B3.2 GEOMETRY OF SURFACES

Dictionary of some terminology from topology and analysis

Comments and corrections are welcome: ritter@maths.ox.ac.uk

Prof. Alexander F. Ritter, Mathematical Institute, Oxford.

1. Topology: a dictionary

◇ A **topological space** is a set X and a collection of subsets of X called *open sets* such that:

(1) the empty set is open,
(2) the whole set is open,
(3) a finite intersection of open sets is open,
(4) an arbitrary union of open sets is open.

Example. A metric[1] space (X, d) is a topological space: the open sets are any union of balls $B_r(x) = \{y \in X : d(x, y) < r\}$ (for centres $x \in X$, radii $r > 0$).

◇ Convention: our spaces are always understood to be topological spaces.

◇ A subset is called **closed** if it is the complement of an open set.

◇ A **neighbourhood**[2] of $x \in X$ is a subset which contains an open set U with $x \in U$.

◇ A map $f : X \to Y$ is **continuous** if f^{-1}(open set) is always open.

(1) A composition of continuous functions is continuous,
(2) f continuous $\Rightarrow f$(compact subset) is compact,
(3) f continuous $\Rightarrow f$(connected subset) is connected.
(4) Continuous bijection from a compact space to a Hausdorff space $\Rightarrow$ homeomorphism.
(5) Continuous surjection from a compact space to a Hausdorff space $\Rightarrow$ quotient map.

◇ $f : X \to Y$ is a **quotient map** if $U \subset Y$ open $\Leftrightarrow f^{-1}(U) \subset X$ is open.

◇ X is **Hausdorff** if any two points can be separated by open sets.[3]

◇ X is **compact** if every open cover by open sets has a finite subcover.[4]

◇ **Heine-Borel theorem:** subsets of $\mathbb{R}^n$ are compact $\Leftrightarrow$ they are closed and bounded.

◇ **Example.** For metric spaces X, Y:

(1) A subset $S \subset X$ is closed $\Leftrightarrow S \ni x_n \to x$ implies $x \in S$.
(2) A map $f : X \to Y$ is continuous $\Leftrightarrow f(x_n) \to f(x)$ whenever $x_n \to x$.
(3) X is automatically Hausdorff.
(4) X is compact $\Leftrightarrow$ any sequence has a convergent subsequence.[5]

◇ X is **connected** if every continuous function $f : X \to \mathbb{Z}$ is constant.

◇ X is **path-connected** if any two points are joined by a continuous path.[6]

◇ X path-connected $\Rightarrow X$ connected, but the converse is false in general.[7]

◇ X is **simply-connected** if it is connected and any loop in X is contractible.[8]

◇ A continuous **deformation** of $f : X \to Y$ is a continuous map $F : X \times [0, 1] \to Y$ with

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[1] So a function $d : X \times X \to \mathbb{R}$ with $d(x, y) = d(y, x) \geq 0$ with equality if and only if $x = y$, and such that the triangle inequality holds: $d(x, y) \leq d(x, z) + d(z, y)$.

[2] We do not require the neighbourhood to be an open set. We say *open neighbourhood* in that case.

[3] For any $x, y \in X$, there are open sets U_x, U_y containing x, y respectively, with $U_x \cap U_y = \emptyset$.

[4] So if $X = \cup U_i$ for some open sets U_i, then $X = U_{i_1} \cup \cdots \cup U_{i_m}$ for some indices $i_1, \ldots, i_m$.

[5] So $x_n \in X$ implies $x_{n_j} \to x \in X$ for some $n_1 < n_2 < \cdots$

[6] For any $x, y \in X$ there is a continuous map $f : [0, 1] \to X$ with $f(0) = x$, $f(1) = y$.

[7] The two notions become equivalent if you assume the space is **locally path-connected**. This means: for any $x \in X$ and any open U containing x, there is an open $V \subset U$ which is path-connected, with $x \in V$.

[8] So for any continuous $f : S^1 \to X$ there is a continuous $F : \mathbb{D} \to X$ with $F|_{S^1} = f$. Here $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a circle, $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ is a disc. By parametrizing $\mathbb{D}$ by sz with $z = e^{it} \in S^1$, $s \in [0, 1]$, you can view F as a family of loops $F : S^1 \times [0, 1] \to X$ from the constant loop $F_0 = F(\cdot, 0)$ to $F_1 = F(\cdot, 1) = f$.

$F(x,0) = f(x)$. So $F_s(x) = F(x,s)$ is a family of maps, $F_0 = f$, and F_1 is the deformed map.

$\diamond$ A map $f : X \to Y$ is **bijective** if there exists a map $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ are the identity maps. Such a g is unique and called the **inverse** $g = f^{-1}$.

$\diamond$ A **homeomorphism** $f : X \to Y$ is a continuous bijection, with continuous inverse f^{-1}.

$\diamond$ X, Y are **homeomorphic** if there exists a homeomorphism $f : X \to Y$.

2. Analysis

$\diamond$ $f : \mathbb{R}^n \to \mathbb{R}^m$ is **continuously differentiable** if all first order partial derivatives exist and are continuous.[1]

Explicitly: in coordinates: $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ maps to $f(x) = (f_1(x), \dots, f_m(x)) \in \mathbb{R}^m$ for some functions $f_i : \mathbb{R}^n \to \mathbb{R}$, called the *components* of f.
So we require that $\frac{\partial f_i}{\partial x_j}$ exist and are continuous for all i, j. We abbreviate $\partial_{x_j} f_i = \frac{\partial f_i}{\partial x_j}$.

$\diamond$ The **Jacobian matrix** of $f : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix $A(x) = (A_{ij}(x)) = (\partial_{x_i} f_j)$ of partial derivatives:

$$A(x) = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_n} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 & \cdots & \partial_{x_n} f_2 \\ \cdots & & & \\ \partial_{x_1} f_m & \partial_{x_2} f_m & \cdots & \partial_{x_n} f_m \end{pmatrix}$$

The linear map given by "multiplication by $A(x)$" is the **derivative map**

$$Df : \mathbb{R}^n \to \mathbb{R}^m, \ v \mapsto D_x f \cdot v = A(x)v.$$

Example. For $f : \mathbb{R} \to \mathbb{R}$, $A(x) = (f'(x))$, $Df : \mathbb{R} \to \mathbb{R}$ is multiplication by $f'(x)$.

$\diamond$ **Chain rule:** Compositions of differentiable maps are differentiable and $D(g \circ f) = Dg \circ Df$:

$$D_x(g \circ f) = D_{f(x)} g \circ D_x f.$$

Example. For $f, g : \mathbb{R} \to \mathbb{R}$ recall $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$.

$\diamond$ Convention: the vector $\partial_{x_j} f$ denotes the j-th column of that matrix.

$\diamond$ **Example.** Linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable with derivative map L. The whole point of the derivative map is to find the best linear approximation to a map: $f(x) = f(p) + D_p f \cdot (x - p) + \text{error}$, where $\frac{\text{error}}{\|x-p\|} \to 0$ as $x \to p$.

$\diamond$ $f : \mathbb{R}^n \to \mathbb{R}^m$ is **smooth** if it has partial derivatives of all orders (they are automatically continuous).[2]

Fact: for smooth functions, partial derivatives commute, e.g. $\partial_{x_1} \partial_{x_2} f = \partial_{x_2} \partial_{x_1} f$.

For open $U, V \subset \mathbb{R}^n$, $f : U \to V$ is a **diffeomorphism** if f is a homeomorphism, and f, f^{-1} are smooth.

$\diamond$ **Integration by substitution (change of variables):** If $f : V \to U$ is a diffeomorphism, for open subsets $U, V \subset \mathbb{R}^n$, and $G = G(x_1, \dots, x_n) : U \to \mathbb{R}$ is a smooth function, then

$$\int_U G(x)\, dx_1 \cdots dx_n = \int_V G(f(y))\, |\det D_y f|\, dy_1 \cdots dy_n$$

Examples.

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[1] The reason for requiring that the partial derivatives are also continuous is necessary to ensure that the derivative map exists, in the sense that $f(x+h) - f(x) = D_x f \cdot h + \text{error}$, where $\frac{\text{error}}{\|h\|} \to 0$ as $h \to 0$.

[2] For example, for the second order, it means: $A : \mathbb{R}^n \to \mathbb{R}^{nm}, x \mapsto A(x)$ is differentiable. As you increase the order, this becomes complicated since you choose the succession of which partial derivatives to take.

(1) Let f be the change of variables from polar coordinates r, θ to (x, y) in $\mathbb{R}^2$. So $f(r,\theta) = (r\cos\theta, r\sin\theta)$, so $Df = \left(\begin{smallmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{smallmatrix}\right)$, so $|\det Df| = r$, hence $\int G(x,y)\,dx\,dy = \int G(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.
(2) If $\gamma = \gamma(t) : [0,1] \to \mathbb{R}^2$ is a smooth curve, and $f = f(s) : [a,b] \to [0,1]$ reparametrizes time (so any strictly increasing smooth function), then the length of the curve, $\int |\text{speed}|\, d(\text{time})$, is well defined independently of the way we parametrize time: $\int_0^1 \|\gamma'(t)\|\, dt = \int_a^b \|\gamma'(f(s))\|\, f'(s)\, ds$.

$\diamond$ $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *local diffeomorphism near* p, if there are open neighbourhoods U, V of $p, f(p)$ respectively such that the restriction $f|_U : U \to V$ is a diffeomorphism.

$\diamond$ *Convention:* we say $f : \mathbb{R}^n \to \mathbb{R}^m$ *is defined near* p to mean: there is an open set $U \subset \mathbb{R}^n$ containing p such that $f : U \to \mathbb{R}^m$ is defined. We say "for x, y close enough to $p, f(p)$" to mean: there are open neighbourhoods U, V of $p, f(p)$ and the statement holds for $x \in U, y \in V$.

$\diamond$ **Inverse function theorem:** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map defined near $p \in \mathbb{R}^n$.

If $D_p f$ is invertible, then f is a local diffeomorphism near p.

Explicitly: the theorem hands us a unique smooth map $g : \mathbb{R}^n \to \mathbb{R}^n$ defined near $f(p)$ such that $f(g(y)) = y$ and $g(f(x)) = x$ (for all x, y close enough to $p, f(p)$ respectively).
Arguably the most important theorem in analysis. It says simple linear algebra (the non-vanishing of the determinant of a matrix) ensures the smooth invertibility of the map, locally.

$\diamond$ Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be smooth, and $n \geq m$. We want to describe the solutions of $f(x) = c$ near a given solution $f(a) = c$, where $x, a \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$.

Implicit function theorem: If m columns of $D_a f$ are linearly independent, then the variables $x_{i_1}, \ldots, x_{i_m}$ corresponding to those columns are redundant. Namely, they can be replaced by unique smooth functions $g_{i_1}, \ldots, g_{i_m}$, depending only on the remaining variables, defined near $x = a$ and satisfying $g_{i_1}(a) = a_{i_1}, \ldots, g_{i_m}(a) = a_{i_m}$, so that

$$f(x)|_{(x_{i_1}=g_{i_1},\, \ldots,\, x_{i_m}=g_{i_m})} = c$$

describes all solutions x near a.

Examples. Below, we seek solutions of $f = 0$ near $x = (0, \ldots, 0)$.

(1) $f(x,y) = y$: $\partial_y f = 1 \neq 0$, so $f(x, g(x)) = 0$ (indeed $g(x) = 0$).
(2) $f(x,y) = x^2 - y$: $\partial_y f = -1 \neq 0$, so $f(x, g(x)) = 0$ (indeed $g(x) = x^2$).
(3) $f(x,y) = (x+1)^2 - 1 + y^2$: $\partial_x f|_{x=0,y=0} = 2 \neq 0$, so $f(g(y), y) = 0$ (indeed $g(y) = -1 + \sqrt{1-y^2}$, which is defined near $y = 0$, and notice $g(0) = 0$).

Proof of the implicit function theorem: *by relabeling coordinates, we may assume the last m columns of $D_a f$ are linearly independent. Abbreviate $k = n - m$. Consider $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x_1, \ldots, x_n) = (x_1, \ldots, x_k, f(x_1, \ldots, x_n))$. Notice that $D_a F$ is invertible (try writing the matrix). Apply the inverse function theorem. Then $F^{-1}(x_1, \ldots, x_k, c_1, \ldots, c_m) = (x_1, \ldots, x_k, g_{k+1}, \ldots, g_n)$ for unique functions $g_{k+1}, \ldots, g_n$ of $x_1, \ldots, x_k, c$.* $\square$

Smooth dependence on c in the implicit function theorem: Notice above $g_{i_1}, \ldots, g_{i_m}$ depend smoothly on c. So there are unique smooth functions $G_{i_1}, \ldots, G_{i_m} : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}$ depending only on non-redundant x_j variables and $y \in \mathbb{R}^m$, defined near $x = a, y = c$ so that

$$f(x)|_{(x_{i_1}=G_{i_1},\, \ldots,\, x_{i_m}=G_{i_m})} = y$$

describes all solutions of $f(x) = y$ for x near a, and y near c.

$\diamond$ A change of coordinates near $x = a$ means a local diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ near $x = a$. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ becomes $\widetilde{f} = f \circ \varphi$ in the new coordinates. So $\widetilde{f}(z) = f(x)$ for $x = \varphi(z)$.

⋄ **Nonlinear coordinates in the implicit function theorem:** There is a change of coordinates of $\mathbb{R}^n$ near $x = a$, and we call the new coordinates $z_1, \ldots, z_n$ *non-linear coordinates*, so that solutions of $\widetilde{f}(z) = y$ near $z = \varphi^{-1}(a)$ are precisely described by the vanishing $z_1 = 0, \ldots, z_m = 0$ of m coordinates (and the other z_j coordinates are free).
Proof. First permute coordinates of $\mathbb{R}^n$ so that we may assume the $i_1, \ldots, i_m$ above are $1, \ldots, m$. Then put $z_1 = x_1 - g_1, \ldots, z_m = x_m - g_m$, and the other $z_j = x_j$. □

3. Complex analysis

⋄ A function $f : \mathbb{C} \to \mathbb{C}$ is **holomorphic** if it is complex differentiable.[1]
⋄ Fact: $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (f_1(x+iy), f_2(x+iy))$ is differentiable with continuous partial derivatives and satisfies

$$DF \circ J = J \circ DF$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (the matrix which rotates by 90°) corresponds to multiplication by i when we identify $\mathbb{R}^2 \equiv \mathbb{C}$, $(x, y) \equiv x + iy$.
Remark. $DF \circ J = J \circ DF \Leftrightarrow$ Cauchy-Riemann equations $\partial_x f_1 = \partial_y f_2$, $\partial_y f_1 = -\partial_x f_2$ hold.

$$DF = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix} = \begin{pmatrix} \partial_x f_1 & -\partial_x f_2 \\ \partial_x f_2 & \partial_x f_1 \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where r, θ are determined by $f'(z) = re^{i\theta}$. Notice $\operatorname{Det} DF = |f'(z)|^2 = r^2$.

⋄ Fact: f holomorphic $\Rightarrow$ the above $F : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth.

⋄ Fact: f holomorphic near $p \Rightarrow f$ has an absolutely convergent Taylor series[2] at p and f is equal to its Taylor series near p.

⋄ **Identity theorem.** If $f, g : \mathbb{C} \to \mathbb{C}$ are holomorphic near p, and there is a sequence $p \neq z_n \to p$ with $f(z_n) = g(z_n)$, then $f = g$ near p.

⋄ $f : \mathbb{C} \to \mathbb{C}$ is a **biholomorphism** if it is bijective and f, f^{-1} are both holomorphic.
Remark. Since the derivative map is a composition of scaling and rotation, it preserves angles between vectors. So biholomorphisms are conformal maps, *meaning they preserve angles.*

⋄ **Inverse function theorem.** For a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ defined near p, if $f'(p) \neq 0$ then f is a local biholomorphism near p.
Explicitly: the theorem hands us a unique holomorphic $g : \mathbb{C} \to \mathbb{C}$ defined near $f(p)$ such that $f(g(w)) = w$ and $g(f(z)) = z$ (for all z, w close enough to $p, f(p)$ respectively).

⋄ **Riemann mapping theorem.** If $U \neq \emptyset, \mathbb{C}$ is a simply connected open subset of $\mathbb{C}$ then there is a biholomorphism $f : U \to D$ onto the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

4. Differential equations

⋄ For smooth smooth $V : \mathbb{R}^n \to \mathbb{R}^n$, a **flowline** $\gamma : [a, b] \to \mathbb{R}^n$ is a solution of $\gamma'(t) = V(\gamma(t))$.
Idea: V is a vector field (a vector at each point of $\mathbb{R}^n$), γ is a curve running in the V-direction.

⋄ **Theorem.** For each point $p \in \mathbb{R}^n$ there is a flowline $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ of V with $\gamma(0) = p$, for small enough $\varepsilon > 0$. Moreover, γ is smooth, unique and depends smoothly[3] on p.

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[1] Meaning $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists. Take $h = t \in \mathbb{R}$, let $t \to 0$: then $f'(z) = \partial_x f = \partial_x f_1 + i\partial_x f_2$. Take $h = it \in i\mathbb{R}$, let $t \to 0$: $f'(z) = -i\partial_y f = \partial_y f_2 - i\partial_y f_1$. Equating gives the Cauchy-Riemann equations.
[2] $\sum_{n=0}^{\infty} a_n(z-p)^n$ with $a_n = f^{(n)}(p)/n!$
[3] Meaning: there is a smooth map $F : (-\varepsilon, \varepsilon) \times U \to \mathbb{R}^n$, called **flow**, defined on a small enough neighbourhood U of p (and $\varepsilon > 0$ depends on U), such that $t \mapsto F(t, q)$ is the flowline of V through $q = F(0, q)$.