Cosmic Galois group and ϕ^4 theory

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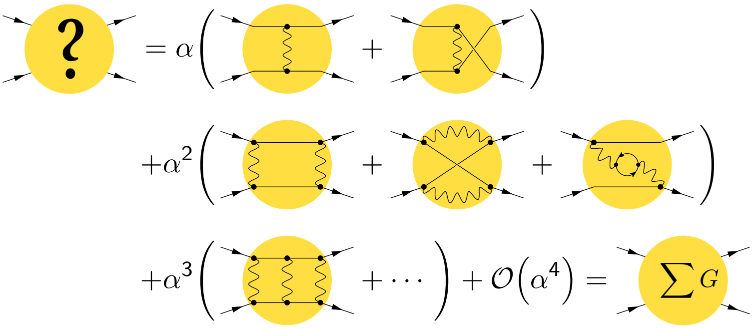
20th July 2022
GROUP34 Straßbourg

Quantum Theory – perturbative



1. more G $\Rightarrow$ higher precision $\leftarrow$ **too many**
2. $\sum$ all G $\Rightarrow$ exact $\leftarrow$ **divergent**
3. graph G $\mapsto$ integral $\mathcal{I}(G)$ $\leftarrow$ **complicated**

Examples

- particle physics
- condensed matter
- deformation quantization

Problems:

1. $\mathcal{I}(G)$ extremely complicated

   ➡ *polylogarithms, iterated elliptic integrals, modular forms, $K3$ surfaces, Calabi-Yau manifolds, ...*

2. $\sum_G \mathcal{I}(G) = \infty$

   ➡ *factorial growth $A \cdot n! \cdot c^n \cdot n^\alpha$, resummation, Borel transformation, resurgence, ...*

Problems:

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Tools for $\mathcal{I}(G)$:

1. topological: Picard-Lefschetz, Landau varieties [Pham]
2. differential: D-modules (integration by parts) [Chetyrkin–Tkachov]
3. arithmetic: c_2 invariant, graph permanent [Schnetz,Crump]
4. geometric: Feynman motives (this talk) [Bloch–Esnault–Kreimer,Brown]
5. tropical: Hepp bound [Panzer]

Period ring: [Kontsevich–Zagier]

$$\mathbb{C} \supset \mathcal{P} = \left\langle \int_{\{P_4,\dots,P_r \geq 0\}} \frac{P_1 + \mathrm{i}P_2}{P_3}\, \mathrm{d}x_1 \dots \mathrm{d}x_n : \begin{array}{l} \bullet\ P_k \in \mathbb{Z}[x] \\ \bullet\ \text{converges} \end{array} \right\rangle$$

Relations:

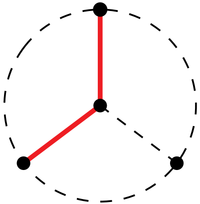
- $\int_\sigma \mathrm{d}\alpha = \int_{\partial\sigma} \alpha$ (Stokes)
- $\int_\sigma f^*\alpha = \int_{f_*(\sigma)} \alpha$ (coordinate change)
- $\int_\sigma (\alpha + \beta) = \int_\sigma \alpha + \int_\sigma \beta$ (linearity in α; similarly σ)

Conjecture

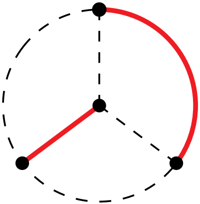
These are all relations.

➡ *motivic (formal) periods $\mathcal{P}^{\mathfrak{m}}$* ➡ *"Galois" group, acts on $\mathcal{P}^{\mathfrak{m}}$*

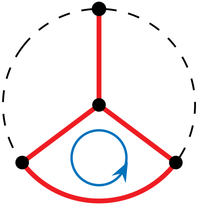
$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & \mathcal{I}(G) & \in & \mathcal{P} \\
& \mathcal{I}^{\mathfrak{m}}(G) & \in & \mathcal{P}^{\mathfrak{m}} \arrow[u, "\mathrm{per}"'] \arrow[loop below, "\mathrm{Gal}^{\mathrm{dR}}"]
\end{tikzcd}$$



not spanning



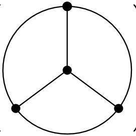
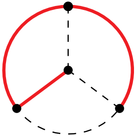
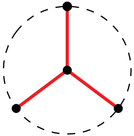
not connected



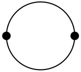
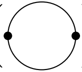
has a loop

Definition

A spanning tree $T \subset G$ is a spanning, simply connected subgraph.

ST() = {  ,  , ... }

#ST() = 16

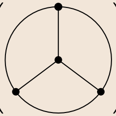
$G =$  $\Rightarrow \quad \mathcal{U} = x_1 + x_2, \qquad \mathcal{I}\Big($  $\Big) = \int_0^\infty \frac{\mathrm{d}x_2}{(1+x_2)^2} = 1$

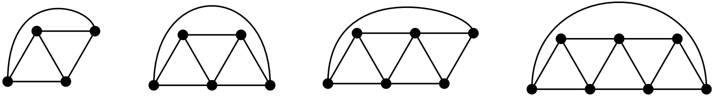
Graph polynomial and Feynman period

$$\mathcal{U} = \sum_{T \in \mathrm{ST}(G)} \prod_{e \notin T} x_e \qquad\qquad \mathcal{I}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{\mathcal{U}^2|_{x_1=1}} \in \mathcal{P}$$

- convergence restricts G (no subdivergences)
- no masses or momenta $(\# \{e\} = 2h_1)$
- hard to compute, even numerically

Example

$\mathcal{I}\Bigg($  $\Bigg) = \int_{\mathbb{R}_+^5} \frac{\mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x_4\, \mathrm{d}x_5\, \mathrm{d}x_6}{(x_1x_2x_3 + 15 \text{ more terms})^2|_{x_1=1}} = 6\zeta(3) = 6 \sum_{n=1}^\infty \frac{1}{n^3}$



ZZ_3 ZZ_4 ZZ_5 ZZ_6

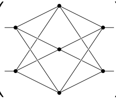
Theorem (Brown & Schnetz 2012)

$$\mathcal{I}(ZZ_n) = 4\frac{(2n-2)!}{n!(n-1)!}\left(1 - \frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3)$$

⇨ only known *infinite* family of ϕ^4 periods [conj. Broadhurst–Kreimer]

Database:

- $>$ 1000 periods known in ϕ^4 [Broadhurst, Kreimer, Schnetz, Panzer]
- among those: mostly multiple zeta values (MZV)

$$\mathcal{I}\left(\right.$$  $$\left. \right) = \frac{288}{5}\left(58\zeta(8) - 45\zeta(3)\zeta(5) - 24\zeta(3,5)\right)$$

$$\zeta(3,5) = \sum_{1\leq n<m} \frac{1}{n^3 m^5}$$

- extension to ϕ^3 [Borinsky, Schnetz]

Theorem (Brown)

$$\{\zeta^{\mathfrak{m}}(n_1, n_2, \ldots)\} \cong \mathbb{Q}[\pi^2] \otimes \mathbb{Q}\langle f_3, f_5, f_7, \ldots\rangle$$

Example

$$\zeta^{\mathfrak{m}}(2n+1) \mapsto f_{2n+1} \qquad \zeta^{\mathfrak{m}}(3,5) \mapsto -5 f_5 f_3$$

⇨ *periods are not just numbers, but have structure*

$$\mathrm{Gal}^{\mathrm{dR}} \ni \delta_k : \quad \delta_k(f_{n_1} \ldots f_{n_r}) := \begin{cases} f_{n_2} \ldots f_{n_r} & \text{if } k = n_1 \\ 0 & \text{else} \end{cases}$$

Example (Galois action)

$$\delta_3 \zeta^{\mathfrak{m}}(3) = 1 \qquad \delta_3 \zeta^{\mathfrak{m}}(3,5) = 0 \qquad \delta_5 \zeta^{\mathfrak{m}}(3,5) = -5\zeta^{\mathfrak{m}}(3) \qquad \delta_k \zeta^{\mathfrak{m}}(2n) = 0$$

- $g\zeta^{\mathfrak{m}}(3,5) = \lambda(g)^8 \cdot \zeta^{\mathfrak{m}}(3,5) - 5 f_5(g)\lambda(g)^3 \cdot \zeta^{\mathfrak{m}}(3) + f_3 f_5(g) \cdot 1$

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- $g\zeta^{\mathfrak{m}}(2n+1) = \lambda(g)^{2n+1} \cdot \zeta^{\mathfrak{m}}(2n+1) + f_{2n+1}(g) \cdot 1$
- $g\zeta^{\mathfrak{m}}(2n) = \lambda(g)^{2n} \cdot \zeta^{\mathfrak{m}}(2n)$

⇨ Galois conjugates are suffixes

$$\mathcal{P}^{\mathfrak{m}}_{\phi^4} = \sum_{G\in\phi^4} \mathbb{Q}\cdot\mathcal{I}^{\mathfrak{m}}(G) \subset \mathcal{P}^{\mathfrak{m}}$$

Coaction conjecture (O. Schnetz)

$$\mathrm{Gal}^{\mathrm{dR}} \quad \text{acts on} \quad \mathcal{P}^{\mathfrak{m}}_{\phi^4}$$

➡ *strong constraint, implies propagation of "holes"*

weight	0	2	3	4	5	6	7
$\mathcal{P}^{\mathfrak{m}}_{\phi^4}$	1		$\zeta^{\mathfrak{m}}(3)$		$\zeta^{\mathfrak{m}}(5)$	$\zeta^{\mathfrak{m}}(3)^2$	$\zeta^{\mathfrak{m}}(7)$
missing MZVs		$\zeta^{\mathfrak{m}}(2)$		$\zeta^{\mathfrak{m}}(4)$	$\zeta^{\mathfrak{m}}(2)\zeta^{\mathfrak{m}}(3)$	$\zeta^{\mathfrak{m}}(2)^3$	$\zeta^{\mathfrak{m}}(2)\zeta^{\mathfrak{m}}(5)$, $\zeta^{\mathfrak{m}}(4)\zeta^{\mathfrak{m}}(3)$

$$\xleftarrow{\quad\delta_3\quad} \qquad\qquad \xleftarrow{\quad\delta_5\quad}$$

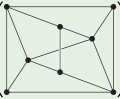
➡ *follows also from **small graphs principle** (theorem) [F. Brown]*

Not all Feynman periods are MZV:

Theorem (Deligne)

$$\left\{\mathrm{Li}^{\mathfrak{m}}_{n_1,n_2,\ldots}(e^{\mathrm{i}\pi/3})\right\} \cong \mathbb{Q}[\mathrm{i}\pi] \otimes \mathbb{Q}\langle f_2, f_3, f_4, f_5, \ldots\rangle$$

Example

$\mathcal{I}^{\mathfrak{m}}\Bigg($  $\Bigg)$

$$= -\frac{332\,262}{43} f_8 f_3 + \frac{54\,918}{55} f_6 f_5 + \frac{1\,134}{13} f_4 f_7 - \frac{1\,874\,502}{3\,485} f_2 f_9$$

$$+ 5\,670 f_2 f_3 f_3 f_3 - \frac{3\,216\,912\,825\,399\,005\,402\,331\,281\,812\,377\,062\,149}{14\,080\,217\,073\,343\,074\,027\,422\,017\,273\,458\,000}\left(\frac{\pi}{\sqrt{3}}\right)^{11}$$

➡ apply δ_{2k} $\Rightarrow$ only odd letters survive $\Rightarrow$ MZV, in ϕ^4

Cohomology of varieties $(X, D \subset P)$ over $\mathbb{Q}$:

- $H^n_{\mathrm{dR}}(P \setminus X, D)$ (*algebraic* differential forms)
- $H^n_{\mathrm{sing}}(P(\mathbb{C}) \setminus X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$
- $H^n_{\mathrm{dR}}(P \setminus X, D) \otimes \mathbb{C} \xrightarrow{\cong} H^n_{\mathrm{sing}}(P \setminus X, D) \otimes \mathbb{C}, \quad \omega \mapsto (\sigma \mapsto \int_\sigma \omega)$

Ring of motivic periods

$$\mathcal{P}^{\mathfrak{m}} = \left\langle [H, \omega, \sigma] : \omega \in H_{\mathrm{dR}}, \sigma \in H^\vee_{\mathrm{sing}} \right\rangle / \sim$$

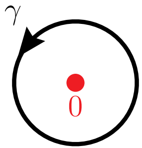
$\sim$ = linearity + Stokes + change of coordinates

Motivic Galois group (de Rham):

$$\mathrm{Gal}^{\mathrm{dR}} = \mathrm{Aut}^{\otimes}(\mathrm{dR})$$

➡ *acts on integrands ω and thus on $\mathcal{P}^{\mathfrak{m}}$*

Lefschetz motive:



$$P = \mathbb{P}^1$$
$$X = \{0, \infty\} \qquad P \setminus X = \mathbb{Q}^\times$$
$$D = \emptyset$$

Cohomology:

$$H^1_{\mathrm{dR}}(P \setminus X) = \mathbb{Q}\left[\tfrac{\mathrm{d}x}{x}\right] \qquad H^1_{\mathrm{sing}}(P \setminus X) = \mathbb{Q}[\gamma]^\vee$$

$$\int \colon H^1_{\mathrm{dR}} \otimes \mathbb{C} \xrightarrow{\cong} H^1_{\mathrm{sing}} \otimes \mathbb{C}, \qquad \left[\tfrac{\mathrm{d}x}{x}\right] \mapsto 2\mathrm{i}\pi \cdot [\gamma]^\vee$$

Motivic $2\mathrm{i}\pi$:

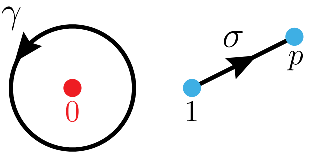
$$(2\mathrm{i}\pi)^{\mathfrak{m}} = \left[H^1(\mathbb{G}_{\mathrm{m}}), \left[\tfrac{\mathrm{d}x}{x}\right], [\gamma]^\vee\right] \in \mathcal{P}^{\mathfrak{m}}$$

Galois group:

$$g \cdot \left[\tfrac{\mathrm{d}x}{x}\right] = \lambda(g) \cdot \left[\tfrac{\mathrm{d}x}{x}\right]$$
$$g \cdot (2\mathrm{i}\pi)^{\mathfrak{m}} = \lambda(g) \cdot (2\mathrm{i}\pi)^{\mathfrak{m}}$$

$$\lambda \colon \mathrm{Gal}^{\mathrm{dR}} \longrightarrow \mathbb{Q}^\times = \mathrm{GL}_1(\mathbb{Q})$$

Kummer motive: $(p \in \mathbb{Q} \setminus \{0,1\})$



$$P = \mathbb{P}^1$$
$$X = \{0, \infty\}$$
$$D = \{1, p\}$$

Cohomology:

$$H^1_{\mathrm{dR}}(P \setminus X, D) = \mathbb{Q}\left[\tfrac{\mathrm{d}x}{x}\right] \oplus \mathbb{Q}\left[\tfrac{\mathrm{d}x}{p-1}\right]$$
$$H^1_{\mathrm{sing}}(P \setminus X, D) = \mathbb{Q}[\sigma]^\vee \oplus \mathbb{Q}[\gamma]^\vee$$

$$\int = \begin{array}{c} \\ {[\sigma]^\vee} \\ {[\gamma]^\vee} \end{array} \begin{array}{c} \begin{array}{cc} \left[\frac{\mathrm{d}x}{p-1}\right] & \left[\frac{\mathrm{d}x}{x}\right] \end{array} \\ \begin{pmatrix} 1 & \log p \\ 0 & 2\mathrm{i}\pi \end{pmatrix} \end{array}$$

Motivic logarithm:

$$\log^{\mathfrak{m}} p = \left[H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, p\}), \left[\tfrac{\mathrm{d}x}{x}\right], [\sigma]^\vee\right] \in \mathcal{P}^{\mathfrak{m}}$$

Galois group:

$$g(\log^{\mathfrak{m}} p) = \lambda(g) \cdot \log^{\mathfrak{m}} p + f(g) \cdot 1, \quad \mathrm{Gal}^{\mathrm{dR}} \longrightarrow \left\{ \begin{pmatrix} 1 & f \\ 0 & \lambda \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Q})$$

[Bloch–Esnault–Kreimer & Brown]

Feynman motives

$$\mathcal{I}^{\mathfrak{m}}(G) = \left[H^{E-1}(P \setminus X, D \setminus X), [\sigma], \left[\frac{\mathrm{d}^{E-1}x}{\mathcal{U}^2}\right] \right] \in \mathcal{P}^{\mathfrak{m}}$$

1. $\pi\colon P \twoheadrightarrow \mathbb{P}^{E-1}$ iterated blowup of $\bigcap_{e\in\gamma} \{x_e = 0\}$ (bridgeless γ)
2. $X = \{\mathcal{U} = 0\}$
3. $D = \pi^{-1}(\{x_1 \cdots x_E = 0\})$
4. $\sigma = \overline{\pi^{-1}(\{[x_1 : \cdots : x_E] \colon\ x_e > 0\})}$ (Feynman polytope)

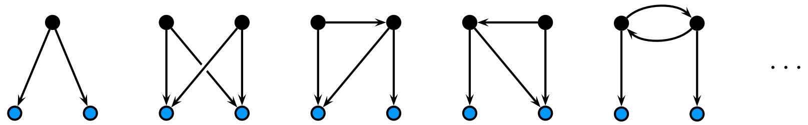
$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \mathcal{I}(G) \in \mathcal{P} \subset \mathbb{C} \\ & \mathcal{I}^{\mathfrak{m}}(G) \in \mathcal{P}^{\mathfrak{m}} \arrow[u, "\text{per}"'] \end{tikzcd}$$

➡ *weights, small graphs principle, ...*

Kontsevich's deformation quantization formula

$$f \star g = \sum_{G} \underbrace{w_G}_{\text{real number}} \cdot \underbrace{B_G(f, g, \{\})}_{\text{differential operator}}$$

Graphs G**:**



Integrals:

$$w_G = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{H}^n} \bigwedge_{i \to j} \mathrm{d} \arg \frac{z_i - z_j}{\bar{z}_i - z_j} \in \mathbb{R}$$

Theorem & Algorithm (Banks–Panzer–Pym)

$$w_G \in \sum_n \mathbb{Q} \frac{\zeta(n_1, \ldots)}{\pi^{n_1 + \ldots}} \subset \mathcal{P}\left[\frac{1}{\mathrm{i}\pi}\right]$$

Goal: lift to $w_G^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}$ gain action by $\mathrm{Gal}^{\mathrm{dR}}$ [with Dupont & Pym]