HUMBOLDT-UNIVERSITÄT ZU BERLIN

Renormalization by kinematic subtraction and Hopf algebras

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from quantum field theory (QFT)

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    - renormalization in QFT formulated as an algebraic Birkhoff-decomposition of characters on a combinatorial Hopf algebra
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- van Baalen, Kreimer, Uminsky, Yeats: study of non-perturbative (analytic) Dyson-Schwinger equations [18, 19, 20, 16]

Aims of the talk

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4. analytic vs. combinatorial descriptions

A model of a single scale

Theorem (Universal property)

To any linear map $L \in \operatorname{End}(\mathcal{A})$ *on an algebra* $\mathcal{A}$ *exists a unique morphism* $\phi : H_R \to \mathcal{A}$ *of algebras (notation* $\phi \in \mathcal{G}_{\mathcal{A}}^{H_R}$*) such that*

$$\phi \circ B_+ = L \circ \phi. \tag{1.1}$$

If $\mathcal{A}$ *is a Hopf algebra and* $L \in HZ_{\varepsilon}^{1}(\mathcal{A})$ *a one-cocycle,* ϕ *is Hopf.*

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$$\phi_s\left(B_+(w)\right) = \int_0^\infty \frac{\mathrm{d}\zeta}{s} f\left(\frac{\zeta}{s}\right) \phi_\zeta(w) \quad \text{for any} \quad w \in H_R. \tag{1.2}$$

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- s is a physical parameter (mass or momentum)
- f is dictated by the graph into which B_+ inserts
- these integrals typically diverge and are understood formally (as a pair of differential form & domain of integration)

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Such logarithmic divergences are independent of the parameter s and thus renormalizable by a subtraction:

Definition

For a *renormalization point* μ, let $R_\mu := \mathrm{ev}_\mu$ denote the evaluation at $s \mapsto \mu$. The BPHZ- or MOM-renormalized character is

$$\phi_{\mathrm{R}} := (R_\mu \circ \phi)^{\star -1} \star \phi = \phi_\mu^{\star -1} \star \phi_s. \tag{1.3}$$

$R_\mu \circ \phi^{\star -1}$ are called the *counterterms*.

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A model of a single scale

Finiteness

Example ($f(\zeta) = \frac{1}{1+\zeta}$ as before)

$$\phi_{\mathrm{R}}(\mathbb{1}) = 1, \quad \phi_{\mathrm{R}}(\bullet) = (\mathrm{id} - R_\mu)\,\phi(\bullet) = \int_0^\infty \mathrm{d}\zeta \left[\frac{1}{\zeta + s} - \frac{1}{\zeta + \mu}\right] = -\ln\frac{s}{\mu}.$$

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Corollary ($B_+ \in \mathrm{HZ}^1_\varepsilon$ is a cocycle: $\Delta B_+ = (\mathrm{id} \otimes B_+)\Delta + B_+ \otimes \mathbb{1}$)

The renormalized character ϕ_R arises from the universal property of H_R:

$$\phi_{R,s}(B_+(w)) = \int_0^\infty \mathrm{d}\zeta \left[\frac{f\left(\frac{\zeta}{s}\right)}{s} - \frac{f\left(\frac{\zeta}{\mu}\right)}{\mu}\right] \phi_{R,\zeta}(w) \quad \textit{for any } w \in H_R. \tag{1.4}$$

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Proof.

Use $S \circ B_+ = -S \star B_+$ and write $L = \int_0^\infty \frac{\mathrm{d}\zeta}{s} f\left(\frac{\zeta}{s}\right) \ldots$ to deduce

$$\begin{aligned}\phi_{\mathrm{R}} \circ B_+ &= \left(R_\mu \phi^{\star -1} \star \phi\right) \circ B_+ = R_\mu \phi^{\star -1} \star \phi B_+ + R_\mu \phi^{\star -1} B_+ \\ &= R_\mu \phi^{\star -1} \star \left[(\mathrm{id} - R_\mu) \circ \phi \circ B_+\right] = (\mathrm{id} - R_\mu) \circ L \circ \phi_{\mathrm{R}}.\end{aligned}$$

□

This is an algebraic *Birkhoff decomposition* along the splitting $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ induced by the character R_μ:

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Lemma (finiteness for logarithmic divergences)

If the kernel $f(\zeta)$ is continuous on $[0, \infty)$ with asymptotic growth

$$f(\zeta) - \frac{c_{-1}}{\zeta} \sim \zeta^{-1-\varepsilon} \quad \textit{at } \zeta \to \infty,$$

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for some $\varepsilon > 0$ and $c_{-1} \in \mathbb{K}$, then ϕ_R is finite. Moreover it is polynomial:

$$\phi_{R,s} = \mathrm{ev}_\ell \circ \phi_R, \quad \phi_R : H_R \to \mathbb{K}[x] \quad \textit{where} \quad \ell := \ln \frac{s}{\mu}. \qquad (1.5)$$

Inserting the polynomial $\phi_{\mathrm{R},\zeta}(w) \in \mathbb{K}[\ln \frac{\zeta}{\mu}]$ into

$$\phi_{\mathrm{R},s}\left(B_+(w)\right) = \int_0^\infty \mathrm{d}\zeta \left[\frac{f\left(\frac{\zeta}{s}\right)}{s} - \frac{f\left(\frac{\zeta}{\mu}\right)}{\mu}\right] \phi_{\mathrm{R},\zeta}(w)$$

actually supplies the algebraic recursion

$$\phi_{\mathrm{R}} \circ B_+ = P \circ F\left(-\partial_x\right) \circ \phi_{\mathrm{R}}, \tag{1.6}$$

where $P := \mathrm{id} - \mathrm{ev}_0$ annihilates the constant terms and the analytic input of the kernel f is captured by the operator

$$F\left(-\partial_x\right) := -c_{-1} \textstyle\int_0 + \sum_{n \geq 0} c_n \left(-\partial_x\right)^n \in \mathrm{End}\left(\mathbb{K}[x]\right) \quad \text{and} \tag{1.7}$$

$$c_{n-1} := \int_0^\infty \mathrm{d}\zeta \left[f(\zeta) + \zeta f'(\zeta)\right] \frac{(-\ln \zeta)^n}{n!}. \tag{1.8}$$

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An algebraic recursion: Examples

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Remark

The Laurent series $F(z) \in z^{-1}\mathbb{K}[[z]]$ is the *Mellin transform*

$$F(z) = \int_0^\infty \mathrm{d}\zeta \; f(\zeta) \cdot \zeta^{-z} = \sum_{n \geq -1} c_n z^n. \tag{1.9}$$

The Hopf algebra of polynomials

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$$\delta(\alpha) = P \circ \sum_{n \geq 0} \alpha\left(\frac{x^n}{n!}\right) \partial_x^n \in \mathsf{HZ}_\varepsilon^1 \subset \mathsf{End}\left(\mathbb{K}[x]\right) \tag{1.11}$$

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- $\mathrm{HZ}_\varepsilon^1\left(\mathbb{K}[x]\right) = \mathbb{K} \cdot \int_0 \oplus \delta\left(\mathbb{K}[x]'\right)$, i.e. the only non-trivial one-cocycle is
$$\int_0 : \mathbb{K}[x] \to \mathbb{K}[x], p = \sum_{n\geq 0} p_n x^n \mapsto \int_0^x p(y)\mathrm{d}y = \sum_{n>0} \frac{p_{n-1}}{n} x^n \tag{1.12}$$

The renormalization group

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Corollary (since $P \circ F(-\partial_x) \in \mathrm{HZ}^1_\varepsilon(\mathbb{K}[x])$ is a cocycle)

$\phi_R : H_R \to \mathbb{K}[x]$ is a morphism of Hopf algebras: $\Delta \circ \phi_R = (\phi_R \otimes \phi_R) \circ \Delta$.

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This means that

$$\phi_{\mathrm{R},a+b} = \mathrm{ev}_{a+b} \circ \phi_{\mathrm{R}} = (\mathrm{ev}_a \star \mathrm{ev}_b) \circ \phi_{\mathrm{R}} = (\mathrm{ev}_a \circ \phi_{\mathrm{R}}) \star (\mathrm{ev}_b \circ \phi_{\mathrm{R}}) = \phi_{\mathrm{R},a} \star \phi_{\mathrm{R},b}.$$

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$\phi_R = \exp_\star(-x\gamma)$ for the anomalous dimension $\gamma := -\partial_0 \circ \phi_R \in \mathfrak{g}_{\mathbb{K}}^{H_R} \subset H_R'$.

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Example ($\widetilde{\Delta}$ () $= 2\bullet \otimes$  $+ \bullet\bullet \otimes \bullet$ and $\widetilde{\Delta}^2$ () $= 2\bullet \otimes \bullet \otimes \bullet$)

ϕ_{R}() $= \left[-\frac{x^3}{6}\gamma^{\star 3} + \frac{x^2}{2}\gamma^{\star 2} - \gamma x\right]$() $= -\frac{x^3}{3}[\gamma(\bullet)]^3 + x^2\gamma(\bullet)\gamma$() $- x\gamma$()

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Using the Mellin transforms, we can calculate γ recursively by

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&= 2c_1\left[\gamma(\bullet)\right]^2 = 2c_{-1}^2 c_1
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Analytic regularization

Regulate divergences by a parameter $z \in \mathbb{C}$, resulting in Feynman rules ${}_z\phi : H_R \to \mathcal{A}$ taking values in Laurent series $\mathcal{A} = \mathbb{K}[z^{-1}, z]]$:

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Renormalizing as before, the finiteness implies the existence of

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Use antipodes $S\left(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\right) = -\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} + \bullet\bullet$, $S(\bullet) = -\bullet$ and $R_\mu \circ {}_z\phi^{\star -1} = R_\mu \circ {}_z\phi \circ S$ in

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The scale dependence ${}_z\phi_s = {}_z\phi_\mu \circ \theta_{-z\ell}$ is dictated by the grading

$$\theta_t := \sum_{n\geq 0} \frac{(Yt)^n}{n!} \in \mathrm{Aut}\,(H_R)\,,\, w \mapsto e^{t|w|}\cdot w \quad \text{where} \quad Yw = |w|\cdot w. \quad (1.16)$$

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The anomalous dimension can be derived from the regularized character by

$$\gamma = -\partial_\ell|_{\ell=0}\phi_{\mathrm{R}} = \lim_{z\to 0}\left[z \cdot {}_z\phi \circ (S \star Y)\right] = \mathrm{Res}\, {}_z\phi \circ (S \star Y). \qquad (1.17)$$

Minimal subtraction

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Minimal subtraction splits $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ into poles $\mathcal{A}_- = z^{-1}\mathbb{K}[z^{-1}]$ and holomorphic $\mathcal{A}_+ = \mathbb{K}[[z]]$ along the projection

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Since R_{MS} is not a character (only Rota-Baxter), the Birkhoff decomposition of a (regularized) character ${}_z\phi \in \mathcal{G}_{\mathcal{A}}^{H_R}$ entails

- Bogoliubov map ($\bar{R}$-operation): $\bar{\phi} = \phi + \left({}_z\phi_- \otimes {}_z\phi\right) \circ \widetilde{\Delta}$
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Finiteness is trivial, let $\phi_+ := \lim_{z\to 0} {}_z\phi_+$ denote the physical limit.

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{}_z\phi_+(\bullet) &= (\mathrm{id} - R_{\text{MS}})\, {}_z\phi_s(\bullet) = (\mathrm{id} - R_{\text{MS}})\, s^{-z}F(z) = s^{-z}F(z) - \frac{c_{-1}}{z} \\
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Dimensional regularization and locality

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$$ {}_z\phi_+\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right) = (\mathrm{id} - R_{\mathrm{MS}}) \left[{}_z\phi_{s/\mu}\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right) + {}_z\phi_-(\bullet)\, {}_z\phi_{s/\mu}(\bullet) \right] $$

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Observation: The counterterms ${}_z\phi_-(\bullet) = -\frac{c_{-1}}{z}$ and ${}_z\phi_-\left(\overset{\bullet}{\underset{\bullet}{|}}\right) = \frac{c_{-1}^2}{2z^2} - \frac{c_{-1}c_0}{2z}$ are independent of ℓ.

Minimal subtraction

local characters and the β-function

Definition

A Feynman rule ${}_z\phi \in \mathcal{G}_{\mathcal{A}}^{H_R}$ is called *local* $:\Leftrightarrow$ its MS counterterm ${}_z\phi_{-,s} = \left({}_z\phi \circ \theta_{-z\ell}\right)_-$ is independent of $\ell \in \mathbb{K}$.

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Theorem

${}_z\phi \in \mathcal{G}_{\mathcal{A}}^{H_R}$ is local $\Leftrightarrow$ the inverse counterterms ${}_z\phi_-^{\star -1} : H_R \rightarrow \mathbb{K}[\frac{1}{z}]$ are poles of only first order on $\operatorname{im}(S \star Y)$*, equivalently*

$$\beta := \lim_{z\rightarrow 0} \left[z \cdot {}_z\phi_-^{\star -1} \circ (S \star Y) \right] = -\operatorname{Res}\left({}_z\phi_- \circ Y \right) \in \mathfrak{g}_{\mathbb{K}}^{H_R} \qquad (1.19)$$

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exists. The physical limit of MS-renormalized local characters is

$$\phi_+ = \exp_\star\left(-\ell\beta\right) \star \left(\varepsilon \circ \phi_+\right). \tag{1.20}$$

Here $\varepsilon \circ \phi_+ = \mathrm{ev}_{\ell=0} \circ \phi_+ \in \mathcal{G}_{\mathbb{K}}^{H_R}$ denote the constant terms.

Minimal subtraction

The scattering formula

Lemma

The vector space $\operatorname{im}(S \star Y)$ *generates* H_R *as a free commutative algebra.*

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Corollary

Counterterms ${}_z\phi_-$ *of local characters are completely determined by their first order poles* ${}_z\phi_-^{\star -1} \circ (S \star Y) = \frac{\beta}{z}$. *Explicitly,*

$$ {}_z\phi_-^{\star -1} = \varepsilon + \frac{\beta \circ Y^{-1}}{z} + \frac{\left[\left(\beta \circ Y^{-1}\right) \star \beta\right] \circ Y^{-1}}{z^2} + \mathcal{O}\left(z^{-3}\right). \qquad (1.21) $$

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From $\beta = -\operatorname{Res}\left({}_z\phi_- \circ Y\right)$ and ${}_z\phi_-(\bullet) = -\frac{c_{-1}}{z}$, ${}_z\phi_-\left(\overset{\bullet}{\underset{\bullet}{|}}\right) = \frac{c_{-1}^2}{2z^2} - \frac{c_{-1}c_0}{2z}$ we know $\beta(\bullet) = c_{-1}$, $\beta\left(\overset{\bullet}{\underset{\bullet}{|}}\right) = c_{-1}c_0$. Now we can check

$$ {}_z\phi_-^{\star-1}\left(\overset{\bullet}{\underset{\bullet}{|}}\right) = \frac{\beta\left(\frac{1}{2}\overset{\bullet}{\underset{\bullet}{|}}\right)}{z} + \frac{\left[\beta(\bullet)\right]^2}{2z^2} = \frac{c_{-1}c_0}{2z} + \frac{c_{-1}^2}{2z^2} = {}_z\phi_-\left(-\overset{\bullet}{\underset{\bullet}{|}} + \bullet\bullet\right). $$

Comparing the MOM and MS schemes

locality and finiteness

We renormalized ${}_z\phi \in \mathcal{G}_{\mathcal{A}}^{H_R}$ in the MOM- and MS-schemes to construct two renormalized characters $\phi_{\text{R}}, \phi_+ : H_R \to \mathbb{K}[x]$:

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Dyson-Schwingerequations (DSEs)

Definition (simplified)

A *perturbation series* $X(\alpha)$ is the solution of a DSE

$$X(\alpha) = \mathbb{1} + \alpha B_+ \left(X^{1+\sigma}(\alpha) \right) =: \sum_{n \geq 0} x_n \alpha^n \in H_R[[\alpha]] \tag{1.22}$$

with *coupling constant* α. The *correlation function* is

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Corollary (in the MOM scheme)

$$G_{a+b}(\alpha) = \left(\phi_{R,a} \otimes \phi_{R,b} \right) \Delta X(\alpha) = G_a(\alpha) \cdot G_b\left(\alpha \cdot G_a^{\sigma}(\alpha) \right)$$

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Consider renormalized characters of the form $\phi_{\mathrm{R}} \circ B_+ = P \circ F\left(-\partial_x\right) \circ \phi_{\mathrm{R}}$:

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Changing the scale s or renormalization point μ is equivalent to an adjustment of the coupling α. Infinitesimally, with $\widetilde{\gamma} := \gamma\left(X(\alpha)\right)$:

$$G_\ell\left(\alpha\right) \cdot \widetilde{\gamma}\left[\alpha G_\ell^\sigma\left(\alpha\right)\right]) = -\partial_\ell G_\ell(\alpha) = \widetilde{\gamma}(\alpha) \cdot \left(1 + \sigma\alpha\partial_\alpha\right) G_\ell(\alpha). \qquad (1.24)$$

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Dyson-Schwinger equations (DSEs)

RGE for correlation functions

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Example (fermion propagator of Yukawa theory, [1, 20])

Summation of all iterated self-insertions of the one-loop-correction amounts to $\sigma = -2$ and

$$F(z) = \frac{1}{z(1-z)} = \sum_{n \geq -1} z^n, \quad \text{thus} \quad \widetilde{\gamma}(\alpha) - \widetilde{\gamma}(\alpha)\left(1 - 2\alpha\partial_\alpha\right)\widetilde{\gamma}(\alpha) = \alpha$$

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Example (photon propagator of quantum electrodynamics, [18, 14])

The setup is analogous, but $\sigma = -1$ yields different solutions in terms of the Lambert W-function.

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