B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 1

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Exercise 1. $\mathbb{CP}^1$ as a quotient of spheres.

Recall that the complex projective space $\mathbb{C}P^1$ is the space of complex lines through 0 in $\mathbb{C}^2$. By thinking about how a complex 1-dimensional vector space intersects the sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$, show that $\mathbb{C}P^1$ as a topological surface can be viewed as a quotient

$$\mathbb{C}P^1 = S^3/S^1$$

where you need to explain how the group S^1 acts on S^3.

Exercise 2. The Möbius band.

The open Möbius band is the quotient

$$M = [0,1] \times (0,1) \,/\, ((0,y) \sim (1, 1-y) \text{ for all } y \in (0,1)).$$

Briefly explain why M is a smooth surface. Find (a homeomorphic copy of) M inside the real projective space $\mathbb{R}P^2$ and inside the Klein bottle K. The Möbius band $\overline{M}$, is obtained by replacing $(0,1)$ by $[0,1]$ above. Show[1] that the boundary of $\overline{M}$ is homeomorphic to S^1. Show that $\mathbb{R}P^2 =$ (closed disc) $\cup\, \overline{M}$ glued along the circular boundary, and $K = \overline{M} \cup \overline{M}$.

Exercise 3. Riemann surfaces arising from polynomial equations.

Briefly explain a natural way to make the sets

(1) $S_1 = \{(z,w) \in \mathbb{C}^2 : w^2 = (z-1)(z-2)\} \cup \{+\infty\} \cup \{-\infty\}$
(2) $S_2 = \{(z,w) \in \mathbb{C}^2 : w^2 = (z-1)(z-2)(z-3)\} \cup \{\infty\}$

into Riemann surfaces. Find homeomorphisms $S_1 \cong$ sphere, $S_2 \cong$ torus. *(Hints in footnote*[2])

Exercise 4. The Klein bottle as a quotient of $\mathbb{R}^2$.

Consider the quotient

$$S = \mathbb{R}^2/G$$

where $G = \mathbb{Z}^2$ acts by $(n,m) \bullet (x,y) = ((-1)^m x + n, y + m)$ on $\mathbb{R}^2$, where $n, m \in \mathbb{Z}$. Briefly explain why S is a smooth surface. Show that S is homeomorphic to the Klein bottle.

Exercise 5. The space of lines in $\mathbb{R}^2$.

Let S be the set of all straight lines in $\mathbb{R}^2$ (not necessarily through 0). Show that there is a natural[3] way to make S into a topological surface. Show that S is homeomorphic to the open Möbius band M.

Date: This version of the notes was created on October 17, 2014.

[1] *Hint. Try cutting out a clever disc from* $\mathbb{R}P^2$.

[2] Hints. It helps if you first ask yourself what local holomorphic coordinate you would use at solutions (z,w) of $w^2 = z$ (recall from lecture notes the discussion of the square root $z^{1/2}$). Then try to build the solution set S_1 by gluing two cut-domains: two copies of $\mathbb{C}$ cut from 1 to 2. Just like for $\operatorname{Log} z$ in lectures, each subset you cut gives rise to *two* copies of that subset in the Riemann surface. In order to be able to draw the Riemann surface inside $\mathbb{R}^3$, it is convenient to reflect one of the cut-domains about the x-axis. Near infinity, try using the coordinates $X = \frac{1}{z}$ and $Y = \frac{w}{z}$ instead of z, w, and ask yourself what happens for $X = 0$ (corresponding to "$z = \infty$"). For S_2 you will need a second cut, from 3 to ∞, and try instead $Y = \frac{w}{z^2}$.

[3] *Hint. For example, lines which are not vertical can be parametrized by 2 numbers: the angle* $\theta \in (-\pi/2, \pi/2)$ *which tells you how much the line is tilted, and* $r \in \mathbb{R}$ *which is the signed distance of the line from the origin* $0 \in \mathbb{R}^2$ *(using the* $+$ *sign if the line passes above* 0*, and the* $-$ *sign if it passes below* 0*).*

B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 2

Comments and corrections are welcome: ritter@maths.ox.ac.uk

Exercise 1. Tori and figure 8 loops.
A figure 8 loop consists of two circles touching at a point. Show that a torus can be obtained by attaching a disc onto a figure 8 loop.

Exercise 2. The Euler characteristic constrains graphs.
Given five points in the plane, show that it is impossible to connect each pair by paths which do not cross. Is it possible for five points in a torus?

Exercise 3. The Euler characteristic constrains Platonic solids.
Using the Euler characteristic, show that there are no more than five Platonic solids.[1]

Exercise 4. The classification of elliptic curves.
Recall from lectures that there are bijections

$$\frac{\left\{\begin{array}{c}\textit{Riemann surfaces}\\ \textit{homeomorphic to a torus}\end{array}\right\}}{\textit{biholomorphisms}} \longleftrightarrow \frac{\left\{\begin{array}{c}\textit{Quotients}\\ \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)\ \textit{with}\ \tau\in\mathbb{H}\end{array}\right\}}{\textit{biholomorphisms}} \longleftrightarrow \mathbb{H}/PSL(2,\mathbb{Z}),$$

where the second map is $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \leftrightarrow [\tau]$, *and where* $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\pm I$ *acts on the upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ *by Möbius maps.*

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, show that $\mathrm{Im}(Az) = \frac{1}{|cz+d|^2} \cdot \mathrm{Im}(z)$. Deduce that, given a constant K, only finitely many $c, d \in \mathbb{Z}$ satisfy $\mathrm{Im}(Az) > K$.

It turns out by some easy group theory that $SL(2,\mathbb{Z})$ *is generated by the matrices* $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, *although we will not need this fact. The corresponding Möbius maps* $S(z) = -1/z$ *and* $T(z) = z + 1$ *are rather useful in this exercise.*

Show that $\mathbb{H}/PSL(2,\mathbb{Z})$ is a topological space homeomorphic to $\mathbb{C}$, by first showing that each point of $\mathbb{H}/PSL(2,\mathbb{Z})$ has a representative inside the "strip"

$$\{\tau \in \mathbb{H} : |\mathrm{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}$$

and then checking that the only remaining identifications are on the boundary of the strip.
Hint. Try to maximize the imaginary part for the orbit of z under the action.

Does $PSL(2,\mathbb{Z})$ act freely[2] on $\mathbb{H}$? Briefly comment on why the natural local coordinates from $\mathbb{H}$ do *not* make $\mathbb{H}/PSL(2,\mathbb{Z})$ into a topological surface (let alone a Riemann surface).

Cultural remark. *By Exercise sheet 1 equations like* $w^2 = 4z^3 - g_2 z - g_3$ *determine Riemann surfaces homeomorphic to a torus. It turns out that this is biholomorphic to* $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ *if we take coefficients* $g_2 = 60\sum(m+n\tau)^{-4}$ *and* $g_3 = 140\sum(m+n\tau)^{-6}$ *summing over all integers* $(m,n) \neq (0,0)$. *It also turns out that* $\mathbb{H}/PSL(2,\mathbb{Z})$ *can be made into a Riemann surface via the biholomorphism* $\mathbb{H}/PSL(2,\mathbb{Z}) \cong \mathbb{C}, \tau \mapsto j(\tau) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2}$, *called the* elliptic modular function *or* Klein's j-invariant.

Date: This version of the notes was created on October 24, 2014.

[1] A Platonic solid is a convex polyhedron with congruent faces consisting of regular polygons and the same number of faces meet at each vertex.

[2] A group G acts freely on X if stabilizers are trivial, explicitly: if $g \bullet x = x$ then $g = 1$.

B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 3

Comments and corrections are welcome: ritter@maths.ox.ac.uk

Exercise 1. Bump functions and embedding the Klein bottle in $\mathbb{R}^4$.

Recall from analysis that $\alpha(x) = e^{-1/x^2}$ is a function $\mathbb{R} \to \mathbb{R}$ (defining it to be zero at $x = 0$) which is infinitely differentiable, so smooth, but all the derivatives at $x = 0$ vanish! (so the Taylor series at 0 is useless)

⋄ Sketch the following functions (no need to justify your sketches):

(1) $\beta(x) = \alpha(x)$ for $x > 0$, and $\beta(x) = 0$ for $x \leq 0$.
(2) $\gamma(x) = \beta(x-a) \cdot \beta(b-x)$, where $0 < a < b$.
(3) $\delta(x) = \int_x^b \gamma(t)\, dt / \int_a^b \gamma(t)\, dt$.

⋄ The function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, $\varepsilon(x) = \delta(\|x\|)$ is called a *bump function compactly supported on the disc* $\|x\| \leq b$. Check that $\varepsilon(\mathbb{R}^n) \subset [0,1]$, that $\varepsilon = 1$ on $\|x\| \leq a$ and $\varepsilon = 0$ on $\|x\| \geq b$.

⋄ A figure 8 loop can be obtained as the image of a continuous map $f : S^1 \to \mathbb{R}^2$ which is injective except at $\pm 1 \in S^1 \subset \mathbb{C}$ where $f(+1) = f(-1)$ (so f is not an embedding). Using a bump function, show that the figure 8 loop can be continuously embedded into $\mathbb{R}^3$.

⋄ Show that the Klein bottle K can be smoothly embedded in $\mathbb{R}^4$.

⋄ **Optional harder question:** Show that all compact surfaces can be embedded in $\mathbb{R}^4$.

Exercise 2. Holomorphic maps between Riemann surfaces.

Using the local form of a holomorphic map between Riemann surfaces, deduce:
Open mapping theorem: any holomorphic map $f : R \to S$ between Riemann surfaces, with R connected, is either constant or an open map, meaning f(any open set) is open.[1]

Deduce the following, for $f : R \to S$ holomorphic, R, S Riemann surfaces:

(1) If f is non-constant, R compact connected, then $f(R) \subset S$ is a connected component.
(2) If f is non-constant, R, S both compact connected, then f is surjective: $f(R) = S$.
(3) If R is compact connected, S non-compact connected, then f is constant.
(4) A holomorphic map $S \to \mathbb{C}$ on a compact connected Riemann surface is constant.
(5) Fundamental theorem of algebra: non-constant complex polynomials have a root.

Exercise 3. Implicit function theorem.

Consider $R = \{(z, w) \in \mathbb{C}^2 : w^3 = z^3 - z\}$. Use the implicit function theorem to check that R is a Riemann surface. Now consider the projection

$$\pi : R \to \mathbb{C},\ \pi(z, w) = z.$$

Find the branch points of π. Find the valency $v_\pi(p)$ at the ramification points.

Next, we seek how many points are “missing” at infinity. Write $z^3 - z = z^3(1 - z^{-2})$ for large $|z|$, and briefly explain that there are three holomorphic solution functions to $w^3 = z^3 - z$. Deduce that $\pi^{-1}(\{z \in \mathbb{C} : |z| > 100\})$ is biholomorphic to three punctured discs.

Compute the Euler characteristic of R using the Riemann-Hurwitz formula. Deduce that R is homeomorphic to a torus with three points removed.

Please turn over.

Date: This version of the notes was created on December 8, 2014.

[1] *Hint. Notice that to show a map is open, it's enough to show that for each p, there are some nice arbitrarily small open neighbourhoods of p which map to open sets.*

Exercise 4. Tangent space and orientability of Riemann surfaces.

For an open set $U \subset \mathbb{R}^2$, the tangent space at $p \in U$ is defined as the vector space $T_pU = \mathbb{R}^2$. You can visualise $v \in T_pU$ as an arrow pointing out of $p \in U$ in the plane. You should view v as the velocity vector $c'(0)$ of a smooth curve $c : (-\varepsilon, \varepsilon) \to U$ through $c(0) = p$. There are many different smooth curves c with $c'(0) = v$, $c(0) = p$, so v is an equivalence class of such.

⋄ Write down the formula for a very simple curve $c : \mathbb{R} \to \mathbb{R}^2$ with $c(0) = p$, $c'(0) = v$. Deduce a simple formula for a curve which represents the vectors $v_1 + v_2$ and λv for $\lambda \in \mathbb{R}$. What does your formula become in components when v is a standard basis vector $e_1, e_2 \in \mathbb{R}^2$?

Let S be an (abstract) smooth surface. Given $p \in S$, we define T_pS as the collection of equivalence classes of smooth curves $c = c(t) : \mathbb{R} \to S$, defined for small $|t|$, with $c(0) = p$. Here $c \sim \widetilde{c}$ are equivalent if in some local coordinates $c'(0) = \widetilde{c}'(0)$.

⋄ Using curves, there is an obvious way to identify $T_uU \equiv T_pS$ when you are given a local parametrization $F : \mathbb{R}^2 \supset U \to S$ with $p = F(u) \in F(U)$. Show that the composite $T_uU \equiv T_pS \equiv T_{\widetilde{u}}\widetilde{U}$, for overlapping parametrizations $F, \widetilde{F}$, becomes the map $D_u(\widetilde{F}^{-1} \circ F) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ (left-multiplication by the matrix of partial derivatives of the transition).

⋄ Show that for S a Riemann surface, T_pS is a vector space with a well-defined linear isomorphism $J : T_pS \to T_pS$ with $J^2 = -\mathrm{Id}$ which in local coordinates is multiplication by i.

⋄ Consider the open Möbius band M from Exercise sheet 1, and consider the equator $f : [0, 1] \to M$, $t \mapsto (t, \frac{1}{2})$. Show that M cannot be a Riemann surface, by considering what J does to the tangent vector $f'(t) \in T_{f(t)}M$ as t varies from 0 to 1.

⋄ Show that for a Riemann surface S, for any vector $v \neq 0 \in T_pS$, the basis v, Jv is a right-handed basis in any local parametrization.

Cultural remark. *For a smooth map $f : S_1 \to S_2$ of surfaces you can now define the derivative map by*

$$D_pf \cdot [\textit{curve } c(t)] = [\textit{curve } f \circ c(t)],$$

and you can check that if you write f in local coordinates, this corresponds to the matrix of partial derivatives of f at p acting by left-multiplication on $c'(0) = v \in \mathbb{R}^2$. So

$$D_pf : T_pS_1 \to T_{f(p)}S_2$$

is a linear map between the tangent spaces. Example: $e_1, e_2 \in \mathbb{R}^2$ map to $\partial_x f, \partial_y f \in \mathbb{R}^2$.

One can also think of vectors as differential operators acting on smooth functions: for example, if $h : S \to \mathbb{R}$ is smooth, then locally $e_1 \cdot h$ means the partial derivative $\partial_x h \in \mathbb{R}$.

B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 4

Comments and corrections are welcome: ritter@maths.ox.ac.uk

Exercise 1. Riemann-Hurwitz formula.
In the following, all spaces are compact connected Riemann surfaces, and all maps are holomorphic maps. Deduce from the Riemann-Hurwitz formula that:

(1) if $f : R \to S$ is not constant, then the genus $g(R) \geq g(S)$.
(2) if $f : \mathbb{C}P^1 \to S$ is not constant, then S is homeomorphic to a sphere.
(3) if $f : R \to S$ has degree 1 then f is a biholomorphism.
(4) if R admits a meromorphic function with only one pole of order 1, then $R \cong \mathbb{C}P^1$.

Exercise 2. Meromorphic functions on Riemann surfaces.
Show that a map $f : S \to \mathbb{C}P^1$ is meromorphic if and only if locally f is expressible as a quotient of holomorphic functions (where the denominator is not identically zero).

Show that if f, g are two meromorphic functions on a compact connected Riemann surface having the same zeros and the same poles (including multiplicities) then $f = \text{constant} \cdot g$.

By comparing Taylor series of $\wp, \wp'$ near ramification points, deduce by the previous part (by viewing the two sides of the equation below as meromorphic functions) that:

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

where $e_1 = \wp(\frac{1}{2}\omega_1)$, $e_2 = \wp(\frac{1}{2}\omega_2)$, $e_3 = \wp(\frac{1}{2}(\omega_1 + \omega_2))$, $\infty = \wp(0)$ are the branch points of $\wp$.

Exercise 3. Elliptic curves and the Weierstrass $\wp$-function.
The goal is to prove that the following is a biholomorphism:

$$\begin{array}{rcl} \mathbb{C}/\Lambda & \to & S = \{(Z, W) \in \mathbb{C}^2 : W^2 = 4(Z - e_1)(Z - e_2)(Z - e_3)\} \cup \{\infty\} \\ z & \mapsto & (\wp(z), \wp'(z)) \end{array}$$

where on the right we compactify as done in Exercise Sheet 1. *Here is a checklist/hints:*

(1) Explain why e_1, e_2, e_3 are distinct,
(2) Show S is a Riemann surface. In particular, what is the local holomorphic coordinate?
(3) Explain why the map is well-defined,
(4) Show that the map is holomorphic (do this carefully, locally),
(5) For very general reasons, explain why the map has to be surjective,
(6) Show that the degree of the map is 1, and use Exercise 1.

Exercise 4. Covering maps.
Find a non-constant holomorphic map from a genus 3 surface to a genus 2 surface with no branch points.

(Hint. What degree must the map have? Start by finding a 2-to-1 map $T^2 \to T^2$, where $T^2 = S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$)

Cultural Remark. *A non-constant holomorphic map $f : R \to S$ between compact connected Riemann surfaces is a* covering map *in the sense that each small enough open set in S is covered via f by a disjoint union of copies of it in R (indeed $\deg(f)$ copies). Think of it as locally looking like a "stack of pancakes" over the "plate" in S. When there are branch points, $f : R \to S$ is called a* ramified covering map*: it fails to be a covering at ramification points.*

Date: This version of the notes was created on November 6, 2014.

B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 5

Comments and corrections are welcome: ritter@maths.ox.ac.uk

Exercise 1. Conformal and area-preserving parametrizations.

In this exercise, the first fundamental form has local matrix $A = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$.

Show that a local parametrization is conformal (i.e. angle-preserving) $\Leftrightarrow$ $f = 0$, $e = g$

Show that a local parametrization is area preserving $\Leftrightarrow$ $\det \begin{pmatrix} e & f \\ f & g \end{pmatrix} = 1$

Show that the stereographic projection is conformal, by considering the parametrization given by the inverse of the stereographic projection:

$$\mathbb{C} \to S^2 \setminus (\text{North Pole}),\ (x, y) \mapsto \frac{1}{1 + x^2 + y^2}(2x, 2y, -1 + x^2 + y^2),$$

and show that the fundamental form for the sphere in these local coordinates is

$$I_F = \frac{4(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}.$$

Cultural remark: That fundamental form on the sphere is called the *chordal metric*. It gives rise to another example of non-Euclidean geometry called *elliptic geometry*.

Exercise 2. Tangential derivatives and Christoffel symbols.

A vector field *is a smooth family of tangent vectors* $v \in TS$, *so locally*

$$v(x, y) = a(x, y)\partial_x F + b(x, y)\partial_y F \in T_{F(x,y)}S,$$

for some smooth functions a, b. *The derivatives of* v *in* x, y *may not lie in* TS, *they lie in* $\mathbb{R}^3 = T_{F(x,y)}S \oplus \mathbb{R}n(x, y)$ *where* n *is the unit normal (Gauss map). If we subtract the normal part, we obtain the* tangential derivative*:*

$$\nabla_x v = \partial_x v - (n \cdot \partial_x v)\, n \qquad \nabla_y v = \partial_y v - (n \cdot \partial_y v)\, n.$$

The symbol ∇ *is called* nabla, *and the operator* ∇ *is called a* connection *for the surface* S.

Using that $v \in TS$ is orthogonal to n, show that $-n \cdot \partial_x v = \partial_x n \cdot v$. Show that ∇ is *compatible* with the Riemannian metric I (the first fundamental form):

$$\partial_x\, I(v, w) = I(\nabla_x v, w) + I(v, \nabla_x w),$$

for any vector fields v, w.

We introduce some helpful notation: $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, $\nabla_1 = \nabla_x$, $\nabla_2 = \nabla_y$. *Abbreviate the basis of* TS *by*

$$X_1 = \partial_x F \qquad \text{and} \qquad X_2 = \partial_y F.$$

Abbreviate $X_{ij} = \partial_i \partial_j F$. *Writing* $\nabla_i X_j \in TS$ *in the basis* X_i, X_j, *yields coefficient functions* $\Gamma^k_{ij}(x, y)$, *called* Christoffel symbols*:*

$$\boxed{\nabla_i X_j = \sum_{k=1}^{2} \Gamma^k_{ij} X_k}$$

Verify the symmetry relations:

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Date: This version of the notes was created on November 26, 2014.

Now abbreviate by II_{ij} the entries of the second fundamental form $I = \left(\begin{smallmatrix} L & M \\ M & N \end{smallmatrix}\right)$, defined by

$$\boxed{II_{ij} = n \cdot X_{ij}}$$

Show that the normal part of the derivatives of X_j determines the second fundamental form:[1]

$$X_{ij} - \nabla_i X_j = II_{ij} n.$$

Let's abbreviate the entries I_{ij} of the first fundamental form $I = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)$ by:

$$\boxed{g_{ij} = I(X_i, X_j)}$$

Notice that $g_{ij} = g_{ji}$. Now prove that

$$X_{ij} \cdot X_\ell = I(\nabla_i X_j, X_\ell) = \sum_{k=1}^{2} \Gamma_{ij}^k g_{k\ell}.$$

Using the earlier compatibility result, and the above equation, show that

$$\partial_i g_{j\ell} = \sum_{k=1}^{2} (\Gamma_{ij}^k g_{k\ell} + \Gamma_{i\ell}^k g_{kj}).$$

Writing g^{ij} for the entries[2] of the inverse matrix $I^{-1} = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)^{-1}$, deduce that

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

Deduce that the Christoffel symbols are determined just by I, and therefore they are invariant under isometries between surfaces.

Exercise 3. Nautical cartography: Mercator's projection.

Using the parametrization

$$F(\theta, \phi) = (\cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi)$$

of the unit sphere $S^2 \subset \mathbb{R}^3$, find the first fundamental form.

To draw maps of the Earth, one often uses Mercator's projection of the unit sphere minus the date line:

$$(X, Y) = (\theta, \log\tan(\frac{\phi}{2} + \frac{\pi}{4})) \in \mathbb{R}^2,$$

where (θ, ϕ) are the longitude and latitude coordinates on the Earth. What does the first fundamental form of the sphere become in the coordinates (X, Y) of the plane? Deduce that Mercator's projection is conformal but not area-preserving. There are pictures of this on Wikipedia: http://en.wikipedia.org/wiki/Mercator_projection

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[1] In case you've lost the plot, an example should clarify:

$$X_{11} = \partial_x \partial_x F = \Gamma_{11}^1 \partial_x F + \Gamma_{11}^2 \partial_y F + L\, n$$

where L is the $(1,1)$-entry II_{11} of the second fundamental form II.

[2] *Hint.* $I^{-1} I = \mathrm{id}$ *becomes the orthonormality equation*

$$\sum_{j=1}^{2} g^{ij} g_{jk} = \delta_k^i$$

where δ_k^i equals 1 for $i = k$ and 0 for $i \neq k$.

B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 6

Comments and corrections are welcome: ritter@maths.ox.ac.uk

Exercise 1. The curvatures for a torus in $\mathbb{R}^3$.

Recall the torus in $\mathbb{R}^3$ given by

$$T^2 = \{((a + b\cos\psi)\cos\theta,\ (a + b\cos\psi)\sin\theta,\ b\sin\psi) :\ \text{all } \theta, \psi \in [0, 2\pi]\}$$

where $a > b > 0$ are fixed constants. Calculate the first fundamental form, the second fundamental form, the principal directions, the principal curvatures, the mean curvature and the Gaussian curvature K.

In a picture, shade the regions where K is positive, zero, and negative.

Exercise 2. Riemann curvature, Ricci curvature, scalar curvature.

Recall in Exercise sheet 5 we defined the tangential derivative, which in the basis $X_1 = \partial_x F$, $X_2 = \partial_y F$ defines the Christoffel symbols:

$$\nabla_i X_j = \Gamma^k_{ij} X_k \tag{*}$$

where from now on we use the Einstein summation convention*: you sum over repeated indices (so above, we sum over k since it appears once as an upper index and once as a lower index).*

The Riemann curvature tensor R^m_{ijk} *measures how much the tangential derivatives fail to commute. It is defined by:*

$$\boxed{\begin{aligned} R(X_i, X_j)X_k &= \nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k \\ &= R^m_{ijk} X_m \end{aligned}} \tag{**}$$

Cultural Remark. *We only defined R on the basis $X_i = \partial_i F$. The general formula is*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where the Lie bracket $[X, Y]$ *is defined by:* $[\sum v^i X_i, \sum w^j X_j] = \sum v^i \partial_i(w^j) X_j - \sum w^j \partial_j(v^i) X_i$, *which measures how much the flow of two vector fields fails to commute, and one studies this in* **C3.5 Lie Groups** *and secretly in* **C2.1 Lie Algebras**. *In our case:* $[X_i, X_j] = \partial_i(1)X_j - \partial_j(1)X_i = 0$ *since 1 is a constant coefficient.*

It's often useful to dot the above with another basis vector X_ℓ, which defines

$$\boxed{R_{ijk\ell} = R(X_i, X_j)X_k \cdot X_\ell = I(R(X_i, X_j)X_k, X_\ell)}$$

You can pass from one to the other by the lowering/raising of indices using the Riemannian metric $g_{ij} = I_{ij} = X_i \cdot X_j$. Explicitly $R_{ijk\ell} = R^m_{ijk} g_{m\ell}$, which you can undo by using the inverse matrix g^{ij} of g_{ij} which satifies $g^{ij} g_{jk} = \delta^i_k$ (summing over j).

By substituting (*) into (**), show that R is determined completely by the Christoffel symbols and the Riemannian metric $g_{ij} = I_{ij}$ (the first fundamental form):

$$\boxed{\begin{aligned} R^m_{ijk} &= \partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^p_{jk} \Gamma^m_{ip} - \Gamma^p_{ik} \Gamma^m_{jp} \\ R_{ijk\ell} &= (\partial_i \Gamma^m_{jk} - \partial_j \Gamma^m_{ik} + \Gamma^p_{jk} \Gamma^m_{ip} - \Gamma^p_{ik} \Gamma^m_{jp}) g_{m\ell}. \end{aligned}}$$

Date: This version of the notes was created on December 2, 2014.

Notice that by Exercise Sheet 5, the Γ_{ij}^k *are determined by* g_{ij}*, so* R *only depends on the Riemannian metric* g_{ij}*. Therefore* R *doesn't change under isometry, even if you pick a different isometric embedding of the surface into* $\mathbb{R}^3$*.*

Explain why $R_{ijk\ell}$ is anti-symmetric in the indices i, j.

Recall by Exercise Sheet 5 that

$$\partial_i I(v, w) = I(\nabla_i v, w) + I(v, \nabla_i w).$$

Since $I(v, w)$ is a smooth function, its partial derivatives commute: $\partial_j \partial_i I(v, w) = \partial_i \partial_j I(v, w)$. Deduce that $R_{ijk\ell}$ is anti-symmetric in k, ℓ.

Deduce that R is determined by just one value: R_{1212}.

Recall by Exercise Sheet 5 that

$$\partial_i X_j = \nabla_i X_j + II_{ij}\, n = \Gamma_{ij}^k X_k + II_{ij}\, n.$$

Since F is smooth, the partial derivatives commute: $\partial_2 \partial_1 \partial_1 F = \partial_1 \partial_1 \partial_2 F$, so $\partial_2 \partial_1 X_1 = \partial_1 \partial_1 X_2$. From this, and the above equation, deduce by brute force calculation that

$$0 = (\partial_2 \partial_1 X_1 - \partial_1 \partial_1 X_2) \cdot X_2 = R_{2112} - \det II_F.$$

Deduce, using $K = \frac{\det II_F}{\det I_F}$ (from lectures), that

$$\boxed{R_{1212} = -K \det I_F}$$

Finally, deduce Gauss' *Theorema Egregium*: the Gaussian curvature K only depends on the Riemannian metric, so it is the same for two isometric surfaces.

The Ricci curvature *is defined as the metric trace of* $R_{ijk\ell}$ *in the* j, ℓ *indices, explicitly:*

$$R_{ik} = R_{ijk\ell} g^{j\ell} = R_{ijk}^j$$

and the scalar curvature *is the metric trace of the Ricci curvature, explicitly:*

$$R = g^{ij} R_{ij}$$

as usual summing over repeated indices.

Show that for surfaces in $\mathbb{R}^3$,

$$\boxed{\mathrm{R}_{ij} = -K g_{ij} \quad \text{and} \quad R = -2K}$$

Cultural Remark. *The above ideas are very important, for example the Einstein field equations for general relativity are*

$$R_{ij} - \frac{1}{2} g_{ij} R + g_{ij} \Lambda = \frac{8\pi G}{c^4} T_{ij}$$

where the left-hand side encodes the geometry of the universe and the right-hand side encodes the physical properties of the universe. The symbols are: G *= Newton's gravitational constant,* c *= speed of light,* Λ *= cosmological constant,* T_{ij} *= stress-energy tensor (which measures the matter/energy content of spacetime). More of this in* **C7.5/C7.6 General Relativity***.*

In a vacuum, these equations become $R_{ij} = 0$ *when the cosmological constant is zero, and* $R_{ij} = \Lambda g_{ij}$ *(so a multiple of the metric) otherwise. Manifolds with a vanishing Ricci tensor are called* Ricci-flat manifolds*, and manifolds with a Ricci tensor proportional to the metric are called* Einstein manifolds*. They are objects of great interest nowadays in geometry.*