Around Functional Transcendence

Andrew Wiles Building
University of Oxford
26-29 June 2018

Abstracts of the talks

- **Vahagn Aslanyan**, Carnegie Mellon University
  Weak Modular Zilber-Pink with Derivatives

  I'll present two weak versions of Pila's Modular Zilber-Pink with Derivatives (MZPD) conjecture. Those can be proven using the Ax-Schanuel theorem and an appropriate Existential Closedness (EC) conjecture for the j-function. Although the EC conjecture in its general form is still open, some special cases are easy to prove which can be used to establish a weak Modular Andre-Oort statement with Derivatives unconditionally. I will also explain how one can prove a special case of one of the aforementioned weak MZPD conjectures unconditionally.

- **Daniel Bertrand**, IMJ-PRG, Université Pierre et Marie-Curie, Paris
  Logarithmic Ax and special curves in a mixed Shimura threefold

  I will report on a recent work with H. Schmidt (ArXiV:1803.04835v1) on unlikely intersections in a family of semi-abelian surfaces over a given CM elliptic curve. This relies on a case-by-case study of the monodromy of the corresponding Betti map. Following a similar path, the result turns into a solution of Pink's conjecture for the mixed Shimura variety attached to the Poincare bi-extension over the elliptic curve.

- **Gal Binyamini**, Weizmann Institute of Science
  Using functional transcendence to sharpen the Pila-Wilkie theorem

  The Pila-Wilkie theorem plays a key role in Pila's proof of the modular Ax-Lindemann theorem and in the proof of many subsequent functional independence statements of Ax-Lindemann and Ax-Schanuel type. In this talk I will discuss an implication in the reverse direction: using functional independence statements to deduce effective bounds on the geometry of foliations defined over number fields, and deriving sharper forms of the Pila-Wilkie theorem (in the spirit of Wilkie's conjecture) as a consequence. If time permits I will discuss some applications of this approach to effectivity questions in problems of unlikely intersections.

- **Paola D'Aquino**, Università degli Studi della Campania
  Spectrum of $\hat{\mathbb{Z}}$ hat and finite adeles over $\mathbb{Q}$

  We study the structure sheaf of $\hat{\mathbb{Z}}$ and of the finite adeles. We give a description of localizations of the previous two rings at primes, and we generalize this to product of valuation rings. We use tools from logic as ultraproduct and Feferman-Vaught theorem. Thisis a join work with A. Macintyre alnd M. Otero.

- **Philipp Habegger**, Unversität Basel
  Variation of the Néron-Tate Height in a Family of Abelian Varieties

  The Néron-Tate height attached to a polarized abelian variety defined over a number field can detect points of finite order. The distribution of points of small height with respect to the Zariski topology is governed by Bogomolov's Conjecture, proved by Ullmo and Zhang. At the center of this talk is the variation of the Néron-Tate height in a family of abelian varieties when the base is a curve defined over a number field. I will explain how the monodromy action on the period lattice can be used to recover a statement that is reminiscent of functional transcendence. Its main application is an inequality between the Néron-Tate height and the Weil height on a subvariety of the family. This inequality extends some aspects of Silverman's Specialization Theorem to higher dimension. I will also discuss an application to the Bogomolov Conjecture over a function field. This is joint work with Ziyang Gao.

- **Jonathan Kirby**, University of East Anglia
  Local interdefinability of elliptic functions

  We consider definability issues around elliptic functions, that is, around the Weierstrass $\wp$, ζ and σ-functions which are associated to the exponential maps of elliptic curves and their multiplicative and additive extensions. We work locally in the real topology in an o-minimal setting. In earlier work with Jones and Servi, we used Ax-Schanuel theorems for $\wp$-functions to show, for example, that one $\wp$-function is locally definable from another if and only if the corresponding lattices are isogenous, possibly after applying complex conjugation to one. In this talk I will discuss the extensions of these results to ζ and σ-functions and some of the complications which occur. This is joint work with Gareth Jones and Harry Schmidt.

- **Bruno Klingler**, Humboldt Universität, Berlin
  Definability of period maps and algebraicity of Hodge loci

  I will explain that period maps for pure variations of Hodge structures are definable in the o-minimal structure . As a corollary we recover the algebraicity of Hodge loci, a classical result of Cattani, Deligne and Kaplan. Joint work with Bakker and Tsimerman.

- **David Masser**, Universität Basel
  On a question of Katz-Oort II

  This talk is a continuation of Umberto Zannier's. A simple example of what we have proved is the existence of many abelian fourfolds, defined over $\mathbb{Q}$ and with endomorphism ring $\mathbb{Z}$, which are not isogenous to any jacobian. We will describe more details of the proof strategy for general g. The use of "Hodge-generic" is essential to the argument, and a particular tool is provided by "endomorphism estimates", which extend isogeny estimates.

- **Ngaiming Mok**, University of Hong-Kong
  Curvature, Rescaling and Uniformization Problems on Bounded Symmetric Domains

  The asymptotic behavior of invariant metrics on bounded domains has been an important tool in several complex variables especially in characterization theorems. A first instance of such a result is the work of B. Wong (1977) showing that a strictly pseudoconvex domain in the Euclidean space is biholomorphic to the complex unit ball $\mathbb{B}^n$ whenever it admits a noncompact group of automorphisms, a result which was obtained by exploiting the asymptotic curvature behavior of the Bergman metric and by the method of rescaling. Here by *rescaling* we will mean the process of composing with a divergent sequence of automorphisms and extracting a convergent subsequence. We will illustrate the method of rescaling by studying special classes of algebraic subsets of bounded symmetric domains Ω in their Harish-Chandra realizations and explain how the method leads to solutions of problems on uniformization and functional transcendence on bounded symmetric domains, including (a) a differential-geometric proof of the hyperbolic Ax-Lindemann Theorem for the rank-1 case and (b) a solution of the characterization of totally geodesic subsets of $X_\Gamma = \Omega/\Gamma$ as the unique bi-algebraic subvarieties without using monodromy results of André-Deligne. These solutions are obtained by studying the asymptotic geometry of a subvariety as it exits $\partial\Omega$ and by the method of rescaling, and they apply without assuming arithmeticity of the lattices.

- **Joel Nagloo**, City University New York
  The Ax-Lindemann-Weierstrass with derivative and the genus 0 *Fuchsian Groups*

  The works of Pila and later Freitag and Scanlon, give the Ax-Lindemann-Weierstrass with derivatives for the Hauptmoduls of arithmetic subgroups of $PSL_2(\mathbb{Z})$. A challenge is to prove similar transcendence results for the Hauptmoduls of all Fuchsian groups of genus zero. The aim of this talk is to discuss some recent progress made in this direction for the cocompact triangle groups (and related 'Galois coverings'). I will also explain what I mean by the differential Ax-Lindemann-Weierstrass in this context and the relevance of the model theory of differentially closed fields of characteristic 0. This is joint work with Guy Casale.

- **Kobi Peterzil**, University of Haifa
  The image of a definable set in a compact nilmanifold
  (joint with Sergei Starchenko)

  We extend our previous work on o-minimal flows inside complex and real tori to compact nilmanifolds, namely quotients of unipotent groups by lattices.

  Theorem. *Assume that G is a real linear unipotent group and X a closed subset of G which is definable in some o-minimal structure M over the reals.*

  For every lattice L in G (i.e. a co-compact discrete subgroup) there exists a definable subset X_L of G, such the closure of $p(X)$ in G/L equals $p(X_L)$ (here $p : G \to G/L$ is the quotient map). Moreover, the set X_L is obtained, uniformly in L, via certain subgroups of G which arise from examining complete types on X.

  When X is a Lie subgroup of G, the above follows from a theorem of Ratner in ergodic theory. We then use our work on mu-stabilizers of types in o-minimal structures, as well as the theory of Tame pairs, to reduce the problem to this result about groups.

- **Thomas Scanlon**, University of California, Berkeley
  Differential algebraic Zilber-Pink theorems

  As is well known, it follows from the Seidenberg embedding theorem that the functional Ax-Lindemann-Weierstrass and Ax-Schanuel theorems on the transcendence of the covering maps for various locally symmetric spaces admit differential algebraic formulations. We will derive effective versions of instances of the Zilber-Pink conjectures from these differential algebraic interpretations. (This is a report on a joint project with Jonathan Pila.)

- **Harry Schmidt**, University of Manchester
  Counting rational points and iterated polynomial equations

  In joint work with Gareths Boxall and Jones we prove a poly-logarithmic bound for the number of rational points on the graph of functions on the disc that exhibit a certain decay. I will present an application of this counting theorem to the arithmetic of dynamical systems. It concerns fields generated by the solutions of equations of the form $P^{\circ n}(z) = P^{\circ n}(y)$ for a polynomial P of degree $D \geq 2$ where y is a fixed algebraic number. The general goal is to show that the degree of such fields grows like a power of D^n.

- **Sergei Starchenko**, University of Notre-Dame
  Remarks on definable fundamental sets in o-minimal structures
  (joint with Kobi Peterzil)

  Assume that Γ is a discrete infinite group acting on a complex manifold M. A goal is to realize the quotient space $\Gamma \setminus M$ as a definable object in an o-minimal structure. The following is sufficient: the existence of a definable (in some o-minimal structure) subset F of M such that (1) $\Gamma.F = M$ and (2) the set of $\{g \in \Gamma \ : \ g.F \cap F \text{ is non empty } \}$ is finite.

  Under additional topological assumptions the quotient of F by Γ, call it M_F, can be endowed with a definable manifold structure, which is naturally biholomorphic to $\Gamma \setminus M$. It turns out that different definable fundamental sets can give rise to definable manifolds which are not definably bi-holomorphic. We consider the basic case of $(\mathbb{C}, +)$ and the group of integers ("the exponential case"), and show that the various fundamental sets give rise to three types of strongly minimal structures: trivial, linear and non-locally modular.

- **Giuseppina Terzo**, Università degli Studi della Campania
  Generic solutions of exponential equations

  Assuming Schanuel's Conjecture, we prove that for every variety V of dimension n contained in $\mathbb{C}^n \times (\mathbb{C}^*)^n$ over the algebraic closure of the rational numbers, and under some natural hypothesis, there exists a generic point in V of the form $(a, \exp(a))$. This result implies some particular case of Zilber's Conjecture.

- **Jacob Tsimerman**, University of Toronto
  o-minimality for Hodge Structures

  We describe joint work with Ben Bakker and Bruno Klingler on how to endow the moduli space of Hodge structures with an $\mathbb{R}_{\text{an, exp}}$ structure, and in doing so give another proof of a classical result of Cattani-Deligne-Kaplan which can be seen as evidence towards the hodge conjecture.

- **Sai-Kee Yeung**, Purdue University
  Revisit of some cases of the conjectures of Coleman and Oort

  With the help of Andre-Oort Conjecture, the conjectures of Coleman and Oort are reduced to showing that a locally Hermitian symmetric space does not parametrize non-trivial families of smooth algebraic curve of sufficiently large genus g. In case that the base has real rank at least 2 as a locally symmetric space, the result is a consequence of a result of Farb-Masur as observed by Hain. The purpose of the talk to explain a more direct approach to such results geometrically.

- **Alex Wilkie**, University of Oxford
  Around Functional Transcendence in o-minimal structures

  I begin by discussing the notion of continuous uniform distribution (c.u.d) for functions definable in an o-minimal expansion, M say, of the real field. Then, in the case that M is polynomially bounded and M* is some elementary extension of M, I discuss a complex version of the valuation inequality for finitely generated (via 0-definable, complex holomorphic functions) substructures of M*[i]. This states that the sum of the valuation rank and (complex) residue dimension of the substructure is bounded by its (complex) dimension. (The dimension here is simply the minimum number of generators needed and the only valuation ring being considered is the ring of finite elements.) It turns out that the valuation rank and residue dimension may be interpreted, using the c.u.d. theory mentioned above, as measuring the complexity of holomorphic, definable functions in the real and imaginary directions respectively. This is then applied to proving a version of Ax-Schanuel for (complex) definable functions.

  There is nothing particularly deep here. It's really just a case of interpreting classical theorems in the o-minimal context. But there is a certain flexibility in the method that might prove useful in other contexts.

- **Umberto Zannier**, Scuola Normale Superiore di Pisa
  On a question of Katz-Oort I

  A question of Katz-Oort asked about the existence of abelian varieties over $\bar{\mathbb{Q}}$, of dimension $g \geq 4$ and not isogenous to any Jacobian. This was treated conditionally by Chai-Oort and shortly afterwards solved by Tsimerman: they produced examples with CM of Weyl type. In recent work with Masser we have used a completely different method, this time to produce examples which are Hodge-generic, complex-dense in A_g, and moreover defined over number fields of bounded degree. In this talk I shall illustrate our strategy for a certain "real" analogue in the case $g = 1$.

- **Boris Zilber**, University of Oxford
  Pseudo-analytic structures and abelian Galois extensions of number fields

  It is a folklore of model theory that if the automorphism group of a finite (sub)structure is commutative then the automorphisms are definable. We use this to give a sufficient condition for an algebraic extension of a field to be abelian. We show how this condition combined with the model theory of pseudo-analytic covers (group covers) quickly gives generators of abelian Galois extensions of $\mathbb{Q}$ and of imaginary quadratic extensions of$\mathbb{Q}$. Then I will discuss the case of real quadratic extensions of $\mathbb{Q}$ in this context.