

Relations among multiple zeta values

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1 Single zeta values

Theorem 1.1 (Euler). *For any $n \in \mathbb{N}$, the even zeta values are*

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!} = -\frac{(2\pi i)^{2n} B_{2n}}{2(2n)!} \in \mathbb{Q} \cdot \pi^{2n} = \mathbb{Q} \cdot \zeta^n(2) \quad (1.1)$$

for the Bernoulli numbers B_n defined by

$$\frac{x}{e^x - 1} = \sum_{0 \leq k} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \dots$$

Hence, as rational multiples of powers of π the even zetas are transcendental.

$2n$	2	4	6	8
B_{2n}	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$
$\zeta(2n)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90} = \frac{2}{5}\zeta^2(2)$	$\frac{\pi^6}{945} = \frac{8}{35}\zeta^3(2)$	$\frac{\pi^8}{9450} = \frac{24}{175}\zeta^4(2)$

For the odd zeta values we do not expect such relations at all:

Conjecture 1.2. *The elements $\pi, \zeta(3), \zeta(5), \zeta(7), \dots$ are algebraically independent over \mathbb{Q} .*

In particular, we expect all zeta values to be transcendental. But we only know very few results on irrationality:

Theorem 1.3 ([1]). $\zeta(3)$ is irrational¹.

Theorem 1.4 ([14]). *Infinitely many of the odd zetas $\zeta(3), \zeta(5), \zeta(7), \dots$ are irrational. In fact, for any $\varepsilon > 0$ exists some $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$,*

$$\dim_{\mathbb{Q}} \operatorname{lin}_{\mathbb{Q}} \{1, \zeta(3), \zeta(5), \dots, \zeta(2n+1)\} \geq \frac{1-\varepsilon}{1+\log 2} \log n. \quad (1.2)$$

Theorem 1.5 ([19]). *At least one of $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ is irrational.*

Theorem 1.6 ([20]). *For any odd $n \in \mathbb{N}$, at least one of $\{\zeta(n+2), \zeta(n+4), \dots, \zeta(8n-1)\}$ is irrational.*

*Misprint in Example 5.5 corrected on 24th November 2017, thanks to Harry Braden!

¹For details on this proof see [13] and its recently updated version.

2 Multiple zeta values

Definition 2.1 ([17]). To $d \in \mathbb{N}$ integers $n_1, \dots, n_d \in \mathbb{N}$ with $n_d > 1$, the multiple zeta value

$$\zeta(n_1, \dots, n_d) := \sum_{0 < k_1 < \dots < k_d} \frac{1}{k_1^{n_1} \dots k_d^{n_d}} \in \mathbb{R}_+ \quad (2.1)$$

assigns a positive real number. We call d its depth and $n_1 + \dots + n_d$ its weight. Be aware of the also common reverse convention [9].

Already Euler [5] knew the case $d = 1$ of

Theorem 2.2 (Sum theorem [7]). For any depth $d \in \mathbb{N}$ and weight $w > 1$,

$$\sum_{n_1 + \dots + n_d = w, n_d > 1} \zeta(n_1, \dots, n_d) = \zeta(w). \quad (2.2)$$

Example 2.3. In weight three, $\zeta(3) = \zeta(1, 2)$ and $\zeta(4) = \zeta(1, 3) + \zeta(2, 2) = \zeta(1, 1, 2)$ in weight four.

Symmetric MZVs are sums of products of single zeta values, as the stuffle identity supplies

Theorem 2.4 ([9]). Let $n_1, \dots, n_d \geq 2$ then

$$\sum_{\sigma \in S_d} \zeta(n_{\sigma(1)}, \dots, n_{\sigma(d)}) = \sum_{\text{partitions } M \text{ of } \{1, \dots, d\}} (-1)^{d-|M|} \prod_{P \in M} (|P| - 1)! \cdot \zeta\left(\sum_{n \in P} n\right). \quad (2.3)$$

Corollary 2.5. Combining (2.3) with (1.1), we deduce

$$E(2n, d) := \sum_{n_1 + \dots + n_d = n} \zeta(2n_1, \dots, 2n_d) \in \mathbb{Q} \cdot \zeta(2n). \quad (2.4)$$

In fact their generating function is computed in [8] to

$$F(t, s) := 1 + \sum_{1 \leq k \leq n} E(2n, k) t^n s^k = \frac{\sin(\pi\sqrt{t(1-s)})}{\sqrt{1-s} \sin(\pi\sqrt{t})}. \quad (2.5)$$

Example 2.6. $E(2n, 2) = \sum_{k=1}^{n-1} \zeta(2n-2k, 2k) = \frac{3}{4} \zeta(2n)$ is already due to Euler [5]. Writing $\{2\}^n$ for the sequence of n consecutive twos, note further

$$E(2n, n) = \zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}. \quad (2.6)$$

Theorem 2.7 ([10]). MZV with $n_1 = \dots = n_{d-1} = 1$ are sums of products of single zetas, namely

$$\sum_{n, m \geq 1} \zeta(\{1\}^{m-1}, n+1) s^n t^m = 1 - \frac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(1-s-t)} = 1 - \exp\left(\sum_{n \geq 2} \zeta(n) \frac{t^n + s^n - (t+s)^n}{n}\right).$$

Again, the case $m = 2$ was known to Euler [5] already.

Theorem 2.8 ([2]). As was originally conjectured by D. Zagier, we have the identity

$$\zeta(\{1, 3\}^n) = \frac{2\pi^{4n}}{(4n+2)!}.$$

3 Shuffles, stuffles and regularization

Definition 3.1 (Hoffman²). *The Hoffman algebra \mathfrak{h} are the non-commutative polynomials*

$$\mathfrak{h} := \mathbb{Q}\langle x, y \rangle \quad (3.1)$$

in two letters x and y . It is graded by the weight (length of a word = number of letters) and depth (number of letters y). On the subspace

$$\mathfrak{h}^0 := \mathbb{Q} \oplus y\mathfrak{h}x \quad (3.2)$$

of admissible words words (beginning with y and ending in x) we define the period map by

$$\zeta : \mathfrak{h}^0 \longrightarrow \mathbb{R}, \quad y_{n_1} \cdots y_{n_d} \mapsto \zeta(n_1, \dots, n_d), \quad (3.3)$$

where $y_n := yx^{n-1}$ for any $n \in \mathbb{N}$. This is extended linearly with $\zeta(1) = 1$ for the empty word. The shuffle product \sqcup and stuffle (also called harmonic) product \star are recursively defined by

$$av \sqcup bw := a(v \sqcup bw) + b(av \sqcup w) \quad \text{and} \quad y_n v \star y_m w := y_n(v \star y_m w) + y_m(y_n v \star w) + y_{n+m}(v \star w) \quad (3.4)$$

for any letters $a, b \in \{x, y\}$ and words $v, w \in \mathfrak{h}^0$. Both of these turn \mathfrak{h}^0 into a commutative, associative and free algebra.

As was shown in Oliver's lectures, the sum representation (2.1) and the iterated integral representation provide

Lemma 3.2 (Dobule-shuffle relations). *ζ is an algebra morphism with respect to both products \sqcup and \star :*

$$\zeta(v \sqcup w) = \zeta(v) \cdot \zeta(w) = \zeta(v \star w) \quad \text{for any } v, w \in \mathfrak{h}^0. \quad (3.5)$$

Example 3.3. *From $y_2 \sqcup y_2 = yx \sqcup yx = 4yyxx + 2yxyx = 4y_1y_3 + 2y_2y_2$ and $y_2 \star y_2 = 2y_2y_2 + y_4$ we deduce $4\zeta(1, 3) + 2\zeta(2, 2) = \zeta^2(2) = 2\zeta(2, 2) + \zeta(4)$ and therefore $\zeta(4) = 4\zeta(1, 3)$.*

Theorem 3.4 (Hoffman relation [9]). *For any $w \in \mathfrak{h}^0$, $w \sqcup y - w \star y \in \mathfrak{h}^0$ (does end in x) and*

$$\zeta(w \sqcup y - w \star y) = 0. \quad (3.6)$$

Example 3.5. *We find $y_2 \star y - y_2 \sqcup y = y_3 + y_1y_2 + y_2y - (2y_1y_2 + y_2y) = y_3 - y_1y_2 \in \mathfrak{h}^0$, thus $\zeta(3) = \zeta(1, 2)$.*

Conjecture 3.6. *All algebraic relations over \mathbb{Q} among MZV are consequences of the so-called regularized double-shuffle relations meaning (3.5) and (3.6).*

4 Weight filtration

Definition 4.1. *Let $\mathcal{Z}_N := \text{lin}_{\mathbb{Q}} \{ \zeta(n_1, \dots, n_d) : n_1 + \dots + n_d = N \}$ denote the space of MZV of weight N and $\mathcal{Z} := \sum_{N \geq 0} \mathcal{Z}_N$ their sum (the \mathbb{Q} -space spanned by all MZV).*

²In the literature the words are written in the reversed way: $y_n = x^{n-1}y$ and $\zeta(y_{n_d} \dots y_{n_1}) := \zeta(n_d, \dots, n_1)$. Here we used this order to be consistent with the first talk of Oliver Schnetz.

From the previous examples, we know that $\zeta(3) = \zeta(1, 3)$, so $\mathcal{Z}_3 = \mathbb{Q} \cdot \zeta(3)$ is one-dimensional.

$$\zeta(4) = \zeta(1, 1, 2) = 4\zeta(1, 3) = \frac{4}{3}\zeta(2, 2) = \frac{2}{5}\zeta^2(2)$$

shows that the four MZV of weight four span only a one-dimensional space $\mathcal{Z}_4 = \mathbb{Q} \cdot \zeta(4)$ as well. Recall the following table of Oliver's talk obtained by using all available relations:

weight N	conjectured basis of \mathcal{Z}_N	d_N	MZV of weight N
0	1	1	1
1		0	0
2	$\zeta(2)$	1	1
3	$\zeta(3)$	1	2
4	$\zeta(2)^2$	1	4
5	$\zeta(5), \zeta(2)\zeta(3)$	2	8
6	$\zeta(2)^3, \zeta(3)^2$	2	16
7	$\zeta(7), \zeta(2)\zeta(5), \zeta(2)^2\zeta(3)$	3	32
8	$\zeta(2)^4, \zeta(2)\zeta(3)^2, \zeta(3)\zeta(5), \zeta(3, 5)$	4	64

Conjecture 4.2 (D. Zagier, [17]). \mathcal{Z} is graded by the weight with Hilbert-Poincaré series

$$\sum_{k \geq 0} d_k t^k = \frac{1}{1 - t^2 - t^3}. \quad (4.1)$$

Equivalently, the following two statements hold:

1. All relations are homogeneous in weight: $\mathcal{Z} = \bigoplus_{N \geq 0} \mathcal{Z}_N$
2. $\dim_{\mathbb{Q}} \mathcal{Z}_N = d_N$ where $d_0 = d_2 = 1$, $d_1 = 0$ and then $d_N = d_{N-2} + d_{N-3}$.

Note that this is a very strong claim as it implies conjecture 1.2 and thus transcendence of all odd zeta values. However, the results of F. Brown on *motivic multiple zeta values* (which will be featured in his upcoming lectures) imply

Theorem 4.3 ([4]). *The Hoffman-elements span \mathcal{Z} in each weight, that is*

$$\mathcal{Z}_N = \text{lin}_{\mathbb{Q}} \{ \zeta(n_1, \dots, n_r) : n_1, \dots, n_r \in \{2, 3\} \text{ and } n_1 + \dots + n_r = N \}. \quad (4.2)$$

In particular this implies (as was also proved independently in [15])

$$\dim_{\mathbb{Q}} \mathcal{Z}_N \leq d_n. \quad (4.3)$$

In fact the results of [4] prove the existence of a surjective algebra morphism

$$\phi : \mathbb{Q}[f_2] \otimes_{\mathbb{Q}} (\mathbb{Q}\langle f_3, f_5, f_7, \dots \rangle, \sqcup) \longrightarrow \mathcal{Z} \quad (4.4)$$

which preserves the weight filtrations. In this picture, conjecture 4.2 is equivalent to ϕ being an isomorphism.

5 Depth filtration

Definition 5.1. Let $\mathcal{Z}_N^{(d)} := \text{lin}_{\mathbb{Q}} \{ \zeta(n_1, \dots, n_r) : n_1 + \dots + n_r = N \text{ and } r \leq d \}$ denote the span of MZV of weight N and depth $\leq d$.

The stuffle relation involves MZV of different depths, so in contrast to the weight, the depth can only be a filtration on \mathcal{Z} . For example recall the example

$$\zeta(\{2\}^n) = \zeta(\underbrace{2, \dots, 2}_{n \text{ twos}}) = \frac{\pi^{2n}}{(2n+1)!} \in \mathbb{Q} \cdot \zeta(2n) = \mathcal{Z}_{2n}^{(1)} \quad (5.1)$$

that is of depth one (not n). Similarly note

Theorem 5.2 ([18]). Setting $H(n) := \zeta(\{2\}^n)$ and $H(a, b) := \zeta(\{2\}^a, 3, \{2\}^b)$,

$$H(a, b) = 2 \sum_{r=1}^{a+b+1} (-1)^r \left[\binom{2r}{2a+2} - \left(1 - \frac{1}{2^{2r}}\right) \binom{2r}{2b+1} \right] H(a+b-r+1) \zeta(2r+1). \quad (5.2)$$

In particular it implies that $H(a, b) \in \mathcal{Z}_{2a+2b+3}^{(2)}$ is of depth at most two, even though it is originally a MZV of high depth $a+b+1$.

Conjecture 5.3 (D. Broadhurst and D. Kreimer, [3]). \mathcal{Z} is graded by the weight and the depth filtration has dimensions

$$d_{n,k} = \dim_{\mathbb{Q}} \left(\mathcal{Z}_n^{(k)} / \mathcal{Z}_n^{(k-1)} \right)$$

given by the generating series

$$\sum_{n,k} d_{n,k} x^k y^n = \frac{1 + \mathbb{E}y}{1 - \mathbb{O}y + \mathbb{S}y^2(1 - y^2)} \quad (5.3)$$

where $\mathbb{E} := \frac{x^2}{1-x^2} = x^2 + x^4 + x^6 + \dots$ and $\mathbb{O} := \frac{x^3}{1-x^2} = x^3 + x^5 + x^7 + \dots$ count the even and odd zetas in each weight, while $\mathbb{S} := \frac{x^{12}}{(1-x^4)(1-x^6)}$ is the generating series of the dimensions of the spaces of cusp forms of $\text{SL}_2(\mathbb{Z})$ in each weight.

Note that this is a refinement of conjecture 4.2 (set $y = 1$ and use $d_n = \sum_k d_{n,k}$). Expanding (5.3) to the first orders in the depth y , observe

$$\sum_{n,k} d_{n,k} = 1 + \underbrace{(\mathbb{E} + \mathbb{O})y}_{\text{single zetas}} + \left(\underbrace{\mathbb{E}\mathbb{O}}_{\text{odd weight}} + \underbrace{\mathbb{O}^2 - \mathbb{S}}_{\text{even weight}} \right) + (\dots)y^3 + \dots \quad (5.4)$$

In particular we see in depth two, that the products $\mathbb{E}\mathbb{O}$ of even and odd single zeta values are the only generators in odd weights. This is known more generally as

Theorem 5.4 ([11, 16]). Every $\zeta(n_1, \dots, n_d)$ with weight $n = n_1 + \dots + n_d$ and depth $d \not\equiv n \pmod{2}$ of different parity is a \mathbb{Q} -linear combination of products of MZV of depth smaller than d .

Example 5.5 (taken from [12, Example 5.5]).

$$\begin{aligned} \zeta(4, 2, 2) &= \zeta(4) \zeta(2, 2) + \zeta(2) [4\zeta(4, 2) + 6\zeta(3, 3) + 7\zeta(2, 4) + 8\zeta(1, 5)] \\ &\quad - 8\zeta(6, 2) - 10\zeta(5, 3) - \frac{23}{2}\zeta(4, 4) - 12\zeta(3, 5) - \frac{15}{2}\zeta(2, 6) \end{aligned}$$

In particular, every depth-two $\zeta(n_1, n_2)$ with odd $n_1 + n_2$ is a sum of products of single zetas. An explicit formula is given as

Theorem 5.6 (Proposition 7 in [18]). *For $m \geq 1$, $n \geq 2$ of odd weight $k = m + n = 2K + 1$,*

$$\zeta(m, n) = (-1)^m \sum_{s=0}^{K-1} \left[\binom{k-2s-1}{m-1} + \binom{k-2s-1}{n-1} - \delta_{n,2s} + (-1)^m \delta_{s,0} \right] \zeta(2s) \zeta(k-2s).$$

On the other hand, for even weights N , the contribution \mathbb{O}^2 to (5.4) counts the $\zeta(n, m)$ with odd entries $n, m \geq 3$ and $n + m = N$. However, these are in general not independent. The first relation (up to $\mathbb{Q} \cdot \zeta(N) = \mathcal{Z}_N^{(1)}$) appears at weight $N = 12$:

$$28\zeta(3, 9) + 150\zeta(5, 7) + 168\zeta(7, 5) = \frac{5197}{691}\zeta(12). \quad (5.5)$$

The origin of these *exotic* relations in depth two correspond [6] to period polynomials for cusp forms of $\mathrm{SL}_2(\mathbb{Z})$ and are counted by \mathbb{S} . These connections might be enlightened by the upcoming lectures of José I. Burgos.

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