Observations of a continuum coupled map model of granular media modified by a sinusoidal mapping term

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August 14, 2003

Abstract

We generalize the continuum coupled map (CCM) model described by Venkataramani and Ott[1,2] by introducing a sinusoidal term into their mapping function. This change yields remarkable patterns whose complete understanding remains and open question. The purpose of this paper is to display the data produced from the sinusoidal mapping term to motivate interest in others to more fully understand this dynamically interesting system.

1 Introduction

Patterned states are a wide ranging phenomenon that occur in then natural world in such forms as convection[3], chemical reactions[3], and granular media dynamics[1,2].

Because we treated the data we modelled as a discrete dynamical system, the method we incorporated in arriving at steady patterns heavily relied upon the repeated iteration of given mapping terms. This repeated iteration was hoped would yield insight into whether or not data from a function will fall into particular orbits, the speed the data might conform to these orbits, and characteristics of the orbits themselves.

Continuum coupled maps consist of discrete maps describing time evolution and continuous field and spatial variables. Venkataramani and Ott[1,2]

Model type	Field variable(s)	Spatial domain	Time
PDE	Continuous	Continuous	Continuous
LODE	Continuous	Discrete	Continuous
CML	Continuous	Discrete	Discrete
CA	Discrete	Discrete	Discrete
CCM	Continuous	Continuous	Discrete

Table 1: Different model types and their qualities.

have used them to study vertically vibrated granular media, to which we now turn our attention.

1.1 Vibrated Granular Media

Granular media is ubiquitous and defines substances ranging from coffee to gravel to grain[4]. Because of their particular size, shape, and surface properties, as well as their dissipative nature, granular media sometimes exhibits complicated dynamics, which are not completely understood[4,5].

The equations we adapted from Venkataramani and Ott[1,2] were descriptions of behavior displayed by experiments done by Umbanhowar, Melo, and Swinney[6]. These experiments made use of brass spheres that were vertically vibrated sinusoidally, and yielded several patterned states.

1.2 Continuum Coupled Map Models (CCM's)

A CCM model consists of continuous variables, a continuous spatial domain, and discrete time intervals of a non-equilibrium system[1,2]. Therefore, the purpose of a CCM is to map a continuous field forward (or backward) in time[1,2]. There are various other model types with different choices of continuity and discreteness for different quantities (Table 1)[1,2], however, the advantage of CCM's is that they allow for both efficiency (through a fast Fourier transform to simulate iteration) and the flexibility to model patterns not regulated by an artificial grid [1,2]. For these reasons, Venkataramani and Ott[1,2] used CCM's to model period-2 vertical forcing in granular media, and it is their CCM model of that forcing that we will later adapt.

1.3 Pattern Formation

Pattern formation is pervasive throughout nature[1,2]. It can be seen on the shells of mollusks, convection, and chemical equations[3].

Fortunately, our understanding of this important phenomenon has dramatically increased under close mathematical scrutiny. However, much still remains to be discovered and seen.

It is hoped that given a situation where a function or functions model the actual behavior of a real system, and a patterned state arises from the successive iteration of that/those function(s), one could gain insight into predicting and/or understanding our physical world.

2 Replication of Prior Results

A crucial first step to understanding the problem to accurately reproduce the results of Venkataramani and Ott[1,2] in order to assure the validity of further investigations into similar phenomena. To set the stage, we now discuss Venkataramani and Ott's CCM formulation.

Venkataramani and Ott utilized an array of values $\xi_n(x)$ to represent the varying heights of the vibrated granular media at time n, where x is a two-dimensional variable representing a value's spatial coordinate in the array. In order to iterate the function to time n+1, they first applied a mapping function M:

$$\xi_n'(x) = M[\xi_n(x), r], \tag{1}$$

where r is a parameter of the chosen map function. Finally, they incorporated coupling of x's neighbors by using a linear spatial operator, ℓ:

$$\xi_{n+1}(x) = \ell[\xi_n'(x)]. \tag{2}$$

Venkataramani and Ott went further in defining ℓ as:

$$\ell[\xi_n'(x)] = f(x) \otimes \xi_n'(x), \tag{3}$$

where $\otimes$ represents convolution.

Venkataramani and Ott employ spatial Fourier transforms of $\xi_n(x)$ and $f(x)$ denoted as $\bar{\xi_n}(k)$ and $\bar{f}(k)$ giving:

$$\bar{\xi}_{n+1}(k) = \bar{f}(k)\bar{\xi}_n'(k), \tag{4}$$

where $\bar{f}(k)$ is:

$$\bar{f}(k) = g(k/k_0, k/k_c). \tag{5}$$

The function g in equation (5) is defined by Venkataramani and Ott as:

$$g(k/k_0, k/k_c) = \phi(k/k_c)\exp[\gamma(k/k_0)], \tag{6}$$

where $\phi(k/k_c)$ and $\gamma(k/k_0)$ are given by:

$$\gamma(k) = \frac{1}{2}\left(\frac{k}{k_0}\right)^2\left[1 - \frac{1}{2}\left(\frac{k}{k_0}\right)^2\right] \tag{7}$$

$$\phi(k) = \mathrm{sgn}(k_c^2 - k^2). \tag{8}$$

For the mapping function, M, in equation (1), Venkataramani and Ott chose:

$$M(\xi, r) = r\exp[-(\xi - 1)^2/2] \tag{9}$$

The most important aspect of this choice is that M is closely related to the logistic map $M(\xi, r) = r\xi(1-\xi)$, but has bounded orbits for all values [1,2].

Figure 1 shows the patterns we obtained that correspond to those in Figure 3 of Ref.'s [1,2].

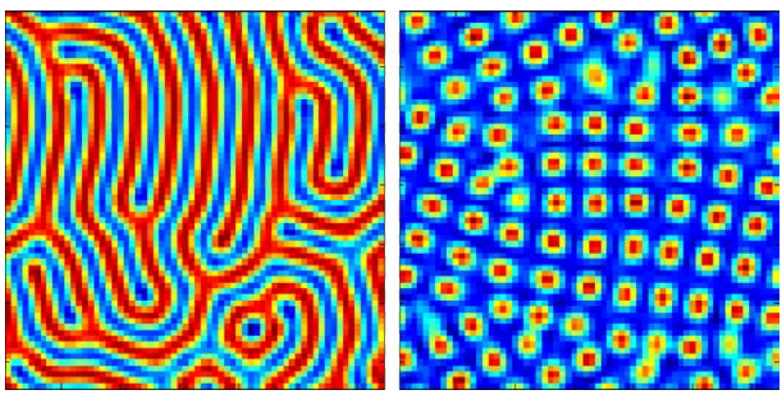

Figure 1: Patterns that correspond to Venkataramani and Ott's[1,2]. Left: r = 1.9, $(k_c/k_0)^2) = 5.0$; Right: r = 1.9, $(k_c/k_0)^2) = 1.5$.

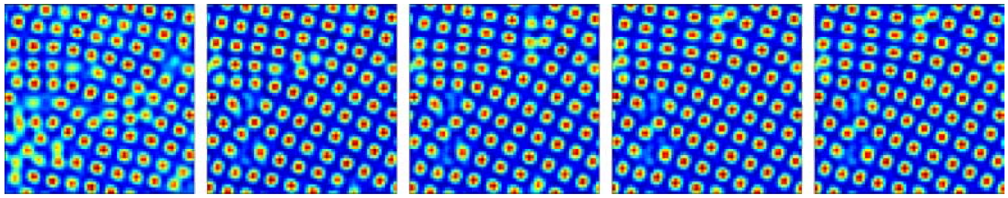

Figure 2: A progression of 100, 500, 1000, 5000, and 10000 iterates for parameters r = 1.9, $(k_c/k_0)^2) = 1.5$.

3 Other Visuals

Figure 2 shows the development of a pattern at successive time iterations. Some of the other interesting patterns that we found that were not included in Ref.'s [1,2] are depicted here in Figure 3.

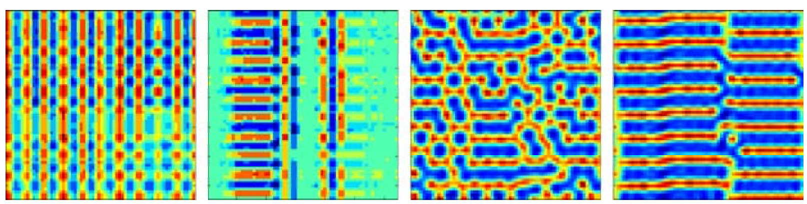

Figure 3: Patterns not mentioned by Venkataramani and Ott[1,2]. Left to right: r = 2.5, $(k_c/k_0)^2) = 2.0$; r = 2.5, $(k_c/k_0)^2) = 3.5$; r = 2, $(k_c/k_0)^2) = 2.5$; r = 3.5, $(k_c/k_0)^2) = 2.5$.

4 Sinusoidal Mapping Function

In an effort to discover new dynamics, we modified the original mapping function (9). Instead of using equation (9), we employed:

$$M(\xi, r) = r\sin[-(\xi - 1)^2/2]. \qquad (10)$$

We studied this map numerically and found some peculiar and interesting phenomena. For instance, Figure 4 illustrates a pattern that seemed unclassifiable with the traditional taxonomy used in Ref.'s [1,2].

Our specific sinusoidal mapping function numerically yielded interesting results. For instance, the data from one pixel after iterating our code using

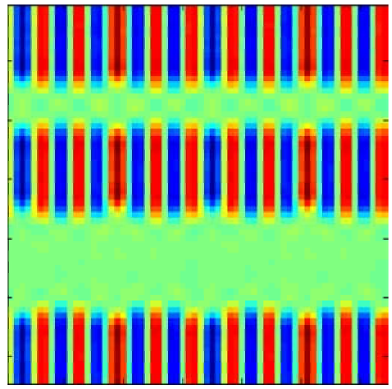

Figure 4: Sinusoidal mapping after 10000 iterates for parameters r = 1.5, $(k_c/k_0)^2) = 4$.

the new sin map after 10000 iterations, setting r to 1.5 and $(k_c/k_0)^2)$ to 4 was: -1.4993 -0.0274 -0.7554 -1.4993 -0.0274 -0.7554 -1.4993. Figure 5 displays the oscillation of this particular pixel for the last 100 and 30 iterations of the 10000 for each respective plot.

Although this data does not look like it is approaching a singular fixed point, it appears that there are three distinct orbits. This implies that these points are in the basin of attraction of a periodic orbit of period 3, i.e. $f(f(f(x_0))) = x_0$.

An interesting area for further study would be an analytical exploration of the sin map. This exploration help explain the period-3 results obtained, which imply chaos in the time evolution of the CCM[7,8]. As a side note, it did seem that the final outcome produced by the code was highly sensitive to the initial random matrix input, which is a hallmark of chaos[7,8].

Figure 6 shows some of the sensitivity we observed. Each frame of the figure represents patterns iterated 10000 times given identical initial parameters except for differing random near-homogeneous seed matrix inputs.

5 Conclusion

There is not a lot that we can firmly conclude other than the fact that pattern formation is a curious phenomenon that deserves more in depth study. Our computational research showed several very interesting patterns, but

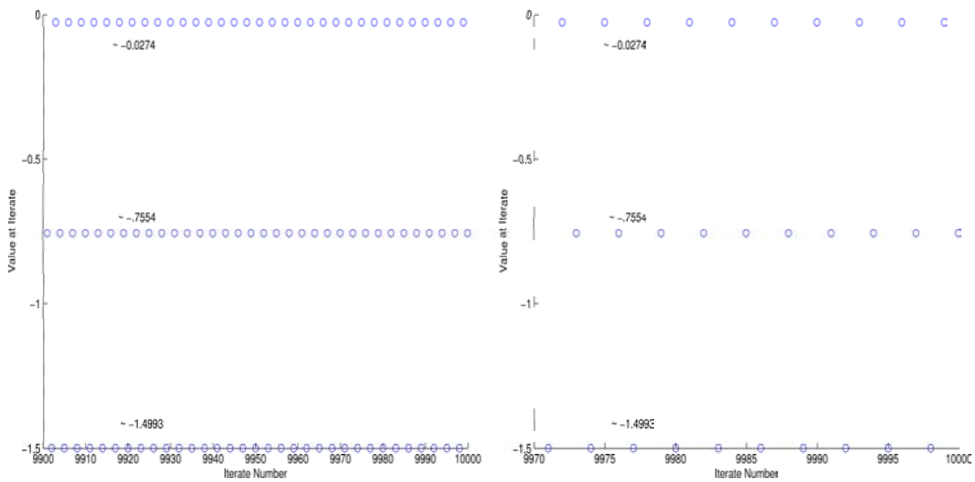

Figure 5: Left: Plot of last 100 iterates of 10000 iterates for for an individual pixel for the sinusoidal map of parameters r = 1.5, $(k_c/k_0)^2) = 4$; Right:Plot of last 30 iterates of 10000 iterates for for an individual pixel for the sinusoidal map of parameters r = 1.5, $(k_c/k_0)^2) = 4$.

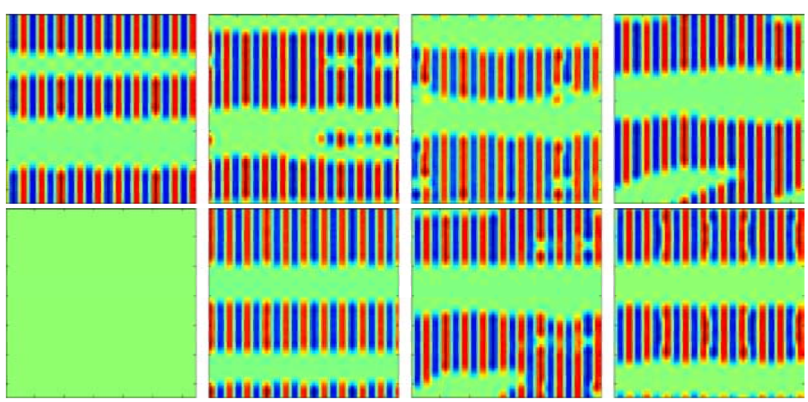

Figure 6: Different pattern observed using near-homogenous random seed matrices but equivalent parameters of r = 1.5, $(k_c/k_0)^2) = 4$ after 10000 iterations.

it is clear that an analytical approach is necessary to complement our numerical studies and achieve a deeper understanding of the phenomenon we considered.

Acknowledgments

This project was advised by Shui-Nee Chow and Mason Porter. Special thanks to Jeremy Corbett for coding most of the equations and explanations. Special thanks to Peter Bucha for help with coding. Special thanks to Shankar Venkataramani for e-mailing us copies of his code and his support.

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