*(\omega_1)(t_1),$$

split the interval

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More generally, the **path concatenation** formula reads

$$\int_{\gamma \star \eta} \omega_n \cdots \omega_1 = \sum_{k=0}^n \int_{\eta} \omega_n \cdots \omega_{k+1} \int_{\gamma} \omega_k \cdots \omega_1.$$

Path concatenation

$G(\vec{\sigma}; z)$ is analytic in $z = \gamma(1)$ on $\mathbb{C} \setminus \Sigma$, but **multivalued**.

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Even though γ is suppressed in the notation $G(\vec{\sigma}; z)$ and $\int_0^z w$, these functions still depend on the homotopy class of γ .

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Similar: change of basepoint

$$\int_b^z \omega_n \cdots \omega_1 = \sum_{k=0}^n \int_0^z \omega_n \cdots \omega_{k+1} \int_b^0 \omega_k \cdots \omega_1$$

Shuffle product

The **shuffle product** of two words

$$w_{n+m} \cdots w_{n+1} \sqcup w_n \cdots w_1 = \sum_{\sigma} w_{\sigma(n+m)} \cdots w_{\sigma(1)}$$

is the sum of all their **shuffles** σ , i.e. permutations which preserve the relative order of letters in both factors:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \quad \text{and} \quad \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

For arbitrary words u and v , we find that ($\int_{\gamma}$ is linearly extended)

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Example

$$\int_{\gamma} \omega_3 \cdot \int_{\gamma} \omega_2 \omega_1 = \int_{\gamma} \omega_3 \omega_2 \omega_1 + \int_{\gamma} \omega_2 \omega_3 \omega_1 + \int_{\gamma} \omega_2 \omega_1 \omega_3$$

$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

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Singularities

Singularities when $\{\gamma(0), \gamma(1)\} \ni z \rightarrow \tau \in \{\infty\} \cup \Sigma$ are logarithmic: There exist (uniquely determined) functions $f_{k,w}(z)$, analytic at $z = \tau$, such that

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Definition (logarithmic regularization)

The **regularized limit** is $\text{Reg}_{z \rightarrow \tau} \int_{\gamma} w := f_{0,w}(\tau)$.

Example

$$\int_{\gamma} \omega_0 = \log \frac{z}{\gamma(0)} \quad G(0; z) = \int_0^z \omega_0 := \text{Reg}_{\gamma(0) \rightarrow 0} \int_{\gamma} \omega_0 = \log(z)$$

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These expansions and limits can be computed algorithmically using shuffles and rescalings like $G(\lambda \vec{\sigma}, \lambda z) = G(\vec{\sigma}; z)$ ($\sigma_1 \neq 0$).

Differentials

By definition, $\partial_z G(\sigma_n, \vec{\sigma}; z) = \frac{1}{z - \sigma_n} G(\vec{\sigma}, z)$. But now consider the σ as variables themselves. Differentiating under the integral sign in

$$G(\sigma_n, \dots, \sigma_1; z) = \int_0^z \frac{dt_n}{t_n - \sigma_n} \int_0^{t_n} \frac{dt_{n-1}}{t_{n-1} - \sigma_{n-1}} \dots \int_0^{t_2} \frac{dt_1}{t_1 - \sigma_1}$$

one finds

$$dG(\vec{\sigma}; z) = \sum_{i=1}^n G(\dots, \phi_i, \dots; z) d \log \frac{\sigma_i - \sigma_{i-1}}{\sigma_i - \sigma_{i+1}} \quad \sigma_0 := z, \sigma_{n+1} := 0$$

Equivalently, in the sum representation of MPL we see

$$d \operatorname{Li}_{\vec{n}}(\vec{z}) = \sum_k \operatorname{Li}_{\vec{n} - \vec{e}_i}(\vec{z}) \frac{dz_k}{z_k}$$

and can replace any occurring zeros (resulting from $n_k = 1$ say) by

$$\operatorname{Li}_{\vec{n}'}(\dots, z_k z_{k-1}, \dots) \frac{dz_k}{1 - z_k} - \operatorname{Li}_{\vec{n}'}(\dots, z_k z_{k+1}, \dots) \frac{dz_k}{z_k(1 - z_k)}.$$

Functional equations

Remember

Proving functional relations between MPL is easy - just differentiate!

Question

Can we write $\text{Li}_2(1 - x)$ as a hyperlogarithm with argument x ?

① Take a derivative:

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Question

Can we write $\text{Li}_2(1-x)$ as a hyperlogarithm with argument x ?

- 1 Take a derivative:

$$\partial_x \text{Li}_2(1-x) = \frac{1}{x-1} \text{Li}_1(1-x)$$

- 2 Express all MPL in the integrand as hyperlogs (**recursion**):

$$\text{Li}_1(1-x) = -\log(x) = -G(0; x)$$

- 3 integrate back:

$$\text{Li}_2(1-x) = C - G(1, 0; x) = C - \log(1-x) \log(x) - \text{Li}_2(x)$$

Functional equations

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- 4 Fix constant $C = \frac{\pi^2}{6} = \zeta_2$ via some limit, for example $x \rightarrow 1$

Products of hyperlogarithms basis

The representation in terms of hyperlogarithms is free of relations:

Lemma

All hyperlogarithms $G(\vec{\sigma}; z)$ are linearly independent functions of z (even with algebraic coefficients).

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The recursive algorithm (differentiation & integration & limits) solves

Problem

Given some MPL $G(\vec{\sigma}(\vec{x}), z(\vec{x}))$ or $\text{Li}_{\vec{n}}(\vec{z}(\vec{x}))$ whose arguments $(\vec{\sigma}, z$ or $\vec{z})$ are rational functions of variables $x_1, \dots, x_n$, write it in the **basis**

$$\sum_{\vec{\sigma}_1, \dots, \vec{\sigma}_n} G(\vec{\sigma}_1(x_2, \dots, x_n); x_1) G(\vec{\sigma}_2(x_3, \dots, x_n); x_2) \cdots G(\vec{\sigma}_n; x_n).$$

- completely algebraic (no numerics), programmed (**HyperInt**)
- in general not the shortest or “simplest” representation
- dependence on order of the variables $x_1, \dots, x_n$

Higher dimensions: multiple polylogarithms

Idea: treat all variables equally; via higher dimensional iterated integrals

Example

$$\begin{aligned} d \operatorname{Li}_{1,1}(z_1, z_2) &= \frac{dz_1}{1-z_1} \operatorname{Li}_1(z_2) - \frac{dz_1}{z_1(1-z_1)} \operatorname{Li}_1(z_1 z_2) + \frac{dz_2}{1-z_2} \operatorname{Li}_1(z_1 z_2) \\ &= \frac{dz_1}{1-z_1} \int_0^{z_2} \frac{dz_2}{1-z_2} - \left(\frac{dz_2}{1-z_2} - \frac{dz_1}{z_1} - \frac{dz_1}{1-z_1} \right) \int_0^{(z_1, z_2)} d \log(1 - z_1 z_2) \\ &= \int_{(0,0)}^{(z_1, z_2)} \left[\frac{dz_1}{1-z_1} \frac{dz_2}{1-z_2} + \left(\frac{dz_1}{z_1} + \frac{dz_1}{1-z_1} - \frac{dz_2}{1-z_2} \right) \frac{z_1 dz_2 + z_2 dz_1}{1 - z_1 z_2} \right] \end{aligned}$$

Corollary (MPL as iterated integrals)

Every $\operatorname{Li}_{\vec{n}}(\vec{z})$ of d arguments is an iterated integral on $\mathbb{C}^d \setminus \mathbf{V}(S_d^{\text{MPL}})$ for the singularities (alphabet)

$$S_d^{\text{MPL}} = \{z_1, \dots, z_d\} \cup \{1 - z_i \cdots z_j : 1 \leq i \leq j \leq d\}$$

Higher dimensions & simplification

Every subset $S \subset \mathbb{Q}[z_1, \dots, z_d]$ of polynomials determines a manifold

$$X = \mathbb{C}^d \setminus \mathbf{V}(S) = \mathbb{C}^d \setminus \bigcup_{f \in S} \{f = 0\}$$

which defines a space of iterated integrals. We consider homotopy invariant (**Chen condition** on w !) integrals $\int_{\gamma} w$ of forms $d \log(f)$, $f \in S$.

Theorem (Kummer 1840)

Every iterated integral of at most three rational forms can be expressed as a sum of products of $\log$, Li_2 and Li_3 .

Conjecture (Duhr, Gangl & Rhodes)

To express every MPL up to weight 4, it suffices to add the functions Li_4 , $\text{Li}_{2,2}$. At weight 5 it suffices to add Li_5 and $\text{Li}_{2,3}$.

Method: basis Ansatz, fit coefficients via symbol (differentiation) or motivic coaction

Basepoints

Remember

The symbol w determines the function $\int_b^Z w$ **completely** if one fixes a choice b for the basepoint.

- when b is forgotten, then the symbol w only captures the highest weight
- with b fixed, $w \mapsto \int_b^Z w$ is injective
- the motivic coaction is not necessary
- different basepoints can make expressions simpler
- physics can sometimes hint a good basepoint

Programs

Symbol techniques are standard now, many people have programs which can do at least some of the work in an automatized way. Ask around; public: **HyperInt**, ... ? ... **yours here!**

Motivic coaction

Idea: abstract from the integral/function and define algebraic objects (**motivic** periods) which have more structure. The construction is complicated. [Brown,Dupont,Goncharov,...]

Hint

In recent lectures at the IHES (available online), Brown constructed motivic Feynman periods in very general cases. Thus, a coaction is defined under very mild assumptions (in particular not restricted to polylogarithmic integrals).

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Upshot: There exist algebraically/geometrically defined objects $I_G^{\text{m}}(\vec{s})$ for a wide class of Feynman graphs G and kinematics $\vec{s}$. These are elements of some algebra $\mathcal{H}^{\text{m}}$ with a period map

$$\text{per}: \mathcal{H}^{\text{m}} \longrightarrow \mathbb{C}$$

sending $I_G^{\text{m}}(\vec{s})$ to the actual Feynman integral (number) $I_G(\vec{s})$.

However, there is more structure: A coaction

$$\Delta: \mathcal{H}^m \longrightarrow \mathcal{H}^{\partial r} \otimes \mathcal{H}^m$$

which is very powerful to obtain relations, in particular for multiple polylogarithms (where differentiation does not buy us anything!).

Example

$$\Delta \zeta_{2,3}^m = 1 \otimes \zeta_{2,3}^m + \zeta_{2,3}^{\partial r} \otimes 1 + 2\zeta_3^{\partial r} \otimes \zeta_2^m + \zeta_2^{\partial r} \otimes \zeta_3^m$$

Note $\zeta_2^{\partial r} = 0$ and that $\{\zeta_5^m, \zeta_2^m \zeta_3^m\}$ is a basis of motivic MZV in weight 5, therefore we conclude $\zeta_{2,3}^m = 2\zeta_2^m \zeta_3^m + c\zeta_5^m$, with c undetermined.

An explicit formula for the coproduct/coaction for MPL is due to Goncharov/Brown.

Caution

If people speak about a **coproduct** $\Delta \longrightarrow \mathcal{H}^{\partial r} \otimes \mathcal{H}^{\partial r}$, they necessarily work with de Rham periods, which have no associated real period ($\zeta_2^{\partial r} = 0$). For multiple polylogarithms, this means that multiples of $i\pi$ are dropped.

Caution

The similarity of both sides, $\mathcal{H}^{\partial\tau}$ and $\mathcal{H}^m$ in the case of MPL is rather misleading. In fact, by construction both sides are very different.

In the case of Feynman integrals, the m -side corresponds to Feynman integrals again, whereas the $\partial\tau$ -side, at least empirically, relates to cuts of integrals, i.e. residues. [talk by Abreu]

Further points

- elliptic MZV and integrals [talk by Bogner, Schlotterer]
- numeric evaluation of MPL, e.g. Ginac, but also optimizations [Manteuffel, Tancredi]
- algebraic functions in the forms (root-valued letters)
- single-valued MPL
- cluster polylogarithms
- special values (number theory): relations between MZV, MPL at roots of unity, ...
- ...

Feynman integrals $\longrightarrow$ polylogarithms

- ① differential equations in normal form [talk by Henn, von Manteuffel]

$$d\vec{f}(\varepsilon, \vec{x}) = \varepsilon \left(\sum_{s \in S} d \log(s) \mathbf{A}_s \right) \vec{f}(\varepsilon, \vec{x})$$

- ② summation: extremely powerful computer algebra tools for sums
[Ablinger, Raab, Schneider, Weinzierl...]
- ③ bootstrap, Ansätze, recursions [talk by Drummond, von Hippel]
- ④ integration of parameter integrals
- ⑤ ...

Definite parameter integrals

They are everywhere [talk by Borowka]:

- hypergeometric functions ${}_pF_q(\cdots; z)$

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt$$

- Appell's functions F_1, F_2, F_3, F_4

$$F_3\left(\begin{matrix} \alpha, \alpha' \\ \beta, \beta' \end{matrix} \middle| \gamma \middle| x, y\right) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \\ \times \int_0^1 \int_0^{1-v} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'} du dv$$

- Feynman integrals in Schwinger parameters
- Phase-space integrals [talk by Dulat]
- ...

Idea: If the representation is simple enough (**linearly reducible**), there is an order of integration for the parameters such that each intermediate partial integral is a MPL of rational arguments.

Schwinger parameters

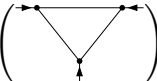
With the *superficial degree of divergence* $\text{sdd} = |E(G)| - D/2 \cdot \text{loops}(G)$,

$$\Phi(G) = \frac{\Gamma(\text{sdd})}{\prod_e \Gamma(a_e)} \int_{(0,\infty)^E} \frac{1}{\psi^{D/2}} \left(\frac{\psi}{\varphi} \right)^{\text{sdd}} \delta(1 - \alpha_N) \prod_e \alpha_e^{a_e-1} d\alpha_e$$

Graph polynomials:

$$\mathcal{U} = \sum_T \prod_{e \notin T} \alpha_e \qquad \mathcal{F} = \sum_{F=T_1 \dot{\cup} T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \mathcal{U} \sum_e m_e^2 \alpha_e$$

Example: massless triangle

$$\Phi \left(\text{triangle diagram} \right) = \int \frac{d\alpha_2 d\alpha_3}{(1 + \alpha_2 + \alpha_3)(\alpha_2 \alpha_3 + z \bar{z} \alpha_3 + (1 - z)(1 - \bar{z}) \alpha_2)}$$


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Singularities of the original integrand: $S = \{\psi, \varphi\}$, i.e. at $\alpha_3 = \sigma_i$ for

$$\sigma_1 = -1 - \alpha_2 \quad \text{and} \quad \sigma_2 = -\frac{\alpha_2(1 - z)(1 - \bar{z})}{\alpha_2 + z \bar{z}}$$

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$$= \int \frac{d\alpha_2}{(z + \alpha_2)(\bar{z} + \alpha_2)} \log \frac{(1 + \alpha_2)(z\bar{z} + \alpha_2)}{(1 - z)(1 - \bar{z})\alpha_2}$$

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After integrating α_1 from 0 to ∞ , the integrand has singularities

$$S_3 = \left\{ \underbrace{1 + \alpha_2}_{\sigma_1=0}, \underbrace{\alpha_2, 1 - z, 1 - \bar{z}}_{\sigma_2=0}, \underbrace{\alpha_2 + z\bar{z}}_{\sigma_2=\infty}, \underbrace{z + \alpha_2, \bar{z} + \alpha_2}_{\sigma_1=\sigma_2 \text{ (pinch)}} \right\}$$

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With the same logic, predict the possible singularities after $\int_0^\infty d\alpha_2$:

$$S_{3,2} = \{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}, z\bar{z} - 1\}$$

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$$= \frac{1}{z - \bar{z}} \int \left(\frac{d\alpha_2}{\alpha_2 + \bar{z}} - \frac{d\alpha_2}{\alpha_2 + z} \right) \log \frac{(\alpha_2 + 1)(\alpha_2 + z\bar{z})}{(1 - z)(1 - \bar{z})\alpha_2}$$

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Polynomial reduction [F. Brown]

Definition

Let S denote a set of polynomials, then S_e are the irreducible factors of

$$\left\{ \text{lead}_e(f), f|_{\alpha_e=0}, D_e(f) : f \in S \right\} \quad \text{and} \quad \{[f, g]_e : f, g \in S\}.$$

Lemma

If the singularities of F are contained in S , then the singularities of $\int_0^\infty F d\alpha_e$ are contained in S_e .

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Improvements

- Fubini: intersect over different orders
- Compatibility graphs

HyperInt: massive triangle

Graph polynomials:

- > E:=[[2,3],[1,3],[1,2]]:
- > M:=[[1,s1],[2,s2],[3,s3]],[m1^2,m2^2,m3^3]:
- > psi:=graphPolynomial(E):
- > phi:=secondPolynomial(E,M):

Polynomial reduction:

- > L[{}]:=[{psi,phi},{psi,phi}]:
- > cgReduction(L,{s1,s2,s3,m1,m2,m3},2):
- > L[{x[1],x[2]}][1];

$$\left\{ s_i + (m_j \pm m_k)^2, \sum_i s_i^2 - \sum_{i \neq j} s_i s_j \right\}$$

$$\cup \left\{ s_1 s_2 s_3 - \sum_i s_i^2 m_i^2 + \sum_i s_i (m_i^2 - m_j^2)(m_i^2 - m_k^2) + \sum_{i < j} s_i s_j (m_i^2 + m_j^2) \right\}$$

Linear reducibility

Definition

If for some order of variables (edges), all $S_{1,\dots,k}$ are linear in α_{k+1} , then S (the Feynman graph G with $S = \{\psi, \varphi\}$) is called **linearly reducible**.

Lemma

If S is linearly reducible, the integral $\prod_e \int_0^\infty d\alpha_e f$ of any rational function f with singularities in S is a MPL with symbol letters in $S_{1,\dots,N}$.

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Integration of linearly reducible integrands in terms of MPL can be automatized via hyperlogarithms.

- open source Maple program
- integration of hyperlogarithms
- transformations of MPL to $G(\cdots; z)$
Note: No further simplification (e.g. rewrite as $\text{Li}_{2,2}$) provided!
- polynomial reduction
- graph polynomials
- symbolic computation of constants (no numerics)

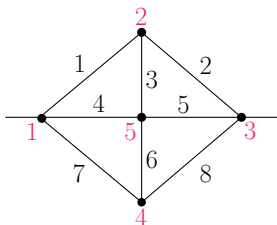
Example

```
> read "HyperInt.mpl":  
> hyperInt(polylog(2,-x)*polylog(3,-1/x)/x,x=0..infinity):  
> fibrationBasis(%);
```

$$\frac{8}{7}\zeta_2^3$$

computes $\int_0^\infty \text{Li}_2(-x) \text{Li}_3(-1/x) dx = \frac{8}{7}\zeta_2^3$.

HyperInt: propagator



```
> E := [[2,1],[2,3],[2,5],[5,1],[5,3],[5,4],[4,1],[4,3]]:
> psi := graphPolynomial(E):
> phi := secondPolynomial(E, [[1,1],[3,1]]):
> add((epsilon*log(psi^5/phi^4))^n/n!,n=0..2)/psi^2:
> hyperInt(eval(%,x[8]=1), [seq(x[n],n=1..7)]):
> collect(fibrationBasis(%), epsilon);
```

$$\left(254\zeta_7 + 780\zeta_5 - 200\zeta_2\zeta_5 - 196\zeta_3^2 + 80\zeta_2^3 - \frac{168}{5}\zeta_2^2\zeta_3\right)\varepsilon^2 \\ + \left(-28\zeta_3^2 + 140\zeta_5 + \frac{80}{7}\zeta_2^3\right)\varepsilon + 20\zeta_5$$

HyperInt: triangle

Graph polynomials:

```
> E:=[[1,2],[2,3],[3,1]] :  
> M:=[[3,1],[1,z*zz],[2,(1-z)*(1-zz)]] :  
> psi:=graphPolynomial(E):  
> phi:=secondPolynomial(E,M):
```

Integration:

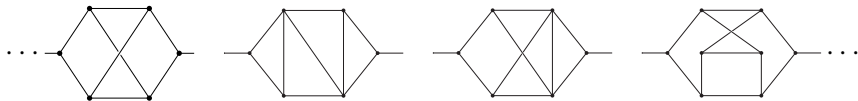
```
> hyperInt(eval(1/psi/phi,x[3]=1),[x[1],x[2]]):  
> factor(fibrationBasis(%,[z,zz]));  
      (G(z;1)G(zz;0) - G(z;0)G(zz;1) + G(zz;0,1)  
      - G(zz;1,0) + G(z;1,0) - G(z;0,1))/(z - zz)
```

Polynomial reduction:

```
> L[{}]:=[{psi,phi},{psi,phi}]:  
> cgReduction(L):  
> L[{x[1],x[2]}][1];  
      {-1 + z, -1 + zz, -zz + z}
```

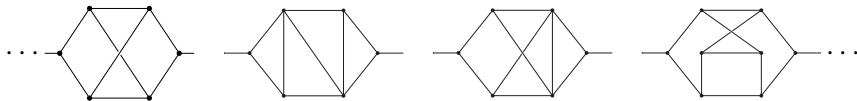
Linearly reducible families (fixed loop order)

- ① all ≤ 4 loop massless propagators (Panzer)

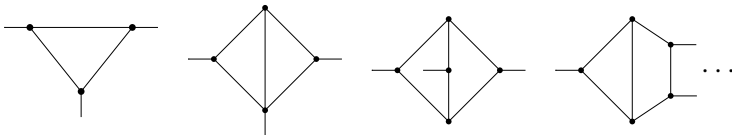


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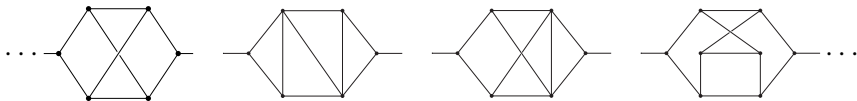


- ② all ≤ 3 loop massless off-shell 3-point (Chavez & Duhr, Panzer)
also in position space; give conformal 4-point integrals

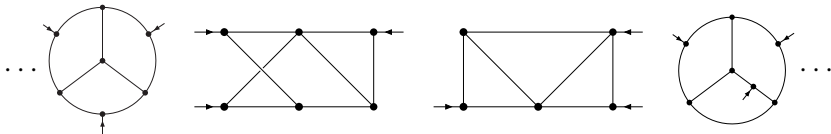


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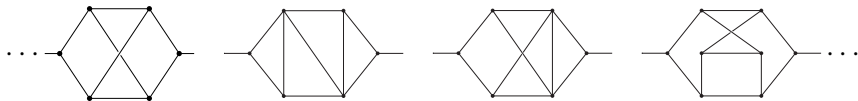


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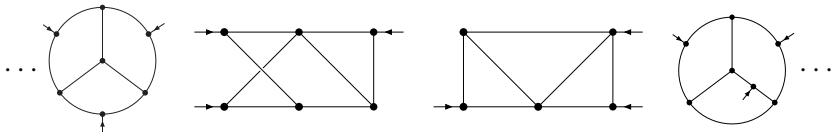


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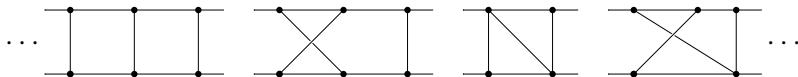
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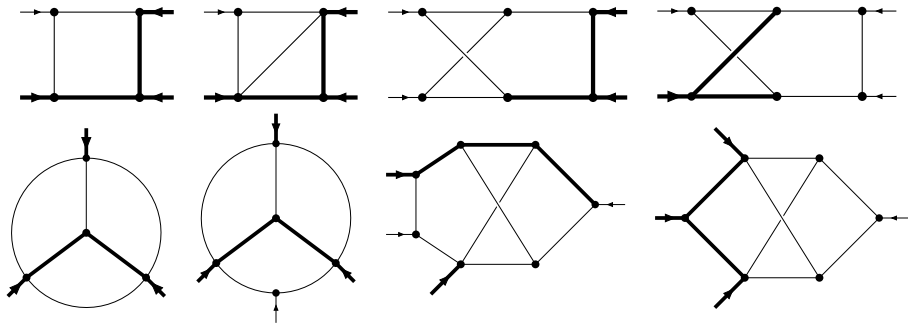
- ② all ≤ 3 loop massless off-shell 3-point (Chavez & Duhr, Panzer)
also in position space; give conformal 4-point integrals



- ③ all ≤ 2 loop massless on-shell 4-point (Lüders)

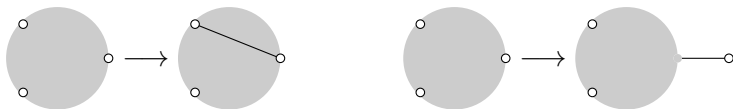


Linearly reducible massive graphs (some examples)



Linearly reducible families (infinite)

- 3-constructible graphs (3-point functions) [Brown, Schnetz, Panzer]

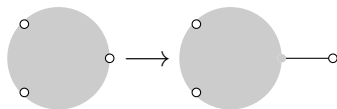
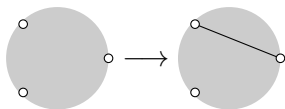


Example



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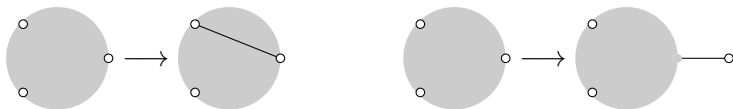


Example

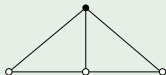


Linearly reducible families (infinite)

- 3-constructible graphs (3-point functions) [Brown, Schnetz, Panzer]

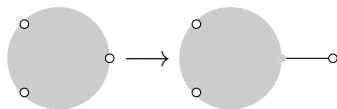
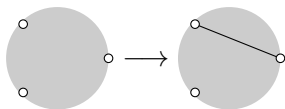


Example

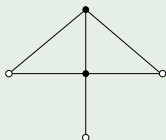


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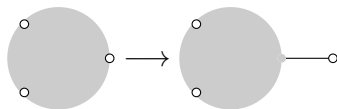
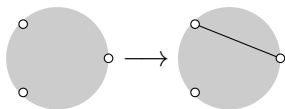


Example

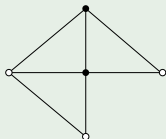


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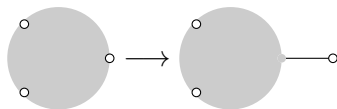
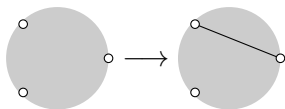


Example

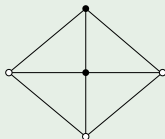


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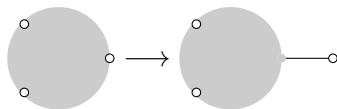
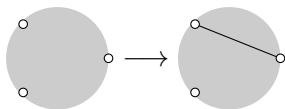


Example

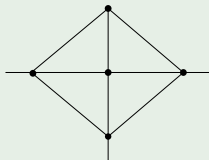


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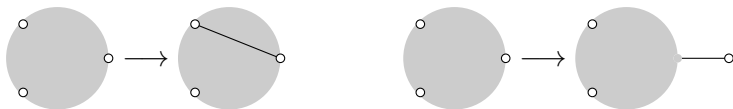


Example



Linearly reducible families (infinite)

- 3-constructible graphs (3-point functions) [Brown, Schnetz, Panzer]

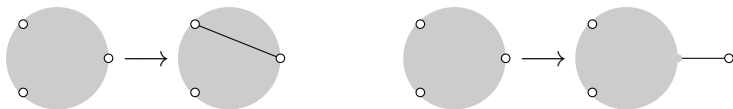


Theorem (Panzer)

All ε -coefficients of these graphs (off-shell) are MPL over the alphabet $\{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}, 1 - z\bar{z}, 1 - z - \bar{z}, z\bar{z} - z - \bar{z}\}$.

Linearly reducible families (infinite)

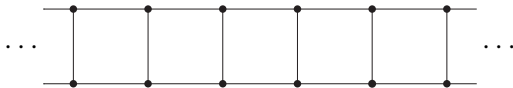
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- minors of ladder-boxes (up to 2 legs off-shell)

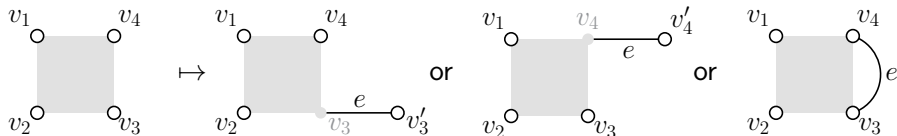


Theorem (Panzer)

All ε -coefficients of these graphs are MPL. For the massless case, the alphabet is just $\{x, 1 + x\}$ for $x = s/t$.

4-point recursions

Start with the box or (double box) and repeat, in any order:



Example

