# B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 2

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**Exercise 1. The classification of elliptic curves.**

*Recall from lectures that there are bijections*

$$\frac{\left\{\begin{array}{c}\textit{Riemann surfaces}\\ \textit{homeomorphic to a torus}\end{array}\right\}}{\textit{biholomorphisms}} \longleftrightarrow \frac{\left\{\begin{array}{c}\textit{Quotients}\\ \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \textit{ with } \tau\in\mathbb{H}\end{array}\right\}}{\textit{biholomorphisms}} \longleftrightarrow \mathbb{H}/PSL(2,\mathbb{Z}),$$

*The second map is $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \leftrightarrow [\tau]$, and $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\pm I$ acts on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by Möbius maps. Although we will not need the following fact, some easy group theory shows that $SL(2,\mathbb{Z})$ is generated by $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. The corresponding Möbius maps $S(z) = -1/z$ and $T(z) = z+1$ are rather useful in this exercise.*

For $A = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, show that $\mathrm{Im}(Az) = \frac{1}{|cz+d|^2}\cdot \mathrm{Im}(z)$. Deduce that, given a constant $K$, only finitely many $c, d \in \mathbb{Z}$ satisfy $\mathrm{Im}(Az) > K$.

Show that $\mathbb{H}/PSL(2,\mathbb{Z})$ is a topological space homeomorphic to $\mathbb{C}$, by first showing that each point of $\mathbb{H}/PSL(2,\mathbb{Z})$ has a representative inside the "strip"

$$\{\tau \in \mathbb{H} : |\mathrm{Re}(\tau)| \le 1/2, |\tau| \ge 1\}$$

and then checking that the only remaining identifications are on the boundary of the strip.[1]

Show that $PSL(2,\mathbb{Z})$ acts freely[2] on $\mathbb{H}$ except at the points in the $PSL(2,\mathbb{Z})$-orbits of $e^{\pi i/3}$ and of $i$, and show that the stabilisers of those points are respectively $\mathbb{Z}/3$ and $\mathbb{Z}/2$.

Briefly comment on why the natural local complex coordinate from $\mathbb{H}$ makes $\mathbb{H}/PSL(2,\mathbb{Z})$ into a Riemann surface except at $e^{\pi i/3}$ and $i$, and explain why it is not a topological surface near those two points (if one uses the coordinates inherited from $\mathbb{H}$).

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**Cultural remark.** *By Exercise sheet one, $w^2 = 4z^3 - g_2 z - g_3$ is Riemann surfaces homeomorphic to a torus. In fact, it is biholomorphic to $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ if we take coefficients $g_2 = 60\sum(m+n\tau)^{-4}$ and $g_3 = 140\sum(m+n\tau)^{-6}$ summing over all integers $(m,n) \neq (0,0)$. There is a holomorphic map $\mathbb{H} \to \mathbb{C}$, $\tau \mapsto j(\tau) = 1728\, g_2^3/(g_2^3 - 27g_3^2)$, which is $PSL(2,\mathbb{Z})$-invariant, so it gives a well-defined map $j : \mathbb{H}/PSL(2,\mathbb{Z}) \to \mathbb{C}$, called the* elliptic modular function *or* Klein's $j$-invariant, *where $\tau = e^{\pi i/3}$, $i$ correspond to $j = 0, 1728$ respectively. The Riemann surfaces corresponding to those two values of $\tau$ have*[3] *automorphism groups $\mathbb{Z}/3$ and $\mathbb{Z}/2$. For example, for $\tau = i$, the generator of the $\mathbb{Z}/2$ automorphism group of $\Lambda/(\mathbb{Z}+\tau\mathbb{Z})$ is multiplication by $i$. So the Riemann surface $\mathbb{C}$ classifies elliptic curves, up to that automorphism ambiguity, and it is called the* moduli space of elliptic curves, $\mathcal{M}_{1,1}$.

*One can give $\mathbb{H}/PSL(2,\mathbb{Z})$ the structure of a Riemann surface $S$ by declaring that $j$ is a biholomorphism, which means that near $i$ and $e^{\pi i/3}$ we use a local holomorphic coordinate $w$ different from the natural coordinate $z$ of $\mathbb{H}$. Namely, $\pi : \mathbb{H} \to S$ is a* branched cover *near those two points, locally $z \mapsto w = z^2$ and $z \mapsto w = z^3$. We will study such maps in the course.*

*One can now prove a big theorem in complex analysis, the* Picard Theorem*:*

The image of a non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ misses at most one value.

*Sketch proof: by translating and rescaling, we may assume by contradiction that $f$ does not attain the values $0, 1728$. Let $g$ be the multi-valued inverse of $\pi : \mathbb{H} \to S$. Then $\varphi = g \circ j^{-1} \circ f$ is locally holomorphic since $g$ is holomorphic except at $0, 1728$. Now analytically continue $\varphi$ to a well-defined holomorphic map $\mathbb{C} \to \mathbb{H}$. But such maps are constant.*[4]

**Exercise 2. Conformal and area-preserving parametrizations.**

*In this exercise, the first fundamental form has local matrix $A = \begin{pmatrix} e & f\\ f & g\end{pmatrix}$.*

Show that a local parametrization is conformal (i.e. angle-preserving) $\Leftrightarrow$ $f = 0$, $e = g$

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[1] *Hint. Try to maximize the imaginary part for the orbit of $z$ under the action.*

[2] A group $G$ acts freely on $X$ if stabilizers are trivial, explicitly: if $g \bullet x = x$ for some $x$, then $g = 1$.

[3] Strictly speaking, an *elliptic curve* is a genus one Riemann surface with a choice of a fixed marked point in the curve (e.g. pick $0 \in \mathbb{C}/\Lambda$). This choice gets rid of the continuous group of automorphisms given by the torus action on itself by translation: $c \in \mathbb{C}/\Lambda$ acts by $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, $z \mapsto z + c$ if we don't require 0 to be fixed.

[4] Compose with the biholomorphism $\mathbb{H} \cong D$ to the open unit disc, to get a bounded holomorphic map $\mathbb{C} \to D$, then Liouville's theorem implies it is constant.

Show that a local parametrization is area preserving $\Leftrightarrow$ $\det \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right) = 1$

Show that the stereographic projection is conformal, by considering the parametrization given by the inverse of the stereographic projection:

$$\mathbb{C} \to S^2 \setminus (\text{North Pole}),\ (x,y) \mapsto \frac{1}{1+x^2+y^2}(2x, 2y, -1+x^2+y^2),$$

and show that the fundamental form for the sphere in these local coordinates is

$$I_F = \frac{4(dx^2+dy^2)}{(1+x^2+y^2)^2}.$$

**Cultural remark:** *That fundamental form on the sphere is called the* chordal metric*. It gives rise to another example of non-Euclidean geometry called* elliptic geometry*.*

**Exercise 3. Nautical cartography: Mercator's projection.**

Using the parametrization

$$F(\theta,\phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$$

of the unit sphere $S^2 \subset \mathbb{R}^3$, find the first fundamental form.

To draw maps of the Earth, one often uses Mercator's projection of the unit sphere minus the date line:

$$(X,Y) = \left(\theta\,,\, \log\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)\right) \in \mathbb{R}^2,$$

where $(\theta,\phi)$ are the longitude and latitude coordinates on the Earth. What does the first fundamental form of the sphere become in the coordinates $(X,Y)$ of the plane? Deduce that Mercator's projection is conformal but not area-preserving.
There are pictures of this on Wikipedia: http://en.wikipedia.org/wiki/Mercator_projection

**Exercise 4. Tangential derivatives and Christoffel symbols.**

*A* vector field *is a smooth family of tangent vectors $v \in TS$, so locally*

$$v(x,y) = a(x,y)\partial_x F + b(x,y)\partial_y F \in T_{F(x,y)}S,$$

*for some smooth functions $a,b$. The derivatives of $v$ in $x,y$ may not lie in $TS$, they lie in $\mathbb{R}^3 = T_{F(x,y)}S \oplus \mathbb{R}n(x,y)$ where $n$ is the unit normal (Gauss map). If we subtract the normal part, we obtain the* tangential derivative*:*

$$\nabla_x v = \partial_x v - (n\cdot\partial_x v)\,n \qquad \nabla_y v = \partial_y v - (n\cdot\partial_y v)\,n.$$

*The symbol $\nabla$ is called* nabla*, and the operator $\nabla$ is called a* connection *for the surface $S$.*

Using that $v \in TS$ is orthogonal to $n$, show that $-n\cdot\partial_x v = \partial_x n\cdot v$. Show that $\nabla$ is *compatible* with the Riemannian metric $I$ (the first fundamental form): for vector fields $v,w$,

$$\partial_x\, I(v,w) = I(\nabla_x v, w) + I(v, \nabla_x w),$$

*We introduce some helpful notation: $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\nabla_1 = \nabla_x$, $\nabla_2 = \nabla_y$. Abbreviate the basis of $TS$ by $X_1 = \partial_x F$ and $X_2 = \partial_y F$. Abbreviate $X_{ij} = \partial_i\partial_j F$. Writing $\nabla_i X_j \in TS$ in the basis $X_i, X_j$, yields coefficient functions $\Gamma_{ij}^k(x,y)$, called* Christoffel symbols*:*

$$\boxed{\nabla_i X_j = \sum_{k=1}^{2} \Gamma_{ij}^k X_k}$$

Verify the symmetry relations:

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

*Now abbreviate by $II_{ij}$ the entries of the second fundamental form $I = \left(\begin{smallmatrix} L & M \\ M & N \end{smallmatrix}\right)$, defined by*

$$\boxed{II_{ij} = n\cdot X_{ij}}$$

Show that the normal part of the derivatives of $X_j$ determines the second fundamental form:[5]

$$X_{ij} - \nabla_i X_j = II_{ij} n.$$

*Let's abbreviate the entries $I_{ij}$ of the first fundamental form $I = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)$ by:*

$$\boxed{g_{ij} = I(X_i, X_j)}$$

Notice that $g_{ij} = g_{ji}$. Now prove that

$$X_{ij} \cdot X_\ell = I(\nabla_i X_j, X_\ell) = \sum_{k=1}^{2} \Gamma_{ij}^k g_{k\ell}.$$

Using the earlier compatibility result, and the above equation, show that

$$\partial_i g_{j\ell} = \sum_{k=1}^{2} (\Gamma_{ij}^k g_{k\ell} + \Gamma_{i\ell}^k g_{kj}).$$

Writing $g^{ij}$ for the entries[6] of the inverse matrix $I^{-1} = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)^{-1}$, deduce that

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

Deduce that the Christoffel symbols are determined just by $I$, and therefore they are invariant under isometries between surfaces.

[5] In case you've lost the plot, an example should clarify:

$$X_{11} = \partial_x \partial_x F = \Gamma_{11}^1 \partial_x F + \Gamma_{11}^2 \partial_y F + L\, n$$

where $L$ is the $(1,1)$-entry $II_{11}$ of the second fundamental form $II$.

[6] *Hint.* $I^{-1} I = \mathrm{id}$ *becomes the orthonormality equation*

$$\sum_{j=1}^{2} g^{ij} g_{jk} = \delta_k^i$$

*where $\delta_k^i$ equals 1 for $i = k$ and 0 for $i \neq k$.*